Paper 263-28

Multilevel Designs and Their Analyses

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ABSTRACT

Multilevel designs are used by researchers in many areas (Raudenbush and Bryk (2002) and Milliken and Johnson (1992)) and those researchers have used different terminologies to describe the designs. Multilevel designs are designs that involve more than one size of experimental unit, thus the name, multilevel. Split-plot designs are multilevel designs and have been used by physical and biological science researchers since the 1930s. Hierarchical designs and now multilevel designs have been used by social science researchers the past few years. The structures are identical and the models needed to describe resulting data are identical. This presentation provides a general frame work within which to identify and then analyze multilevel designs whether called split-plot, hierarchical, or multilevel. Repeated measures designs are multilevel designs. The strip-plot is also a multilevel design that is not a hierarchical design and is a structure not generally considered by those using multilevel designs in the social sciences. A collection of examples is used to demonstrate the similarities and differences of various designs and the needed analyses. All of the models required to provide appropriate analyses of these designs are members of the class of mixed models. Proc Mixed of the SAS® system is used to fit all of the models.

INTRODUCTION

Different terminologies have evolved for describing designs that involve more than one size of experimental unit or sampling unit. The goal of this presentation is to provide a unified structure that can be used to bring the varying descriptions together. The three main terminologies correspond to those used to describe (1) split-plot or repeated measures types of designs, (2) hierarchical types of designs and (3) multi-level types of designs. A unified structure is first described and then each of these designs are fit into that structure,

BASICS OF DESIGNED EXPERIMENTS USING BLOCKS

The discussion centers abound the design of experiments or observational studies. The unified structure involves classifying the factors in an experiment or study as belonging to the treatment structure or design structure (Milliken and Johnson (1992)). The treatment structure consists of those factors in the experiment that were selected to be studied and should have an influence on the response of interest. The design structure of a study consists of the factors used to form groups or blocks of the experimental units. Examples of treatment structures are one-way, two-way factorial arrangement, three-way factorial arrangment, 2^n factorial arrangement, 2^{n-p} fractional factorial, a set of combinations obtained for an optimal design, a two-way factorial plus a control, etc. Examples of design structures are completely randomized (CR) design, randomized complete block (RCB) design, incomplete block design (ICB), etc. Two important assumptions about the relationship between elements of the treatment structure and elements of the design structure are (1) there are no interactions between the factors in the treatment structure and the factors in the design structure and (2) the levels of the factors in the design structure are random effects. The first assumption is very important and is often overlooked by most researchers when factors are selected for forming blocks. Most text books define blocking factors as those that are not of interest in the current study. But one has to be more careful and make sure the blocking factors will not interact with the factors in the treatment structure. The second assumption implies there is a population of experimental units to which inferences are to made. As a consequence of the second assumption, all necessary models are mixed models unless the design structure is CR.

DESIGNS WITH ONE SIZE OF EXPERIMENTAL UNIT

Designs with one size of experimental unit can be formed with any of the design structures, i.e., CR, RCB, or ICB, and are called one level designs. Suppose the treatment structure consists of four treatments and the design structure consists of twelve experimental units. Each of the four treatments can be assigned to three experimental units, thus there can be three replications of each treatment. The CR design is constructed by randomly assigning each of the treatments to three of the experimental units, as demonstrated in Figure 1. A model one could use to describe data from a CR design is

$$y_{ij} = \mu + \tau_i + \varepsilon_{ij}, \quad i = 1,2,3,4\,, j = 1,2,3, \qquad \text{where } \varepsilon_{ij} \sim N(0, \sigma_\varepsilon^2), \tag{1}$$

μ denotes the mean of the response, τ_i denotes the effect of the ith treatment and ϵ_{ij} is the experimental unit error.

For the RCB design structure, construct three blocks of four experimental units and then randomly assign the treatments to an experimental unit within each of the blocks. A graphic demonstrating the construction of a RCB design structure in displayed in Figure 2. A model one could use to describe data from a RCB design is

$$y_{ij} = \mu + \tau_i + b_j + \varepsilon_{ij}, \quad i = 1,2,3,4\,, j = 1,2,3, \qquad \text{where } b_j \sim N(0, \sigma_{blk}^2),\ \varepsilon_{ij} \sim N(0, \sigma_\varepsilon^2), \tag{2}$$

and in addition to the terms described for model 1, b_j represents the random block effect. The estimate of σ_ϵ^2 is computed from the treatment by block interaction. This computation implies it is very important that the elements of the treatment structure must not interact with the elements of the design structure.

Incomplete block design structures occur when the block size is smaller than the number of treatments in the treatment structure, i.e., not all treatments can occur in each block or an incomplete set of treatments occur in each block, thus, the name. Figure 3 contains a graphic demonstrating the construction of an incomplete block design structure with blocks of size 3 and Figure 4 contains the graphic for an incomplete block design structure with blocks of size 2. An incomplete block design is constructed by selecting a block size and determine the number of blocks needed in the study. Treatment patterns, one for each possible block, are determined to provide the appropriate relationship among the treatments. One such relationship is to select a pattern such that each pair of treatments occur together within a block an equal number of times and such an arrangement is called a balanced incomplete block design structure. A model one could use to describe data from a ICB design is

$$y_{ij} = \mu + \tau_i + b_j + \varepsilon_{ij}, \quad i,j \in Q(i,j)$$
$$where \ b_j \sim N(0, \sigma^2_{blk}) \ and \ \varepsilon_{ij} \sim N(0, \sigma^2_{\varepsilon}). \tag{3}$$

