Forecasting the Volatility of Financial Markets: ARCH/GARCH Models and the AUTOREG Procedure
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ABSTRACT
Investment decisions involve considerations of both expected return and risk; the study of financial markets is concerned both with changes in security values and with changes in market volatility over time. The autoregressive conditional heteroscedasticity (ARCH) model and its generalizations (GARCH) are one approach to modeling and forecasting both the mean and the variance of a time series. This paper discusses the use of the SAS/ETS® AUTOREG procedure, with a particular emphasis on forecasting the volatility of future returns.

INTRODUCTION
Economic or financial time series often show time changing variances. The problem of handling this heteroscedasticity is specifying the functional form of the time changing variance. The GARCH-type modeling provides a general approach, since the GARCH-type variances depend on available information.

The volatility is a risk measure in financial market analysis and is represented by the variability (standard deviation) of asset returns. The GARCH-in-mean model can be used to analyze the asset pricing model. Nelson (1991) proposed the exponential GARCH model to allow the negative relationship between the stock returns and the change in returns variability. Schwert (1989) studied the relationship between business cycles and stock volatility and found that the stock volatility could be used to evaluate the state of the economy.

When you use the Black-Scholes formula, the volatility of security returns should be estimated or predicted to calculate the price of derivative securities — options, futures, interest caps, and flexible forward contracts. The GARCH-type model is useful in analyzing the current and future risk characteristics, since it efficiently captures the information that is contained in asset returns.

This paper will address the estimation and forecasting of the GARCH-type models and will investigate the performance of the forecasted volatility from the GARCH model. It will also show how SAS® procedures can be used to analyze the option valuation model.

GARCH-TYPE MODELING USING THE AUTOREG PROEDURE
The AUTOREG procedure currently supports ARCH, GARCH, integrated GARCH (IGARCH), and exponential GARCH (EGARCH), AR-GARCH, and GARCH-in-mean (GARCH-M). The maximum likelihood method is employed to estimate the family of GARCH models assuming the conditional normality of error terms. PROC AUTOREG also produces the forecasted values of conditional error variances.

The GARCH regression model with autoregressive errors is written

\[ h_t = \alpha + \sum_{i=1}^{p} \alpha_i \epsilon_{t-i}^2 + \sum_{j=1}^{q} \gamma_j h_{t-j} \]

\[ \epsilon_t \sim N(0,1) \]

This model combines the mth-order autoregressive error model with the GARCH(p,q) variance model. It is denoted as the AR(m)-GARCH(p,q) regression model. For example, the simple ARCH(1) model is specified when \( m = 0, p = 0, \) and \( q = 1. \)

An artificial linear model with ARCH(2) errors is generated in order to show the characteristic of the ARCH process (Figure 1). The simulated ARCH errors are plotted in Figure 2. The data series explicitly shows the property that large (small) changes are followed by large (small) changes.

First, the ARCH(2) model is estimated and is shown in Figure 3. The estimates for ARCH parameters are statistically significant. The parameter of ARCH(2) is overestimated (ARCH2 = 0.337), while the ARCH(1) parameter is underestimated (ARCH1 = 0.466). The unconditional error variance \((1/(1-0.5-0.25)) = 4\) is also underestimated at 4.795.
**Figure 3. ARCH(2) Estimates**

When the GARCH(1,1) model is estimated, the GARCH parameter estimates (ARCH1 and GARCH1) are indicative of statistical significance (Figure 4). The unconditional variance is also overestimated (4.293) but is smaller than that of the ARCH(2) model. The normality test does not indicate that GARCH(1,1) is a poor approximation to the true model. However, the SBC and AIC statistics of GARCH(1,1) are larger than those of the ARCH(2) model, and the total R-square indicates that ARCH(2) is a better fit than GARCH(1,1).

**Figure 4. GARCH(1,1) Estimates**

The following statements fit an AR(1)-GARCH(1,1) model for the \( Y \) series regressed on the variable \( X \). The GARCH=(P=1,Q=1) option specifies the GARCH(1,1) conditional variance model. The NLAG=1 option specifies the AR(1) error process. The CEV= option on the OUTPUT statement stores the estimated conditional error variance at each time period in the variable \( \text{VHAT} \) in an output data set.

