

A Proportional Odds/Hazards Approach to Analyzing Likert Scale Data

Lynette Duncan, University of Arkansas, Fayetteville, AR

James E. Dunn, University of Arkansas, Fayetteville, AR

ABSTRACT

Linear scoring coupled with parametric regression or ANOVA often is used to analyze responses measured on a Likert scale. Alternatively, we have had success in numerous cases by modeling the same data under relaxed assumptions of the proportional odds model. This is implemented conveniently by procedure LOGISTIC, where a test of the proportional odds parallelism assumption is displayed automatically. When this test fails, a satisfactory model sometimes results by switching from the default cumulative logit link to use of the complementary log-log link, implemented by option LINK=CLOGLOG. Model fit by means of the log-log link also can be assessed using the CLOGLOG option, after minimal data manipulation. Either link induces the equivalent of a proportional hazards model. Interpretation of proportional odds and hazards models in the context of a Likert scale response is illustrated using data obtained from a survey of 849 teachers and administrators involved in implementing a new teaching program in their schools. Of 8 items relating to the perceived degree of implementation of the program, the proportional hazards model represents responses for seven items very well. This paper is directed at survey specialists who employ Likert scale instruments and who are familiar with logistic regression.

INTRODUCTION

Recently, our assistance was requested to aid in the analysis of a survey in which each of eight Likert-scale questions was considered a response variable. Because there were only four levels of the scale, we did not want to treat it as linear, nor did we want to lose the ordinal nature of the data by treating it as strictly nominal. This led us to use the proportional odds model available in SAS[®] procedure LOGISTIC.

However, when we fitted separate models to the eight questions, the proportional odds assumption, i.e., parallelism assumption, was met in only one case. As a consequence, we decided to try the complementary log-log (CLL) link, which, as Agresti (1984) pointed out, is equivalent to fitting a proportional hazards model. This proved successful for seven of the eight questions.

Since these appear to be new innovations in the analysis of Likert-scale data, we present an example where the parallelism assumption failed using the logit link but did not fail with the CLL link, and we will also present an example where the parallelism assumption did not fail with either link.

TEACHING PROGRAM DATA

Data were collected from surveys given to teachers and administrators in schools that had implemented a new teaching program. The survey asked eight questions about components of the program. The teachers rated the degree of desirability on a 1 to 4 Likert scale (1=undesirable and 4=highly desirable), and they also rated the degree of present implementation on the same scale. The number of years the teacher had been involved in the program was also asked in the survey. This paper will focus on N=804 teacher's responses excluding responses from administrators.

The question of interest was: Does the perception of degree of present implementation of each of the eight components vary significantly between years of implementation (1,2,3+) or among degree of desirability (1 to 4)? Note: Because of the low counts, levels 1 and 2 from the Likert-scale were combined as 2.

METHOD

Initially, each component was analyzed with a traditional proportional odds model with logit link. However, since it gave an adequate fit in only one case, we were forced to use a more complex proportional hazards model based on a complementary log-log link. In this model, the response was the answer to the degree of implementation, and the explanatory variables were years of implementation (categorical), degree of desirability (linear), and their interaction. In the case where the parallelism assumption failed for both links, logistic regression was used with the response defined to be 0 if the answer was 1 or 2, i.e., "failure" in some sense, and 1 if the answer was 3 or 4, corresponding to "success".

For all types of models, the initial model allowed tests of significance of the main effects and the interaction. If the interaction was non-significant at $\alpha=0.05$, it was removed from the model.

COMPARISON OF PROPORTIONAL ODDS AND HAZARDS MODELS

When modeling ordinal response data, either the proportional odds model or the proportional hazards model could be used. The proportional odds model is used most frequently, probably because it is a natural extension of the logistic regression model from binary response to ordinal response with more than two categories. However, we have found that when the parallelism assumption fails for the proportional odds model, it is sometimes met by using the proportional hazards model. The two models are shown below, and Table 1 shows a brief comparison of the models.