where Q(i,j) contains the set of indices indicating which treatments occur in each block. The next step is to randomize the groups of treatment patterns or assignments to the blocks of experimental units. The final step is to randomly assign the specified treatments to the experimental units within the block. Table 1 contains the pattern of treatments to be assigned to four blocks of size three (ICB(3)) and the block number assigned to each pattern. The graphic in Figure 3 represents the assignment process. Table 2 contains the pattern of treatments to be assigned to six blocks of size 2 (ICB(2)) and the block number assigned to each pattern. The graphic in Figure 4 represents the assignment process for six blocks of size two. It is important to remember that the treatment structure is the same for each of these design structures and each treatment is assigned to three experimental units providing three replications of each. The differences among the designs are in the number of observations per block. The CR, RCB, ICB(3) and ICB(2) design structures have block sizes of twelve, four, three and two respectively. The effects of the different numbers of blocks in the designs is seen in the resulting analysis of variance tables. Table 3 contains the analysis of variance tables for the four designs where the degrees of freedom for error are 8, 6, 5, and 3 respectively for the CR, RCB, ICB(3) and ICB(2) design structures. The eight degrees of freedom for error from the CR design structure are obtained by pooling the variability of experimental units treated alike across the four treatments. Each treatment is assigned to three experimental units, thus there are two degrees of freedom for error available from each of the treatments. If the variances are equal, these four sets of two degrees of freedom can be pooled to provide eight degrees of freedom for error. When the design structure consists of more than one block, the error is computed from the block by treatment interaction or design structure by treatment structure interaction. Since the error term is computed as an interaction, it is very important that the design structure factors not interact with the treatment structure factors. The error degrees of freedom for the RCB design structure are computed as the block by treatment interaction which has (4-1)*(3-1)=6 degrees of freedom. For the incomplete block designs, the error degrees of freedom are computed by (number of treatments - 1)(number of blocks - 1) - number of empty cells. The number of error degrees of freedom for the design with 4 blocks of size 3 is (4-1)(4-1)-4=5 degrees of freedom. The number of error degrees of freedom for the design with 6 blocks of size 2 is (4-1)(6-1)-12=3 degrees of freedom. Each analysis of variance table has three degrees of freedom for the treatment structure and eight degrees of freedom for the error and design structures. The models for the designs with more than one block are mixed models when the levels of the treatments are considered to be fixed effects.

MUTLILEVEL - PART 1

Suppose data were collected on junior high school students where the response variable is the number of minutes of exercise at a moderate level performed each week. The grade point average during the past term was also measured with the thought that the amount of exercise could be linearly related to a students GPA. A model that could be used to describe this relationship is

$$EXER_i = \alpha + GPA_i \ \beta + \varepsilon_i, i = 1,2,\ldots,n \tag{4}$$
$$\varepsilon_i \sim N(0, \sigma^2_{person})$$

where α is the intercept and β is the slope of the regression model. This model might be appropriate, but it was discovered that the students in the sample were from several schools. Because there are different elements concerning whether to do exercise at each of the schools, it was thought that possibly the intercept and slope of the regression line might be different for each of the schools. If the schools in the study represent a random sample of schools from a population of schools, the intercepts and slopes from each of the schools could be expressed as models,

$$a_k = \alpha + u_k, k = 1,2,\ldots,S$$
$$b_k = \beta + v_k, k = 1,2,\ldots,S \tag{5}$$
$$for \begin{bmatrix} a_k \\ b_k \end{bmatrix} \sim N\left[\begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \begin{bmatrix} \sigma^2_a & \sigma_{ab} \\ \sigma_{ab} & \sigma^2_b \end{bmatrix}\right]$$

or

$$\begin{bmatrix} a^*_k \\ b^*_k \end{bmatrix} = \begin{bmatrix} a_k - \alpha \\ b_k - \beta \end{bmatrix} \sim N\left[\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma^2_a & \sigma_{ab} \\ \sigma_{ab} & \sigma^2_b \end{bmatrix}\right] \tag{6}$$

Using the relationships in equations 5 and 6,the model can be expressed as

$$EXER_{ij} = \alpha + a^*_i + GPA_{ij} \ \beta + GPA_{ij} \ b^*_i + \varepsilon_{ij},$$
$$i = 1,2,\ldots,S, \ j = 1,2,\ldots,r_i, \tag{7}$$
$$\begin{bmatrix} a^*_i \\ b^*_i \end{bmatrix} \sim N\left[\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma^2_a & \sigma_{ab} \\ \sigma_{ab} & \sigma^2_b \end{bmatrix}\right] and \ \varepsilon_{ij} \sim N(0, \sigma^2_{perspm})$$

The above model is called a random coefficients regression model and is a 2 level model since the a_i* and b_i* represent school effects and the ε_{ij} represents the student effect. The following PROC MIXED code can be used to fit the model:

```
proc mixed cl covtest;
class school;
model exer = gpa / ddfm=kr solution;
random int gpa/type=un subject=school;
```

The fixed effects are in the model statement and in this case represent the population regression model, α + β GPA. The random statement is used specify to the covariance structure among the random slopes and intercepts (a_i* and b_i*) , which is accomplished by using Type=UN. This model was constructed in an observational study frame work. The next multilevel model is constructed in a design of experiment setting.

MULTILEVEL - PART 2

Consider a design with two treatments in the treatment structure and 8 blocks of size two experimental units in the design structure. This setup enables one to carry out a RCB design structure and the graphical representation of the assignment of treatments to the two experimental units within each block is displayed in Figure 5. The model that can be used to analyze this data is a model for a one-way treatment structure in a RCB design structure, as displayed in equation (2).

Now suppose that this study is set up to investigate the effect of teaching methods where the treatments are the two teaching methods and the blocks are different schools. The response variable is the class room average on a specific standardized test. A model that can be used to describe this data is

$$y_{jk} = \mu + s^*_j + \tau_k + c_{jk}, \tag{8}$$
$$j = 1,2,\ldots 8, \quad k = 1,2$$
$$s^*_j \sim N(0, \sigma^{*2}_{School}) \ and \ c_{ijk} \sim N(0, \sigma^2_{class})$$