The results for the GARCH model are shown in the following:

**Figure 5. Conditional Variance Estimate**

The plot in Figure 5 indicates that conditional variance estimates of both the ARCH(2) and GARCH(1,1) models are close to the true conditional variance.

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**Heteroscedasticity**

The GARCH process can be a general approach to modeling time series regression models with heteroscedastic errors. Heteroscedasticity causes the OLS estimates to be inefficient and makes the OLS prediction error variance inaccurate. GARCH-type models that take into account the changing variance can make more efficient use of the data.

To illustrate heteroscedastic time series, the simulated series \( Y \) with a changing error variance is created. The error variance is the function of an explanatory variable \( X \). The error disturbances are also autocorrelated.

```plaintext
data one;
  ul = 0;
  pi = 3.14159;
  do time = -10 to 200;
    x = cos(2*pi*time/200);
    v = exp(.1*x + .7*x^2 + .2*x^3);
    u = 0.8*ul + agtr(v)*rannor(34578);
    y = 7 + 5*x + u;
  if time > 0 then output;
  ul = ul;
run;
```

The following statements fit an AR(1)-GARCH(1,1) model for the \( Y \) series regressed on the variable \( X \). The GARCH=(P=1,Q=1) option specifies the GARCH(1,1) conditional variance model. The NLAG=1 option specifies the AR(1) error process. The CEV= option on the OUTPUT statement stores the estimated conditional error variance at each time period in the variable \( \text{VHAT} \) in an output data set.

The results for the GARCH model are shown in the following:

The normality test is not significant (p = .981), which is consistent with the hypothesis that the residuals, \( \epsilon \), from the GARCH model are normally distributed. The sum (.987) of GARCH parameter estimates, ARCH1 and GARCH1, are close to 1. The plot of \( \text{VHAT} \) together with the true variance \( V \) used to generate the simulated data is shown in Figure 6. The GARCH model shows a reasonably good performance of approximating the error variance in this example.
Figure 6. Comparing the True and GARCH(1,1) Variances

Asset Pricing Model

It is often reported that the variance of stock return is time varying. The market model with heteroscedastic disturbances is specified as

\[ R_t = \alpha_t + \beta_t R_{mt} + \epsilon_t \]

where \( R_t \) is a return on stock \( i \) at time \( t \); \( R_{mt} \) is a market return; and \( \epsilon_t \) is a random disturbance with mean 0 and variance \( \sigma^2_t \).

The market model for PANAM is estimated using the OLS and ARCH(1) models. The ARCH estimate of \( \beta_t \) of the PANAM common stock is significantly positive but is almost the same as the OLS estimate (.73).

The following SAS statements create the new return data with heteroscedastic error disturbances (Figure 7). The artificial market model is generated with the heteroscedasticity function:

\[ \text{data simul; set sasuser.capmi; sig2 = .1 + .25*R^2; ri = -.0067 + .7301*rm + \sqrt{\text{sig2}}*\text{ranor}(1234567); run; } \]

Figure 7. Generated Heteroscedastic Returns

The OLS estimate (.548) of \( \beta \) is much smaller than the parameter value and has the insignificant t statistic (1.184). The Durbin-Watson test might be significant because of heteroscedasticity.

You can estimate the heteroscedastic data with the subset ARCH model.

\[ \text{proc autoreg data=simul; model ri = rm / garch=(q=(3)); run; } \]

The estimate (.723) of \( \beta \) with the subset ARCH(3) model is very close to the parameter value. Here is the part of the PROC AUTOREG output.

The excess return \( R^e_t \) can be modeled as

\[ R^e_t = \alpha^e + \alpha^e R^e_{t-1} + \sigma_t \sqrt{R^e} + \epsilon_t \]

where \( \epsilon_t = \sqrt{R^e} \), and \( \sigma_t \sim N(0, 1) \).