PROPORTIONAL ODDS MODEL WITH LOGIT LINK

For three levels of response with answers 1 and 2 combined, the proportional odds model requires two equations, namely

$$\text{Logit}(P[Q_i \leq 2 | \mathbf{x}]) = \ln \left[\frac{P[Q_i \leq 2 | \mathbf{x}]}{1 - P[Q_i \leq 2 | \mathbf{x}]} \right] = \alpha_2 + \beta' \mathbf{x}$$

$$\Rightarrow P[Q_i \leq 2 | \mathbf{x}] = \frac{1}{1 + e^{-(\alpha_2 + \beta' \mathbf{x})}}$$

and

$$\text{Logit}(P[Q_i \leq 3 | \mathbf{x}]) = \ln \left[\frac{P[Q_i \leq 3 | \mathbf{x}]}{1 - P[Q_i \leq 3 | \mathbf{x}]} \right] = \alpha_3 + \beta' \mathbf{x}$$

$$\Rightarrow P[Q_i \leq 3 | \mathbf{x}] = \frac{1}{1 + e^{-(\alpha_3 + \beta' \mathbf{x})}}$$

where Q_i is the i^{th} implementation question and \mathbf{x} is a vector of values of the explanatory variables. Clearly, $P[Q_i \leq 4] = 1$ so that it need not be modeled.

As its name suggests, the proportional odds model can be interpreted in terms of odds and odds ratios. The odds that the response $Q_i \leq j$, where $j=2$ or 3 , are

$$\frac{P[Q_i \leq j | \mathbf{x}]}{1 - P[Q_i \leq j | \mathbf{x}]} = e^{\alpha_j + \beta' \mathbf{x}},$$

and the odds ratio for $Q_i \leq j$ corresponding to two different conditions of the explanatory variables, represented by vectors \mathbf{x}_1 and \mathbf{x}_2 , is

$$\frac{\frac{P[Q_i \leq j | \mathbf{x}_2]}{1 - P[Q_i \leq j | \mathbf{x}_2]}}{\frac{P[Q_i \leq j | \mathbf{x}_1]}{1 - P[Q_i \leq j | \mathbf{x}_1]}} = \frac{e^{\alpha_j + \beta' \mathbf{x}_2}}{e^{\alpha_j + \beta' \mathbf{x}_1}} = e^{\beta'(\mathbf{x}_2 - \mathbf{x}_1)}.$$

Thus, the odds of $Q_i \leq j$ vary only as the exponentiated difference

between x_1 and x_2 , regardless of which level j of response is specified.

PROPORTIONAL HAZARDS MODEL WITH COMPLEMENTARY LOG-LOG LINK

Likewise, for three levels of response, the proportional hazards model requires two equations, namely

$$CLL(P[Q_{2l} \leq 2]) = \ln[-\ln(1 - P[Q_{2l} \leq 2])] = \alpha_2 + \beta'x$$

$$\Rightarrow P[Q_{2l} \leq 2] = 1 - e^{-e^{\alpha_2 + \beta'x}}$$

$$CLL(P[Q_{2l} \leq 3]) = \ln[-\ln(1 - P[Q_{2l} \leq 3])] = \alpha_3 + \beta'x$$

$$\Rightarrow P[Q_{2l} \leq 3] = 1 - e^{-e^{\alpha_3 + \beta'x}}$$

where Q_{2l} is the i^{th} implementation question and x is a vector of values of the explanatory variables. Again, $P[Q_{2l} \leq 4] = 1$, so that it need not be modeled.

Proportional hazards models are used in survival analysis, so we can consider the model with that notation.

Specifically,

$P[Q_{2l} \leq j | x] = F(j, x)$, the cumulative distribution function, while

$S(j, x) = 1 - F(j, x) = P[Q_{2l} > j | x]$ defines the associated survivor function. Thus,

$$\ln[-\ln(1 - P[Q_{2l} \leq j | x])] = \alpha_j + \beta'x$$

$$\Rightarrow \ln[1 - F(j | x)] = \ln[S(j, x)] = -e^{\alpha_j + \beta'x},$$

$$\text{or } S(j, x) = S_0(j) e^{\beta'x},$$

which characterizes a proportional hazards model where

$$S_0(j) = e^{-\alpha_j} \text{ assumes the role of a baseline survivor function.}$$