The analysis of variance table is in Table 4, where ϕ^2 denotes the non-centrality parameter, which is zero when the treatment effects are equal. In addition, suppose that four of the schools are from large school districts and four are from small school districts. Schools cannot be randomly assigned to a size of school district, instead they are selected from within a size of school district. Assume that the four large school districts were randomly selected from the population of large school districts and the four small school districts were randomly selected from the population of small school districts. Next, randomly select one school from each district. At this point there is a design associated with the schools included in the study. Figure 6 is a graphic demonstrating that the eight schools represent a study with size as a factor with two levels. Measure the response as the average of the classes standardized test scores. A model that can be used to provide the analysis of this school level data, like model (1), is

$$s^*_{ij} = \mu + \varphi_i + s^+_{ij} \;, 1 = 1,2, \quad k = 1,2,3,4 \qquad (9)$$
$$s^+_{ij} \sim N(0, \sigma^{2+}_{School})$$

where Φ_i represents the effect of the district size. Table 5 contains the analysis of variance table for the analysis involving the school level data. For this part of the analysis, the school is the experimental unit for comparing the two levels of size of district. It is the variation among schools within a size of district pooled across districts that is used as the gauge to evaluate the effect of the different school sizes on the response (the equal variance assumption should be verified before the variances are pooled). Finally, randomly assign the two treatments to two classes within each school as shown in Figure 7. At this point, there are two "LEVELS" of analysis. The first level is that associated with the class within a school and its analysis is in Table 4, that is, the class is the experimental unit for comparing teaching methods. The second level of analysis is that of the school and its analysis is in Table 5, that is, the school is the experimental unit for comparing the sizes of districts.
It is desired to have one analysis that incorporates the two levels of data or two sizes of experimental units. The two level model can be constructed by substituting the model for s^*_{ij} in equation (9) into the model in equation (8). These models are called multilevel or hierarchical design models with a two-way treatment structure. The hierarchical structure occurs because of the nesting in the design structure, i.e., classes are nested within schools. Also, since there are two factors in the study, there is the possibility that the two factors interact, i.e., the treatment by size interaction needs to be included in the model. The interaction effect is a within school or between class comparison. A two level model that can be used to represent this data is

$$y_{ijk} = \mu + \varphi_i + s_{ij} + \tau_k + (\varphi\tau)_{ik} + c_{ijk}\,,$$
$$i = 1,2, \quad j = 1,2,3,4, \quad k = 1,2 \qquad (10)$$
$$s_{ij} \sim N(0, \sigma^2_{School}) \text{ and } c_{ijk} \sim N(0, \sigma^2_{class})$$

where $(\Phi\tau)_{ik}$ represents the interaction between the levels of size of district and the levels of treatment. Table 6 contains the analysis of variance table for model (10), which can be obtained using the following PROC MIXED code:

```
proc mixed cl covtest;
class size treatment school;
model y=size treatment
        size*treatment/ddfm=kr;
random school(size);
```

This model has two levels and thus has two levels of analysis. The level 1 analysis corresponds to the analysis based on the classes as the experimental units. Level 1 models are for the analysis of data from the smallest experimental unit in the study. The level 2 analysis corresponds to the analysis based on the schools as experimental units. The level 2 models are for the analysis of data from the next to smallest size of experimental unit. The experimental units for level 2 models are the blocks of level 1 experimental units. So, level 1 experimental units are nested within the level 2 experimental units. For the current study, classes are the level 1 units, which are nested within the school, the level 2 experimental unit.

SPLIT-PLOT DESIGNS

The construction of a split-plot design uses some what of a different approach than that used by the construction of multilevel designs. As with the two-level design described in the previous section, a two-way treatment structure is used for the split-plot. Construct the two-way treatment structure by crossing two levels of factor A and three levels of factor B. The design structure used in this part of the discussion consists of six blocks of size three. The randomization process starts with randomly assigning each level of A to three blocks of experimental units. The second step of the randomization process is to randomly assign the levels of B to the three experimental units within each of the blocks. The randomization process is graphically demonstrated in Figure 8. The design structure consists of blocks of size three, but there are six treatment combinations, so the resulting design structure is an incomplete block. The block of three experimental units are the entities to which the levels of A are randomly assigned and observed and are the experimental units for comparing the levels of A. The blocks of three experimental units are generically called whole-plots and the levels of A are called the whole-plot treatments. The individual experimental units are the entities to which the levels of B are randomly assigned and observed. The individual experimental units are call the sub-plots or split-plots and the levels of B are called the sub-plot treatments. A model that can be used to described the split-plot data is

$$y_{ijk} = \mu + \varphi_i + wp_{ij} + \tau_k + (\varphi\tau)_{ik} + sp_{ijk}\,, \qquad (11)$$
$$i = 1,2, \quad j = 1,2,3, \quad k = 1,2,3$$
$$wp_{ij} \sim N(0, \sigma^2_{wp}) \text{ and } sp_{ijk} \sim N(0, \sigma^2_{sp})$$

where wp_{ij} represents the whole plot error and sp_{ijk} represents the sub plot error. Table 7 contains the analysis of variance table for the model (11). There are two error terms, one for the whole plots and one for the sub-plots. The entries in Table 7 can be constructed by carrying out two analyses, whole plot analysis and sub-plot analysis, and then combining the two. The whole plot part of the analysis starts by considering the structure in Figure 8, but ignore the individual experimental units and the levels of B. Figure 9 contains a graphical representation of the whole plot analysis, which is a representation of a one-way treatment structure in a CR design structure. A model that can be used to describe the block means is

$$y_{ijk} = \mu + \varphi_i + wp^*_{ij} \;, i = 1,2, \quad j = 1,2,3 \qquad (12)$$
$$wp^*_{ij} \sim N(0, \sigma^2_{wp^*})$$

Table 8 contains the analysis of variance table for model (12) where the whole plot error is a measure of the variability of the blocks within a level of A, pooled over the levels of A. Thus there are 4 degrees of freedom associated with the whole plot error. The sub-plot analysis is accomplished by using the structure in Figure 8, but ignore the application of the levels of A. Figure 10 contains the graphical representation of the sub-plot part of the model, which represents a one-way treatment structure in a RCB design structure. A model to describe the sub-plot part of the design is

$$y_{ijk} = \mu + wp^*_{ij} + \tau_k + sp^*_{ijk}\,, \qquad (13)$$
$$i = 1,2, \quad j = 1,2,3, \quad k = 1,2,3$$
$$sp^*_{ijk} \sim N(0, \sigma^2_{sp^*})$$