First, it is assumed that \( \epsilon_t \) follows the GARCH(\( p, q \) process. The monthly market excess returns are estimated with GARCH(1,4)-M model over the period of 1977 to 1987. The estimate of \( \delta \) is positive (.474).

\[ \text{proc autoreg data=one; model r = /nlag=1 garch=(q=1,p=(4),mean); run; } \]
The conditional variance of EGARCH\((p,q)\) is specified as

\[ \log(h_t) = \omega + \sum_{i=1}^{p} \alpha_i g(z_{t-i}) + \sum_{j=1}^{q} \gamma_j g(h_{t-j}) \]

\[ g(z_t) = \beta z_t + \gamma E[z_t | - \ldots | E(z_t)] \]

\[ z_t = \sqrt{h_t} \]

The EGARCH\((1,1)\)-M estimation produces a negative relation \((-1.582)\) between the market excess return and the volatility. This showed that the volatility specification of GARCH(1,1) is less than the estimates of Nelson's (1991), though Nelson's used the conditional variance instead of the standard deviation.

\[ \text{proc autoreg data=one;}
\text{model r = lag1;}
\text{garch(q=1,p=1,type=exp,mean);}
\text{run;}
\]

### EGARCH\((1,1)\) Estimates

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<tr>
<th>Variable</th>
<th>DF</th>
<th>t Value</th>
<th>Std Error</th>
<th>t Ratio</th>
<th>Appr. Prob</th>
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<td>0.0022</td>
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<td>0.03397</td>
<td>3.7204</td>
<td>0.00006</td>
</tr>
</tbody>
</table>

\[ \text{Variable DF B Value Std Error t Ratio Appr. Prob}
\text{Intercept 1 0.09759674 0.03984 -2.4781 0.0136}
\text{AT 1 -0.151502 0.003467 -4.7272 0.00001}
\text{BARCH1 1 -0.09759674 0.03202 -3.0721 0.0022}
\text{BARCH2 1 0.0061805 0.03984 0.1572 0.0600}
\text{BARCH3 1 0.0051805 0.01306 3.8723 0.0001}
\text{BARCH4 1 0.0572316 0.03959 1.4263 0.0148}
\text{BARCH5 1 -0.1260559 0.03397 -3.7204 0.00006}
\]

### Figure 8. EGARCH\((1,1)\) Estimates

The inclusion of the volatility term in the risk premium equation is not justified theoretically. However, there have been statistical evidences on a significant positive relation between risk premia and the conditional variance. French, Schwert, and Stambaugh (1987) showed that the volatility specification of the GARCH-M model performed better than the conditional variance specification.

### Option Valuation

To study the market of security options, some basic concepts are defined. A call option is a contract for its owner to purchase the underlying asset at a predetermined strike (or exercise) price to a given expiration date. On the other hand, a put option is a contract for its owner to sell the underlying asset at a predetermined strike price to a given expiration date. This study will focus on the European option that can only be exercised on the expiration date.

When you have bought the option (have taken the long position), the payoff \( (C_p) \) from a long position in an European call option is

\[ C_p = \begin{cases} S - K & \text{if } S > K \\ 0 & \text{if } S \leq K \end{cases} \]

where \( S \) is the underlying security price on the expiration date, and \( K \) is the strike price. The payoff \( (P_p) \) of a European put option is \( P_p = \max(0, K - S) \). For example, a call option on the stock is contracted at the strike price of $60 on the second Thursday of July, and transaction costs are assumed to be 0. The payoff from your long position in a call option is $30 if the underlying stock price is $90 on the expiration date, while it is worthless if the stock price is less than the strike price on the second Thursday of July.

The value of security options can be estimated using the Black-Scholes model:

\[ C = S \Phi(d_1) - Ke^{-rt} \Phi(d_2) \]

\[ P = Ke^{-rt} - S \Phi(-d_2) - D \Phi(-d_1) \]

\[ d_1 = \frac{\ln(S/K) + rt}{\sqrt{t}} \]

\[ d_2 = d_1 - \sqrt{t} \]

where \( C \) is the value of a call option while \( P \) is the value of a put option; \( D \) is a discount factor for the security price; \( \Phi(\cdot) \) is the distribution function of \( \mathcal{N}(0,1) \); \( r \) is the risk-free annual interest rate; \( t \) is the time until the expiration date; and \( \sigma \) is the volatility of the underlying security. The discount factor \( D \) is formulated as \( e^{-rt} \) where \( r \) is a discount rate. This discount rate \( r \) is a dividend rate for the stock options, while the foreign interest rate is used as the discount rate for foreign currency options. It is assumed that the discount rate is nil, that is, \( D = 1 \). To estimate the Black-Scholes model, you need to know the values of parameters, such as the price of the underlying security, the strike price of the option, the expiration time, the risk free interest rate, and the security price volatility. The security price, exercise price, and time to expiration are determined by the terms of option contract. The market price of Treasury bill rate that has the same maturing time as the option can be used for a proxy of risk free interest rate. Only the volatility needs to be estimated.