Even though the proportional hazards model based on the use of the CLL link does not lend itself to estimates of odds ratios, a parallel development proceeds by considering the ratio of log-survivor functions, that is

$$\frac{\ln[S(j | x_2)]}{\ln[S(j | x_1)]} = \frac{e^{\beta'x_2} \ln[S_0(j)]}{e^{\beta'x_1} \ln[S_0(j)]} = e^{\beta'(x_2 - x_1)}$$

so that

$$S(j | x_2) = S(j | x_1) e^{\beta'(x_2 - x_1)}$$

Thus, exceedence probabilities under conditions described by x are doubly exponentiated proportional to $x_2 - x_1$ as conditions move to that give by x_2 .

Table 1. The proportional odds and hazards models have the same underlying assumption, but the models use different link functions to model ordinal response data.

	Proportional Odds Model	Proportional Hazards Model
Link	Logit	Complementary Log-Log
Assumption	Equal slopes across levels of response variable	Equal slopes across levels of response variable
Model	$P[Q_{2l} \leq j] = \frac{1}{1 + e^{-(\alpha_j + \beta'x)}}$	$P[Q_{2l} \leq j] = 1 - e^{-e^{\alpha_j + \beta'x}}$

EXAMPLE 1

In this example, we fit the proportional odds model and the proportional hazards model to the second implementation question: "To what degree do you presently implement the component 'Teacher knowledge and skills are the foundation of progress in the classroom'?" Although the parallelism assumption is not met for the proportional odds model, the results from this model are shown for the purpose of comparison.

Table 2. A proportional odds model and a proportional hazards model were both fit to the second implementation question. The p-values for the parallelism test and for the effects are shown.

	Proportional	Proportional
--	--------------	--------------

	Odds Model	Hazards Model
Full Model:		
Test of Equal Slopes	p=0.0034	p=0.1026
Type III Tests:		
Q _{2d}	p<0.0001	p<0.0001
Year	p=0.9695	p=0.1603
Q _{2d} x Year	p=0.8423	p=0.0860
Model without interaction:		
Test of Equal Slopes	p=0.0015	p=0.2093
Type III Tests:		
Q _{2d}	p<0.0001	p<0.0001
Year	p=0.0122	p=0.0880

PROPORTIONAL ODDS MODEL WITH LOGIT LINK

The parallelism assumption was not met for the proportional odds model that included the interaction ($\chi^2=17.65$, $df=5$, p -value=0.0034). Since the interaction term was far from significant ($p=0.8423$), it was removed. The parallelism assumption still failed when only the main effects were included in the model ($\chi^2=15.4048$, $df=3$, p -value=0.0015). In the subsequent model, both main effects were significant at $\alpha=0.05$ (see Table 2). The parameter estimates are shown in Table 3.

Table 3. The parameter estimates for the proportional odds model for Q_{2l} are shown here.

Parameter	DF	Estimate	Standard Error	Wald Chi-Square	P-value
Intercept	2 1	3.5484	0.7061	25.2504	<0.0001
Intercept	3 1	6.9811	0.7682	82.5856	<0.0001
Q _{2d}	1	-1.9174	0.1973	94.4161	<0.0001
Year	1 1	0.6854	0.2544	7.2587	0.0071
Year	2 1	0.3458	0.2193	2.4870	0.1148
Year	3 0	0	.	.	.

PROPORTIONAL HAZARDS MODEL WITH CLL LINK

The parallelism assumption was met for the proportional hazards model that included the interaction ($\chi^2=9.17$, $df=5$, p -value=0.1026). The interaction term was not significant ($p=0.0860$), and it was removed. The parallelism assumption was still met when only the main effects were included in the model ($\chi^2=4.5337$, $df=3$, p -value=0.2093). In the subsequent model, Q_{2d} was significant at $\alpha=0.05$, but year was not significant at $\alpha=0.05$ (see Table 2). The parameter estimates are shown in Table 4.

Table 4. The parameter estimates for the proportional hazards model for Q_{2l} are shown here.