Table 9 contains the analysis of variance table for model (13). The five degrees of freedom for the sum of squares due to blocks is the five degrees of freedom for the whole plot analysis in Table 8. The residual sum of squares consists of the block by B interaction. If all blocks were identical, the residual sum of squares would be the sub-plot error sum of squares. But, some blocks are assigned to level A1 and other blocks are assigned to level A2. Thus, the A by B interaction is contained in the residual sum of squares. Subtracting the A by B interaction sum of squares from the residual sum of squares provides the sub-plot error sum of squares. Another way to obtain the sub-plot error sum of squares is to consider that part of the design to which A1 has been assigned and provide an analysis to compare the levels of B at A1. Figure 11 is a graphical representation of the part of the design with just those blocks assigned to A1. The data in Figure 11 consists of a one-way treatment structure in a RCB design structure and a model to represent this part of the data is

$$y_{1jk} = \mu + wp_{1j} + \tau_k + sp_{1ijk}, \quad j = 1,2,3, \quad k = 1,2,3 \qquad (14)$$

$$sp_{1jk} \sim N(0, \sigma^2_{sp})$$

Table 10 contains the analysis of variance table for model (10). The sub-plot error sum of squares is the block by B interaction and is based on four degrees of freedom. Next compute the block by B interaction for each of the levels of A. After justifying the equal variance assumption, pool these sub plot error sums of squares. Thus, the sub-plot error sum of squares is obtained by pooling the B by block interaction pooled across the levels of A and is denoted by Block*B(A). Combining the results from Tables 8, 9 and 10 provides the analysis of variance in Table 7. The following Proc Mixed Code can be used to obtain the analysis of variance table in Table 7:

```
proc mixed cl covtest;
class A wp B;
model y=a b a*b;
random wp(a);
```

The whole plot error term is obtained from the variation among whole plots within a level of A and is computed by using WP(A). The split-plot designs have two sizes of experimental units and the sub-plots are nested within the blocks or whole plots, which is the structure of the hierarchical design or multilevel design discussed above. The whole plot analysis corresponds to the level 2 analysis and the sub-plot analysis corresponds to the level 1 analysis. Thus the structures of the two designs are identical. Generally the multilevel designs are the result of a stratified sampling process where the smallest strata correspond to the level 1 part of the model and the next size strata the level 2 part of the model. The split-plot designs are generally constructed in a designed experiment setting where experimental units can be grouped into blocks to form the whole plots.

REPEATED MEASURES DESIGNS

Repeated measures designs have structures identical to the split-plot and the multilevel designs. The main difference is that at some point in the process of assigning treatments to the experimental units, the levels of some factor cannot be randomly assigned. When a person is measured each week for 10 weeks and it is of interest to study the effect of some treatment over time, then time becomes a factor in the treatment structure. A person corresponds to the whole plot and a one-week time interval corresponds to the sub-plot. But, the levels of time cannot be randomly assigned to the 10 one-week time intervals for a person. Because of this non randomization of assignment, this design is called a repeated measures design. The importance of the repeated measures analysis is that the correlation structure of the repeated measurements within a person needs to be estimated (Littel, et al(1996)). For the split-plot and multi-level designs, the assumption that the observations within a whole plot are equally correlated is adequate for many situations. The analysis of variance tables were constructed using the equal correlation assumption.

MULTILEVEL - PART 3

Now suppose for the study involving class rooms from schools from two school district sizes that the data consist of the individual student test scores. It is also of interest to determine if sex of student has an effect on the resulting test scores. A level 1 model is

$$y_{ijkmn} = W_{ijk} + \rho_m + p_{ijkmn}, i = 1,2, \ j = 1,2,3,4, \\ k = 1,2, m = 1,2, n = 1,2,\ldots, r_{ijkm}, \qquad (15)$$

$$p_{ijkmn} \sim N(0, \sigma^2_{person})$$

where W_{ijk} is the mean response from size i, school j, and treatment k, ρ_m is the effect of sex m and p_{ijkmn} is the person random error. A model for the W_{ijk} or the level 2 model is

$$W_{ijk} = M_{ij} + \tau_k + c_{ijk}, i = 1,2, j = 1,2,3,4, k = 1,2 \qquad (16)$$

$$c_{ijk} \sim N(0, \sigma^2_{class})$$

where M_{ij} is the mean response for size i and school j, τ_k is the effect of treatment k and c_{ijk} is the class random error. A model for M_{ij} or a level 3 model is

$$M_{ij} = \mu + \theta_i + s_{ij}, i = 1,2, j = 1,2,3,4. \qquad (17)$$

$$s_{ij} \sim N(0, \sigma^2_{school})$$

where μ is the overall mean, θ_i is the effect of district size i, and s_{ij} the school random error. The first step in constructing the three level model is to replace M_{ij} in the class or level 2 model (16) with the school or level 3 model (17). The interaction between size and treatment needs to be added to provide the class level or level 2 model

$$W_{ijk} = \mu + \varphi_i + s_{ij} + \tau_k + (\varphi\tau)_{ik} + c_{ijk}, i = 1,2, j = 1,2,3,4, k = 1,2 \qquad (18)$$

$$s_{ij} \sim N(0, \sigma^2_{school}) \text{ and } c_{ijk} \sim N(0, \sigma^2_{class})$$

The final step is to replace W_{ijk} in the person or level 1 model (15) with the representation in the level 2 model (18) and add the interactions between the factors associated with the person part of the model and the factors of the new level 2 model. Thus, add the sex by treatment, sex by size, and sex by treatment by size interactions to obtain the final 3 level model :

$$y_{ijkmn} = \mu + \varphi_i + s_{ij} + \tau_k + (\varphi\tau)_{ik} + c_{ijk} + \rho_m + (\theta\rho)_{im} \\ + (\tau\rho)_{km} + (\theta\tau\rho)_{ikm} + p_{ijkmn}, \\ i = 1,2, j = 1,2,3,4, k = 1,2, m = 1,2, n = 1,2,\ldots, r_{ijkm} \\ s_{ij} \sim N(0, \sigma^2_{school}), c_{ijk} \sim N(0, \sigma^2_{class}) \\ \text{and } p_{ijkmn} \sim N(0, \sigma^2_{person}) \qquad (19)$$

This model can be fit using the following Proc Mixed code:

```
proc mixed cl covtest;
class size school treatment class sex;
model y= size treatment size*treatment
        sex size*sex treatment*sex
        size*treatement*sex / ddfm=kr;
random school(district)
        treatment*school(district);
```

The factorial effects for size, treatment and sex and all of their interactions are included in the model as fixed effects. The random statement is used to specify the school error (`school(district)`) and the class error (`treatment*school(district)`). The person error is relegated to the residual part of the model. This model involves three sizes of experimental units. The schools are the large size experimental units, the classes are the middle size experimental units, and the students are the small size experimental units. The students are nested within the classes which are nested within the schools. Thus this is a hierarchical design or a three level design. The split-split-plot design is also a three level design as the following construction demonstrates.