The sample standard deviation of the security returns is used as an estimate for the volatility by assuming that the mean level of the security volatility will not have any dramatic changes over the long period of time. However, it is not realistic to believe that the sample standard deviation calculated over the recent past represents the current and future volatility. Therefore, you can use the predicted values of conditional standard deviation of the GARCH model as a reasonable estimate for the volatility.

In the following example, you can see how to value a call option using the ARCH(1) model. With a DATA statement, four missing values are added to the original series to produce the 1- to 4-step (one month) ahead conditional variance forecasts.

The data are the weekly closing prices of the national semiconductor stock (Cox and Rubinstein, 1985 p 254). The semiconductor call option that expires May 16, 1981 has the exercise price of $40 with today's (Jan 2, 1981) underlying semiconductor stock price of $40.76.
The ARCH(1) model is estimated and the conditional prediction variance $HT$ is created by the OUTPUT statement. The sum of the four predicted variances becomes the variance forecast for the coming four weeks, conditional on currently available information.

The value of a European (call or put) option is computed in the following SAS/IML® module `OPTIONP`. Two volatilities are computed: the annual conditional standard deviation of the ARCH(1) model is obtained by multiplying by the monthly volatility; the weekly sample standard deviation is computed and multiplied by $\sqrt{52}$ to get the yearly standard deviation.

You might estimate the volatility using the current option price observed in the market. The SAS/ETS MODEL procedure can produce the implicit volatility. With a DATA step, the option value `CALLPR` is generated in the range of 4 to 10. Since the Black-Scholes formula contains the volatility in the denominator, you need to initialize the value, which is greater than zero. The calculated volatility is shown in Figure 9.
Figure 9. Implicit Volatility vs Call Price

Risk Analysis of Options

When the call option price is considered as a function of underlying security price and the volatility, the sensitivity of an option value to the two parameters is shown in Figure 10. You can observe that the relationship between the call option price and the stock price tends to be linear when the volatility is high.

For this analysis, a European call option on the common stock is considered; the exercise price is $300, the annual risk free interest rate is 6%, and the time to expiration is .1667.

Figure 10. Call Option as a Function of Stock Price and Volatility

Based on the function of a call or a put option value, some useful measures can be derived. First, \( \delta \) is the rate of change of the option value with respect to the security price. For example, the delta of a call option is written as

\[
\Delta = \frac{\partial C}{\partial S}
\]

The SAS/IML module DELTA computes the delta of a call or a put option:

```sas
start delta(s,k,r,vol,time,call); /* call = 0 dp/ds */ /* put = 1 dC/dS */ pi = 4*atan(1); /* 3.14159 */ root_t = sqrt(time); di = (log(s/k) + r*time)/(vol*root_t) + .5*vol*root_t; delta_c = probnorm(di); if call then val = delta_c; else val = delta_c - 1; return(val); finish;
```

The 3D-plot shows that the call option value is insensitive to the change of the stock price when the stock price is far below the exercise price (out of the money) while the value of \( \delta \) is close to 1 when the stock price is much higher than the strike price (in the money). When the volatility is greater than .1, the value of \( \delta \) is lower than 1 in the given range of stock prices — 260 to 340.

Since the purpose of hedging protects investors from the uncertain security price movements, the delta neutral hedge is to keep the option value change with respect to security price 0.

For example, if we have two options with their values, \( V_1 \) and \( V_2 \), and their numbers, \( n_1 \) and \( n_2 \), then the value of the current option position is

\[
V = n_1 V_1 + n_2 V_2
\]

Then the delta neutral position satisfies

\[
n_1 \Delta_1 + n_2 \Delta_2 = 0
\]

where \( \Delta_i = \frac{n_i \delta_i}{n_1 \delta_1 + n_2 \delta_2} \).