Parameter	DF	Estimate	Standard Error	Wald Chi-Square	P-value
Intercept	2 1	1.4676	0.4654	9.9449	0.0016
Intercept	3 1	4.3353	0.4678	85.9019	<0.0001
Q _{2d}	1	-1.3015	0.1236	110.8880	<0.0001
Year	1 1	0.3030	0.1899	2.5479	0.1104
Year	2 1	0.2811	0.1634	2.9599	0.0854
Year	3 0	0	.	.	.

Table 5 displays the observed and predicted fractions with each response to Q_{2l}. Predicted values were obtained for both the proportional odds and the proportional hazards model. The individual probabilities in Table 5 are defined by as follows:
 $P[Q_{2l} = 1 \text{ or } 2] = P[Q_{2l} \leq 2]$
 $P[Q_{2l} = 3] = P[Q_{2l} \leq 3] - P[Q_{2l} \leq 2]$
 $P[Q_{2l} = 4] = 1 - P[Q_{2l} \leq 3]$.
 The individual probabilities were calculated because the

researcher found them more appropriate than the cumulative probabilities for explaining her results. (Note that these probabilities are available from the OUTPUT statement in PROC LOGISTIC by including PREDPROBS=I expression along with the usual PRED=variable.) Figure 1 shows the same data for year 3.

The predicted probabilities based on the proportional hazards model relate as follows. Given Q_{2d} , the proportion of year-1 teachers scoring higher than any fixed response equals the proportion of year-2 teachers scoring higher than that response after it is raised to the $e^{0.3030 \cdot 0.2811} = 1.022$ power, e.g., $P[Q_{2l} > 3 \mid Q_{2d} = 1 \text{ or } 2 \text{ and Year} = 1] =$

$$= P[Q_{2l} > 3 \mid Q_{2d} = 1 \text{ or } 2 \text{ and Year} = 2]^{1.022}$$

$$= P[Q_{2l} = 4 \mid Q_{2d} = 1 \text{ or } 2 \text{ and Year} = 1]^{1.022}$$

$$= 0.00056^{1.022} = 0.00047.$$

Table 5. The cells show the observed and predicted probabilities for each level of Q_{2l} . The first number in each cell is the observed probability, the second number is the predicted probability based on the proportional odds model, and the last number is the predicted probability based on the proportional hazards model without the interaction.

		Individual Probability: $Q_{2l} =$		
Q_{2d}	Year	1 or 2	3	4
1 or 2	1	0.50000	0.00000	0.50000
		0.59846	0.38033	0.02121
		0.35277	0.64675	0.00047
	2	0.50000	0.50000	0.00000
		0.51487	0.45560	0.02953
		0.34662	0.65282	0.00056
	3	0.42857	0.57143	0.00000
		0.42890	0.52987	0.04123
		0.27481	0.72169	0.00350
3	1	0.11111	0.88889	0.00000
		0.17971	0.69181	0.12848
		0.11165	0.76384	0.12451
	2	0.00000	0.90909	0.09091
		0.13495	0.69353	0.17152
		0.10936	0.76036	0.13028
	3	0.02299	0.82759	0.14943
		0.09942	0.67424	0.22635
		0.08373	0.70161	0.21466
4	1	0.06897	0.41379	0.51724
		0.03120	0.46807	0.50073
		0.03171	0.40104	0.56725
	2	0.01149	0.41379	0.57471
		0.02242	0.39280	0.58479
		0.03103	0.39468	0.57429
	3	0.03142	0.29575	0.67283
		0.01597	0.31844	0.66559
		0.02351	0.31860	0.65788

EXAMPLE 2

In this example, we fit the proportional odds model and the proportional hazards model to the third implementation question: "To what degree do you presently implement the component 'Self respect is basic to learning?'" In this example, the parallelism assumption is met for both models at $\alpha=0.01$.

Table 6. A proportional odds model and a proportional hazards model were both fit to the third implementation question. The p-values for the parallelism test and for the effects are shown.