SPLIT-SPLIT-PLOT DESIGN

Assume there are 64 students and they are assigned to 16 classes of four students where each class consists of two female and two male students (small class sizes are used here to enable the graphic to be constructed). Combine two classes of students to form a school. At this point there are 16 classes or blocks of four students and eight pairs of classes to form schools. Figure 7 is a graphical representation of the arrangements of students into classes and then into schools. At this point, students cannot be randomly assigned to schools and schools cannot be randomly assigned to size of school district, but assume such a process could happen. Then the randomization process is to randomly assign students to the two classes within a school and randomly assign schools to school district size. Finally, randomly assign the two treatments to the two classes within each school, as shown in Figure 7. The school is the experimental unit for size of school district. The class is the experimental unit for the level of treatment. The student is the experimental unit for the level of sex. In the split-split-plot terminology, the school is called the whole plot and sizes of districts are the levels of the whole plot treatment, the classes is called the sub-plot and the two treatments are the levels of the sub-plot treatments, and the students are called the sub-sub-plots and the two sexes are called the levels of the sub-sub-plot treatment. The model for this data set is identical to that of the three level model discussed in the previous section. The corresponding analysis of variance table is in Table 11, which is obtained using the Proc Mixed code for the above 3 level model (19).
Each level of the multilevel design corresponds to a unit in the split-plot design where the level 1 units correspond to the smallest experimental units, level 2 units correspond to the next size of experimental unit, level 3 units correspond to the next size of experimental unit, etc. Thus, the structure of the three level design is identical to that of a split-split-plot design.

STRIP-PLOT DESIGN

The final design structure is the strip-plot which begins by grouping the experimental units into rectangular arrays. The rectangular arrays are blocks of experimental units. A two-way treatment structure needed and the one for this example has factor A with three levels and factor B with four levels. The rectangles need to have three rows and four columns (or four rows and three columns). Randomly assign the levels of A to the rows of each rectangle and then randomly assign the levels of B to the column of each rectangle. The randomization process is graphically displayed in Figure 12 where two three by four rectangles are used in the demonstration. This design has three sizes of experimental units. The rows of a rectangle are the experimental units for comparing the levels of A. The columns of a rectangle are the experimental units for comparing the levels of B. An interaction comparison is free of the row effects and free of the column effects, thus the individual cell is the experimental unit for measuring the effect of the A*B interaction. The following process can be used to construct an appropriate model. If you ignore the application of the levels of B, the resulting design involving the levels of A is a one-way treatment structure in a RCB design structure where the rows are the experimental units as displayed in Figure 13. The analysis of variance table is displayed Table 12 The rectangle by A interaction measures the row error. If you ignore the application of the levels of A, the resulting design involving the levels of B is a one-way treatment structure in a RCB design structure where the columns are the experimental units as displayed in Figure 14. The analysis of variance table is displayed in Table 13. The rectangle by B interaction provides the sum of squares for the column error. The remaining part of the analysis corresponds to the A*B interaction. The rectangle by A by B interaction provides the cell error. The complete analysis of variance table is obtained by combining Tables 12 and 13 to provide Table 14. A model that can be used to describe data from a strip-plot design structure is:

$$
\begin{aligned}
&y_{ijk} = \mu + b_i + \alpha_j + r_{j(i)} + \beta_k + c_{k(i)} + (\alpha\beta)_{jk} + \varepsilon_{ijk} \\
&i = 1,2,\ldots,R,\ j = 1,2,3,\ \ k = 1,2,3,4, \\
&b_i \sim N(0,\sigma^2_{rect}),\ r_{j(i)} \sim N(0,\sigma^2_{row}),\ c_{k(i)} \sim N(0,\sigma^2_{col}), \\
&and\ \varepsilon_{ijk} \sim N(0,\sigma^2_{cell}),
\end{aligned}
\tag{20}
$$

The following Proc Mixed Code can be used to fit the strip-plot model:

```
proc mixed cl covtest;
class rect a b;
model y=a b a*b / ddfm=kr;
random rect a*rect b*rect;
```

The fixed effects of A, B, and A*B are specified in the model statement and the random statement is used to specify the row error and the column error. The strip-plot model is not a hierarchical model, but it is a multilevel design. The experimental units are such that the rows are nested within rectangles and the columns are nested within rectangles, but the rows and columns within a rectangle are cross, i.e., one in not nested within the other, a requirement for the model to be of the hierarchical structure.

REPEATED MEASURES – PART 2

Any of the above design structures can involve repeated measures, i.e., for any design structure one or more sizes of experimental units could have the treatments assigned in a non random fashion. For the split-split-plot design structure the repeated measures could occur on the whole-plot, the sub-plot and/or the sub-sub-plot. When modeling the correlation structure among the repeated measurements with Proc Mixed, the `REPEATED` statement is used when the repeated measurements are on the smallest size of experimental unit. When the repeated measurements are on an experimental unit other than the smallest size, the `RANDOM` statement is used to specify the correlation structure. An example of an experiment where the repeated measurement is on an intermediate size experimental unit is included in Chapter 16 of Milliken and Johnson (2002).

CONCLUSION

Researchers from different areas have developed terminology to describe studies that involve more than one size of experimental unit. The really important aspect of these designs is to be able to identify the fact that more than one size of experimental unit is involved in the study. Once the different sizes of experimental units are identified and the design structure associated with each is recongized, an appropriate model can be constructed. At this point it does not make any difference if you call the design a split-plot design, a multilevel design, a hierarchical design or a between subjects design. You just incorporate the information you have about the different sizes of experimental units into the analysis.

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Milliken, George A. And Johnson, Dallas E. (1992) *Analysis of Messy Data: Vol I, Design Experiments*, Chapman & Hall/CRC, London.
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Raudenbush, S.W. and Bryk, A. A. (2002) *Hierarchical Linear Models-Applications and Data Analysis Methods, 2nd Edition*. Sage Publications, Thousand Oaks, CA.