Figure 11. \( \Delta \) as a Function of Stock Price and Volatility

Another useful measure related with the delta is the gamma. The gamma is the rate of change of the delta concerning the security price. Here is the SAS/IML module GAMMA.
start gamma(s,k,r,vol,time);
pl = 4*atan(1); /* 3.14159... */
root_t = sqrt(time);
d1 = (log(s/k) + r*time)/(vol*root_t)
+ .5*vol*root_t;
d1_ds = 1/(s*vol*root_t); /* d(d1)/ds */
phi_d1 = exp(-.5*d1*d1)/sqrt(2*pi);
return(phi_d1*d1_ds);
finish;

If the absolute value of the gamma is large, the variation of the delta is high with security price movements. The size of a hedging error depends on the curvature of the option value as a function of the security price. Therefore, the delta neutral portfolio should be changed to reduce the risk when you detect the large value of the gamma in absolute terms. In the following plot, you can observe that the gamma has a peak when the stock price is at the exercise price.

Figure 12. Gamma as a Function of Stock Price and Volatility

The time decay is measured by theta, which is the rate of change of the option value with respect to time. The following SAS/Ml code calculates the theta of the option:

start theta(s,k,r,vol,time,call);
/*-----------------------------------------*/
/* call = 0 d(-c)/dt */
/*-----------------------------------------*/
pl = 4*atan(1); /* 3.14159 */
root_t = sqrt(time);
d1 = (log(s/k) + r*time)/(vol*root_t)
+ .5*vol*root_t;
d2 = d1 - vol*root_t;
phi_d1 = exp(-.5*d1*d1)/sqrt(2*pi);
if call then
  val = - s*phi_d1*vol/(2*root_t)
    + k*exp(-r*time)*probnorm(d2);
else
  val = - s*phi_d1*vol/(2*root_t)
    + k*exp(-r*time)*probnorm(-d2);
return(val);
finish;

The theta of the derivative security becomes 0 when the stock price and the volatility are very low. However, the high volatility makes theta decline. When the theta is large and negative, the gamma should be large and positive assuming that the delta is 0. Here is a plot of the theta as a function of stock price and volatility.

Figure 13. Theta as a Function of Stock Price and Volatility

FORECASTING THE VOLATILITY

In this section, the Dow Jones weighted index is analyzed over the Eisenhower-Kennedy period of 1953 to 1963. The stock index (i) is the weighted average of the 30 industrial and 20 transportation closing price indices. The market return is defined as

\[ R_t = \log(\frac{P_t}{P_{t-1}}) \]

Consider the autoregressive process:

\[ R_t = \varphi_0 - \varphi_1 R_{t-1} - \ldots - \varphi_p R_{t-p} + \varepsilon_t \]

where \( \varepsilon_t \) follows GARCH(\( p,q \)) process. To obtain the predicted error variance, the GARCH(\( p,q \)) process can be written

\[ \varepsilon_t^2 = \omega + \sum_{j=1}^{\infty} (\alpha_j + \gamma_j) \varepsilon_{t-j}^2 \]

where \( \eta_t = \varepsilon_t^2 - h_t \) and \( n = \max(p,q) \). The variance \( h_t \) becomes \( \omega \) if the disturbances \( \varepsilon_t \) are homoscedastic. Then for any \( d > 0 \), you take the conditional expectation:

\[ E(\varepsilon_{t+d}^2 | W_t) = \omega + \sum_{j=1}^{d} (\alpha_j + \gamma_j) E(\varepsilon_{t+j-d}^2 | W_t) - \sum_{j=1}^{n} \gamma_j E(\eta_{t+d-j} | W_t) \]

where \( W_t \) denotes the available information at time \( t \). The \( d \)-step ahead prediction error, \( \xi_{t+d} = R_{t+d} - R_{t+d} - h_t \), has the conditional variance

\[ V(\xi_{t+d} | W_t) = \sum_{j=0}^{d-1} \gamma_j E(\varepsilon_{t+j-d}^2 | W_t) \]
Coefficients in the conditional d-step prediction error variance are calculated recursively using the following formula:

\[ g_j = \phi_j g_{j-1} - \cdots - \phi_m g_{j-m} \]

where \( g_0 = 1 \) and \( g_j = 0 \) if \( j < 0 \); \( \phi_1, \ldots, \phi_m \) are autoregressive parameters. Since the parameters are not known, the conditional variance is computed using the estimated GARCH and autoregressive parameters. The d-step ahead prediction error variance is simplified when there are no autoregressive terms:

\[ \nu(t+\delta|\Omega_t) = \nu_0 e^{\nu_t} \]