	Proportional Odds Model	Proportional Hazards Model
Full Model:		
Test of Equal Slopes	p=0.0751	p=0.3296
Type III Tests:		

Q_{2d}	p<0.0001	p<0.0001
Year	p=0.3261	p=0.1557
$Q_{2d} \times \text{Year}$	p=0.2977	p=0.1487
Model without interaction:		
Test of Equal Slopes	p=0.1475	p=0.8439
Type III Tests:		
Q_{2d}	p<0.0001	p<0.0001
Year	p=0.8353	p=0.8953
Model without year effect:		
Test of Equal Slopes	p=0.0298	p=0.6906
Type III Tests:		
Q_{2d}	p<0.0001	p<0.0001

PROPORTIONAL ODDS MODEL WITH LOGIT LINK

The parallelism assumption was met for the proportional odds model that included the interaction ($\chi^2=10.0062$, $df=5$, $p\text{-value}=0.0751$). The interaction term was not significant ($p=0.2977$), and it was removed. The main effect of year was also non-significant ($p=0.8353$) and was removed. The parallelism assumption still held when only the main effect of Q_{2d} was included in the model ($\chi^2=4.7231$, $df=1$, $p\text{-value}=0.0298$). The test for parallelism and the Type III tests are shown in Table 6. The parameter estimates are shown in Table 7.

Table 7. The parameter estimates for the proportional odds model for Q_{3l} are shown here.

Parameter	DF	Estimate	Standard Error	Wald Chi-Square	P-value	
Intercept	2	1	3.1782	0.7069	20.2150	<0.0001
Intercept	3	1	6.1360	0.7505	66.8529	<0.0001
Q_{3d}	1	-1.7069	0.1929	78.3097	<0.0001	

PROPORTIONAL HAZARDS MODEL WITH CLL LINK

The parallelism assumption was met for the proportional hazards model that included the interaction ($\chi^2=5.7661$, $df=5$, $p\text{-value}=0.3296$). The interaction term was not significant ($p=0.1487$), and it was removed. The main effect of year was also non-significant ($p=0.8953$) and was removed. The parallelism assumption still held when only the main effect of Q_{2d} was included in the model ($\chi^2=0.1584$, $df=1$, $p\text{-value}=0.6906$). The test for parallelism and the Type III tests are shown in Table 6. The parameter estimates are shown in Table 8.

Table 8. The parameter estimates for the proportional hazards model for Q_{3l} are shown here.

Parameter	DF	Estimate	Standard Error	Wald Chi-Square	P-value	
Intercept	2	1	1.7942	0.4762	14.1974	0.0002
Intercept	3	1	4.3630	0.4830	81.5819	<0.0001
Q_{3d}	1	-1.3194	0.1272	107.5484	<0.0001	

Table 9 displays the observed and predicted fractions with each response to Q_{3l} . Predicted values were obtained for both the proportional odds and the proportional hazards model. Figure 2 shows the same data.

Similarly to Table 5, in Table 9 the predicted probabilities based on the proportional hazards model relate as follows. The proportion of teachers with $Q_{3d} = 1$ or 2 scoring higher than any fixed response equals the proportion of teachers with $Q_{3d} = 3$ scoring higher than that response after it is raised to the $e^{1.3194} = 3.7412$ power, e.g., $P[Q_{3l} > 3 \mid Q_{3d} = 1 \text{ or } 2] = P[Q_{3l} > 3 \mid Q_{3d} = 3]^{3.7412}$
 $= P[Q_{3l} = 4 \mid Q_{3d} = 3]^{3.7412}$
 $= 0.22334^{3.7412} = 0.00367.$

Table 9. The cells show the observed and predicted probabilities for each level of Q_3 . The first number in each cell is the observed probability, the second number is the predicted probability based on the proportional odds model, and the last number is the predicted probability based on the proportional hazards model.

Individual Probability: Q_3 =			
Q_3d	1 or 2	3	4
1 or 2	0.45455	0.54545	0.00000
	0.44139	0.49694	0.06167
	0.34931	0.64702	0.00367
3	0.06522	0.70652	0.22826
	0.12538	0.60869	0.26592
	0.10851	0.66815	0.22334
4	0.03414	0.29730	0.66856
	0.02535	0.30836	0.66629
	0.03024	0.29991	0.66985

CONCLUSION

Both of the approaches demonstrated here provide an innovative method to use in analyzing Likert-scale data. We saw from these examples that by using the CLL link in proc LOGISTIC, we were able to find a satisfactory model for ordinal response data when the traditional proportional odds model may not be appropriate. We believe that this provides a viable alternative to analyzing ordinal response data as strictly nominal when the parallelism assumption fails. These approaches should be added to a practicing statistician's "toolbox".