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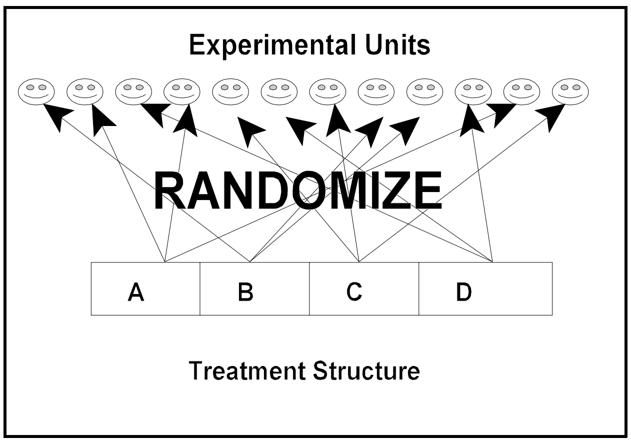

Figure 1 Graphic of CR design structure.

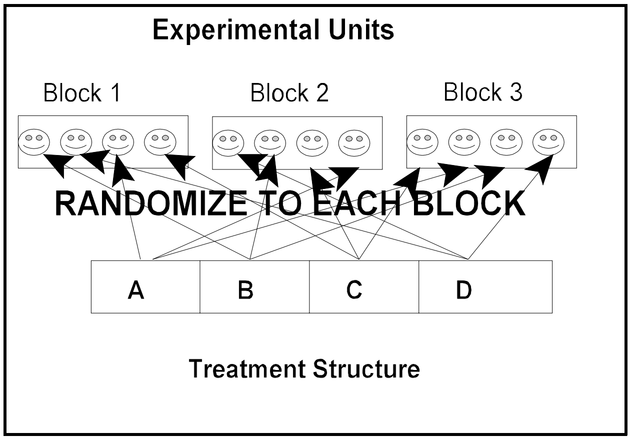

Figure 2 Graphic of RCB Design Structure

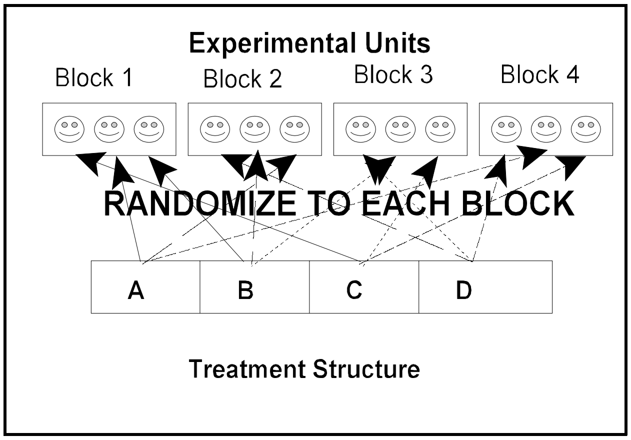

Figure 3 Graphic of ICB Design Structure with blocks of size 3.

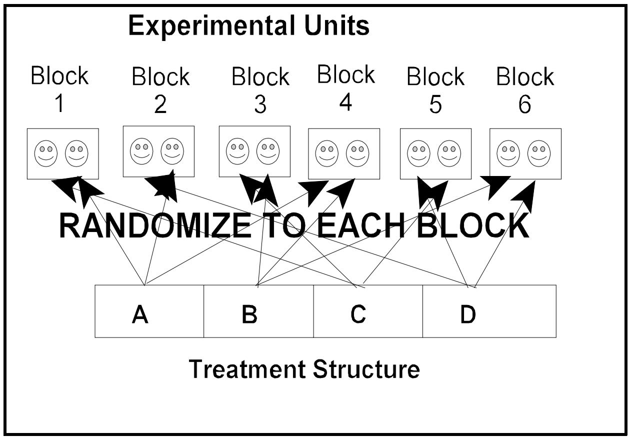

Figure 4 Graphic of ICB design structure with blocks of size 2.

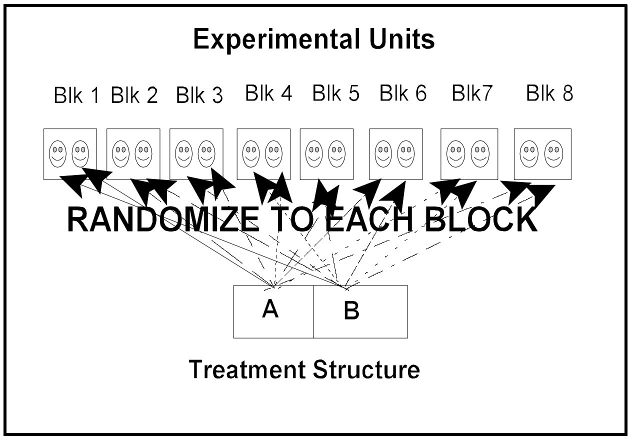

Figure 5 Graphic of RCB with two treatments and eight blocks.

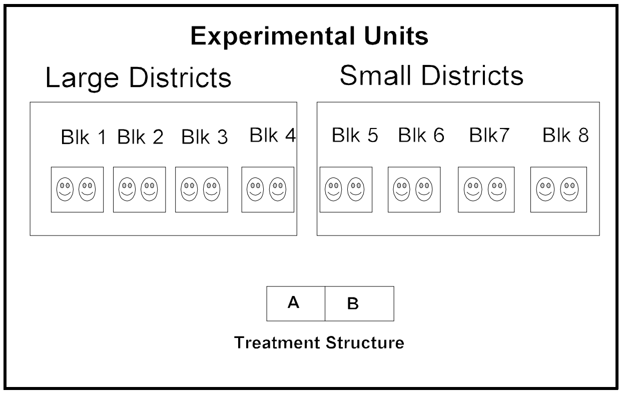

Figure 6 Graphic of CR with two districts and four schools per district.

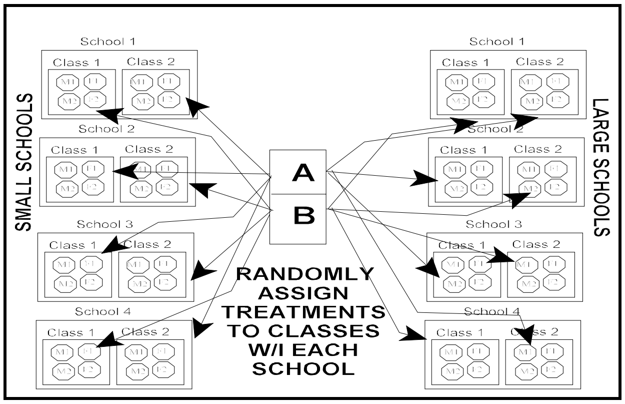

Figure 7 Graphic of assignment process of treatments to classes within each school.