REFERENCES

Agresti, A. (1984), *Analysis of Ordinal Categorical Data*. New York: Wiley.
 SAS Institute, Inc. (2000), *SAS OnlineDoc Version 8*, Cary, NC: SAS Institute, Inc.

CONTACT INFORMATION

Your comments and questions are valued and encouraged. Contact the author at:

Lynette Duncan
 University of Arkansas
 Department of Mathematical Sciences
 538 HOTZ
 Fayetteville, AR 72701
 (501) 575-6429
 duncan@uark.edu

SAS and all other SAS Institute Inc. product or service names are registered trademarks or trademarks of SAS Institute Inc. in the USA and other countries. ® indicates USA registration.

Other brand and product names are registered trademarks or trademarks of their respective companies

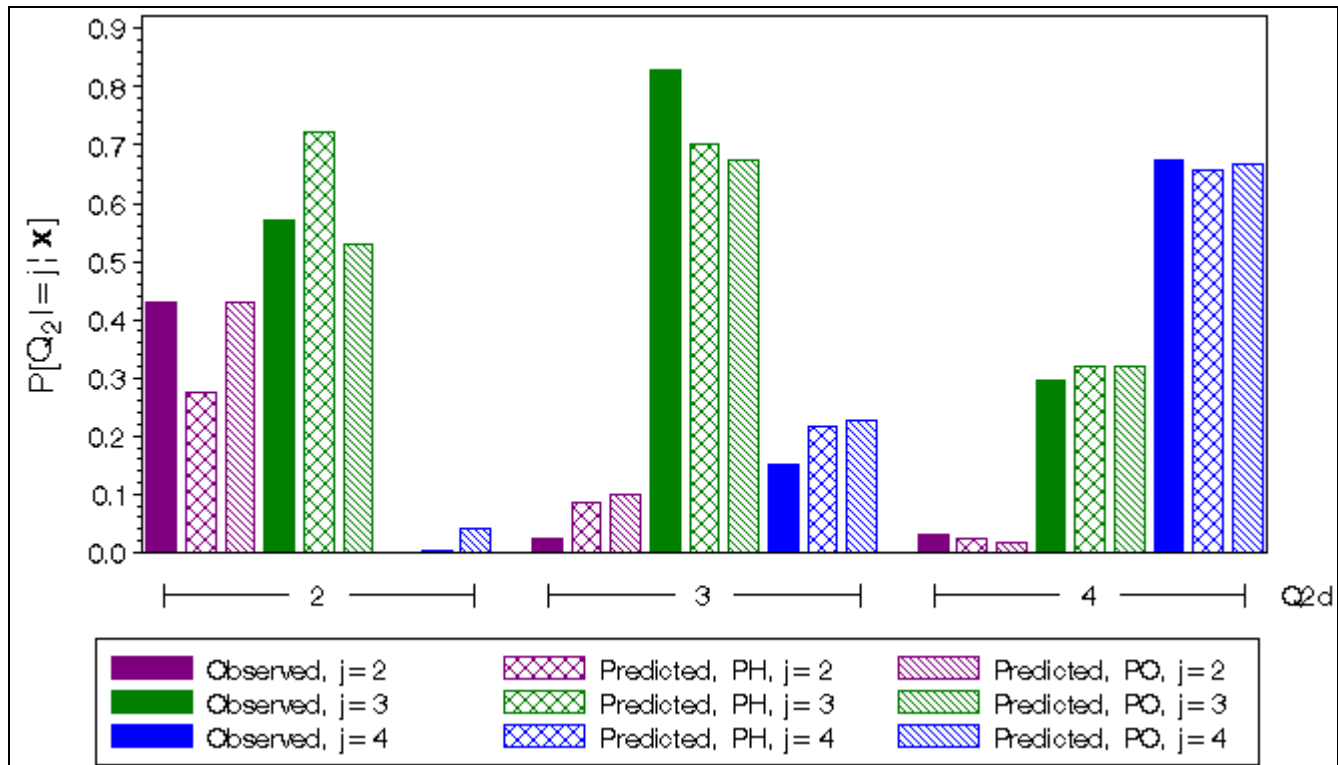


Figure 1. This chart illustrates the distribution of response from the observed data and from the models shown in Example 1 where year=3. (PH = Proportional Hazards and PO = Proportional Odds).