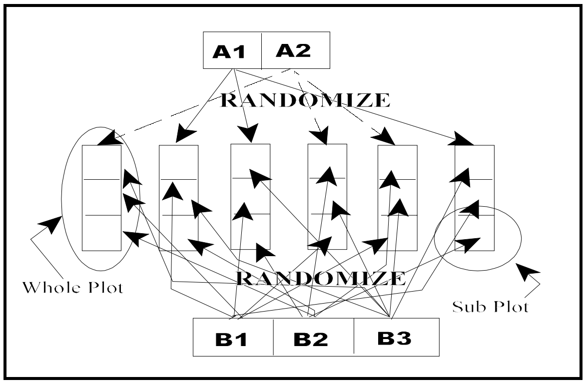

Figure 8 Graphic of assignment of the treatment structure to experimental units of a split-plot design.

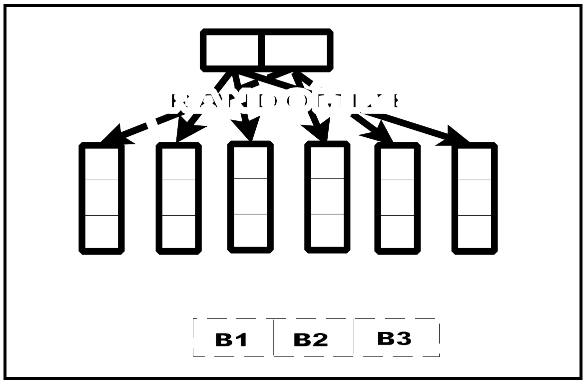

Figure 9 Graphic of assignment of the levels of A to the whole plots for a CR design structure.

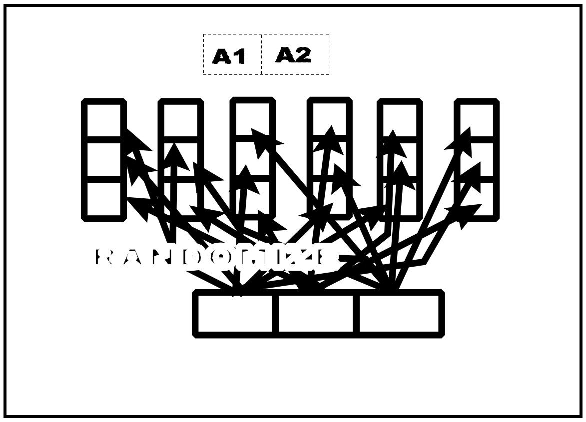

Figure 10 Graphic of assignment of the levels of B to the sub plot experimental units in a RCB.

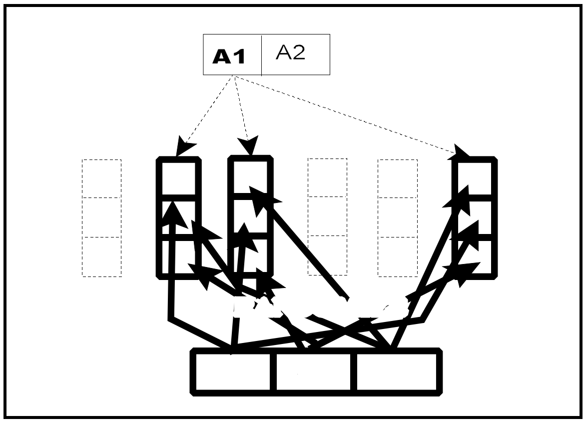

Figure 11 Graphic of assignment of the levels of B to the sub plots in the whole plots assigned to level A1, a RCB.

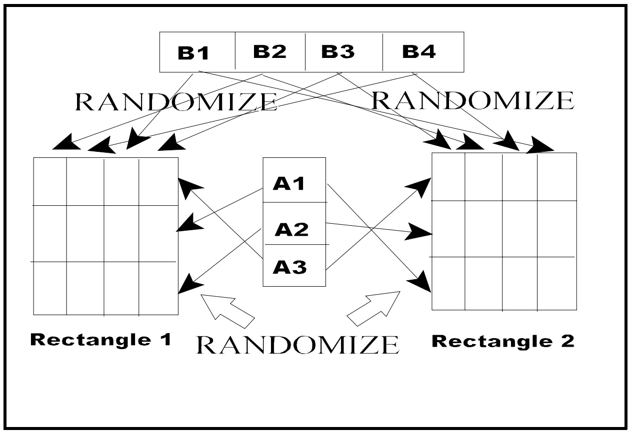

Figure 12 Graphic of the assignment of the levels of A to the rows and the levels of B to the columns for the strip-plot design structure.

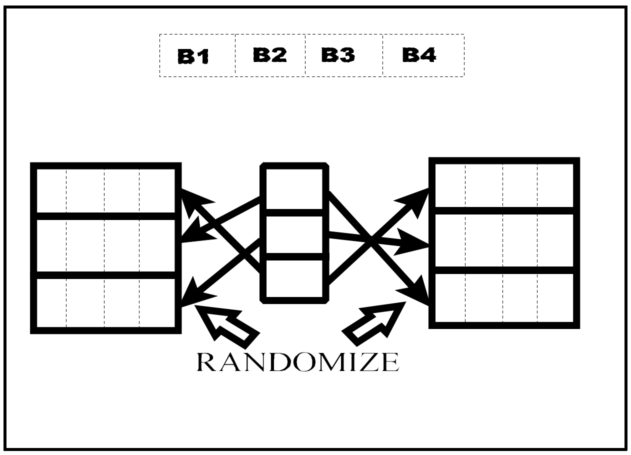

Figure 13 Random assignment of the levels of B to the rows of each of the rectangles, ignoring A, a RCB.