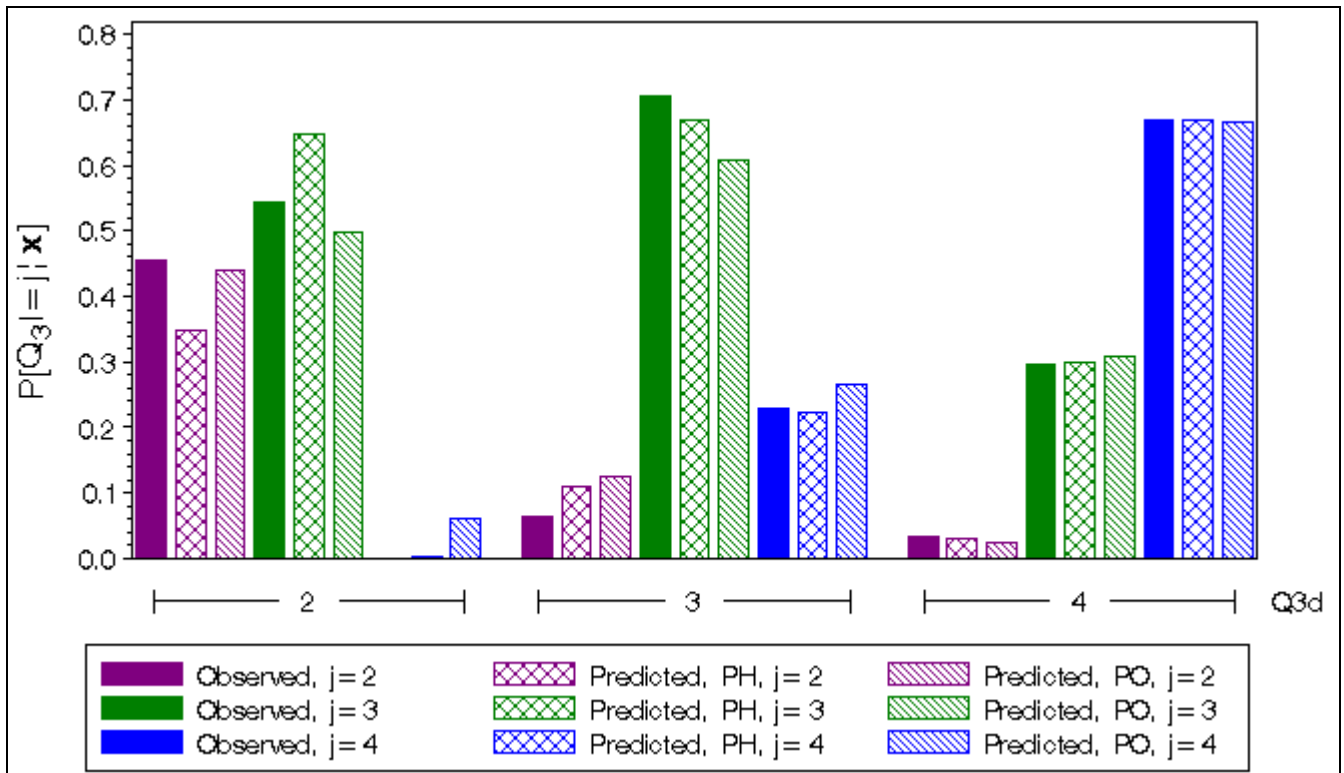


Figure 2. This chart illustrates the distribution of response from the observed data and from the models shown in Example 2. (PH = Proportional Hazards and PO = Proportional Odds).