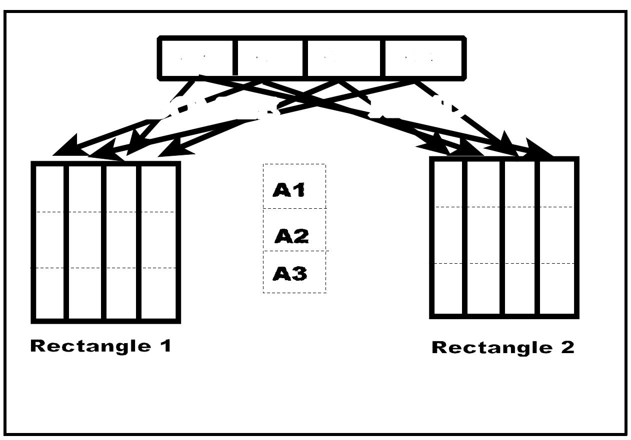

Figure 14 Graphic of the assignment of the levels of B to the columns for the strip-plot design structure, a RCB.

Table 1. Treatment patterns for block assignments for ICB with blocks of size 3

Pattern	Pattern Number	Assigned Block Number
A B C	1	1
A B D	2	2
A C D	3	4
B C D	4	3

Table 2. Treatment patterns for block assignments for ICB with blocks of size 2

Pattern	Pattern Number	Assigned Block Number
A B	1	4
A C	2	1
A D	3	2
B C	4	3
B D	5	6
C D	6	5

Table 3. Analysis of variance tables for four design structures involving four treatments.

Source	CR df	RCB df	ICB(3) df	ICB(2) df
Blocks	0	2	3	5
Treatments	3	3	3	3
Error	8	6	5	3

Table 4. Analysis of variance table for design with eight schools and two treatments

Source	DF	EMS
Schools	7	$\sigma^2_{class} + 2\ \sigma^{2*}_{School}$
Treatments	1	$\sigma^2_{class} + \phi^2$
Error	7	σ^2_{class}

Table 5. Analysis of variance table for school level design with eight schools and from two district sizes.

Source	DF	EMS
Size	1	$\sigma^{s+}_{School} + \phi_S^{\ 2}$
Error(Schools)	6	σ^{2+}_{School}

Table 6. Two-level analysis of variance table for the school district size study.

Source	DF	EMS
Size	1	$\sigma^2_{class} + 2\,\sigma^2_{School} + \phi^2(\theta)$
Error(Schools)	6	$\sigma^2_{class} + 2\,\sigma^2_{School}$
Treatment	1	$\sigma^2_{class} + \phi^2(\tau)$
SizeXTreatment	1	$\sigma^2_{class} + \phi^2(\theta\tau)$
Error(Class)	6	σ^2_{class}

Table 7. Split-plot analysis of variance table

Source	DF	EMS
A	1	$\sigma^2_{sp} + 2\,\sigma^2_{wp} + \phi^2(\theta)$
Error(whole-plot)	4	$\sigma^2_{sp} + 2\,\sigma^2_{wp}$
B	2	$\sigma^2_{sp} + \phi^2(\tau)$
AXB	2	$\sigma^2_{sp} + \phi^2(\theta\tau)$
Error(Sub-plot)	8	σ^2_{sp}

Table 8. Whole plot analysis for the split-plot design.

Source	DF	EMS
A	1	$\sigma^{2*}_{wp} + \phi^2(\theta)$
Error(whole-plot)	4	σ^{2*}_{wp}

Table 12. Analysis of variance table for the row analysis of the strip-plot design structure.

Source	DF	EMS
Rectangle	1	$\sigma^2_{row\ mean} + 3\,\sigma^2_{rect}$

Table 9. Sub-plot analysis for split-plot design.

Source	DF	EMS
Blocks	5	???
B	2	???
Residual(Sub-plot)	10	???

Table 10. Sub-plot analysis of levels of B for blocks assigned to A1.

Source	DF	EMS
Blocks	2	$\sigma^2_{sp} + 2\,\sigma^2_{wp}$
B	2	$\sigma^2_{sp} + \phi^2(\tau)$
Residual(Sub-plot)	4	σ^2_{sp}

Table 11. Analysis of variance table for the split-split-plot and three level design.

Source	df	EMS
Size	1	$\sigma^2_{person}+4\sigma^2_{class}+8\sigma^2_{school}+\Phi^2(Size)$
Error(School)	6	$\sigma^2_{person} + 4\sigma^2_{class} + 8\sigma^2_{school}$
Treatment	1	$\sigma^2_{person} + 4\sigma^2_{class} + \Phi^2(T)$
Size*Treatment	1	$\sigma^2_{person} + 4\sigma^2_{class} + \Phi^2(Size*T)$
Error(Class)	6	$\sigma^2_{person} + 4\sigma_{2class} + \Phi^2(Size*T*Sex)$
Sex	1	$\sigma^2_{person} + \Phi^2(Sex)$
Size*Sex	1	$\sigma^2_{person} + \Phi^2(Size*Sex)$
Treatment*Sex	1	$\sigma^2_{person} + \Phi^2(T*Sex)$
Size*Treat*Sex	1	$\sigma^2_{person} + \Phi^2(Size*T*Sex)$
Error(Student)		σ^2_{person}

A	3	$\sigma^2_{row\ mean} + \phi^2(\alpha)$
Error(row) = A*Rectangle	3	$\sigma^2_{row\ mean}$

Table 13. Analysis of variance table for the column analysis of the strip-plot design structure.

Source	DF	EMS
Rectangle	1	$\sigma^2_{col\ mean} + 4\,\sigma^2_{rect}$
B	4	$\sigma^2_{col\ mean} + \phi^2(\beta)$
Error(column) = B*Rectangle	4	$\sigma^2_{col\ mean}$

Table 14. Analysis of variance table for the strip-plot design structure.

Source	DF	EMS
Rectangles	1	$\sigma^2_{cell}+3\ \sigma^2_{row}+4\ \sigma^2_{col} + 12\ \sigma^2_{rect}$
A	3	$\sigma^2_{cell}+3\ \sigma^2_{row}+\phi^2(A)$
Error(row)	3	$\sigma^2_{cell}+3\ \sigma^2_{row}$
B	4	$\sigma^2_{cell}+4\ \sigma^2_{col}+ \phi^2(B)$
Error(column)	4	$\sigma^2_{cell}+4\ \sigma^2_{col}$
A*B	12	$\sigma^2_{cell}+ \phi^2(A*B)$
Error(cell)	12	σ^2_{cell}