The daily stock returns are estimated with AR(1)-GARCH(1,1) model, and the monthly conditional variances of the AR-GARCH model is calculated in the following SAS statements:

```sas
data one: set sasuser.djind;
retain mon 1:
  r = dif(log(djind));
  if _n_ > 1 then do;
    time = _n_ - 1;
    if lag(month) ^= month
      then mon = mon+1;
    output;
  end;
run;
proc autoreg data=one;
model r = / nlag=1 garch=(q=1,p=1) 1
  out=aa outout=a eeV=cv;
run;
proc: means data=a noprint; var r CV
  output out=b mean=returns mcv
    sr
  by mon;
run,
```

The monthly stock return volatility is calculated by taking the square root of the conditional variance. The daily stock returns for the period of 1953 to 1963 are plotted in Figure 14, and the monthly stock volatilities are shown in Figure 15.

Figure 14. Daily Stock Return Series (1953 - 1963)

To study the forecasting performance, the return series is estimated for the 6 month and 12 month periods, and the subsequent one-month and three-month-ahead forecasts are produced. The estimation sample is shifted forward by one month regardless of the prediction time span. Since the sample consists of 132 months, you can get 126 one-month-ahead forecasts when the model is estimated over the 6 month period.

To compare with the GARCH model, two more volatility measures are computed. The simple measure is the sample standard deviation:

\[ sd_{t+1} = \left[ \sum_{i=1}^{T} (R_i - \bar{R})^2 / (T-1) \right]^{1/2} \]

where \( \bar{R} = \sum_{i=1}^{T} R_i / T \) and \( T \) is the sample observations. For example, \( T = \sum_{i=1}^{T} N_i \) for the six month period estimation, where \( N_i \) is the number of observations of the \( i \)th month.

Taifor (1986) suggested that the next day standard deviation be predicted with the exponentially weighted moving average (EWMA) method, since the EWMA prediction requires less parameters estimated and makes the nonstationary forecasting possible. The EWMA standard deviation forecast is calculated as follows:

\[ \hat{\nu}_{t+1} = (1 - \gamma) \hat{\nu}_t + \gamma | R_t - \bar{R}| \]

\[ \hat{\nu}_{t+1} = \hat{\nu}_{t+1} / \sqrt{\gamma^2} \]

where \( \bar{R} \) is the mean value of return. The weight parameter \( \gamma \) is estimated minimizing the function

\[ \sum_{i=1}^{T} [m_i - \hat{\nu}_i]^2 \]

where \( m_i = | R_i - \bar{R}| \).

The forecasted volatility of the sth month for the sample and EWMA standard deviations are computed as \( \sqrt{\nu_s} sd_{t+1} \) and \( \sqrt{\nu_s} \nu_{t+1} \), respectively, since \( sd_{t+1} \) and \( \nu_{t+1} \) are daily volatility forecasts. For example, the total number of observation \( T \) for the monthly forecast based on the past 12 months is \( \sum_{i=1}^{12} N_i \).
The one month forecast of future conditional variance for the GARCH model is calculated as the sum of the daily forecasts of conditional variances. For example, the monthly variance forecast $\hat{\sigma}_L^2$ using the daily conditional variance forecast is written

$$\hat{\sigma}_L^2 = \sum_{i=1}^{L} \nu(t_i, j|\nu_i)$$

The three month GARCH forecast variance is computed as $\hat{\sigma}_L^2$. For each estimation period, the AR(1)-GARCH(p,q) model is estimated. Since variances for the forecast period are not observed, the following variance estimate is used as a proxy for the actual variance:

$$\nu^2 = \sum_{i=1}^{T_l} (R_i^2 - \bar{R})^2 \left[ 1 + \sum_{j=1}^{T_l-1} (T - j)/\nu_i \right]$$

where $T_l = N_l$ for one month forecast while $T_l = \sum_{j=1}^{T} \sum_{i=1}^{N_l}$ for three month forecasts where the autocorrelation coefficient $\nu$ is estimated over the whole sample period.

The standard deviation of forecasts are evaluated using the following statistics:

$$MSE = \sum_{i=1}^{N} \left( \nu_{i} - \nu \right)^2 / N$$

$$MAE = \sum_{i=1}^{N} | \nu_{i} - \nu | / N$$

$$MAPE = \sum_{i=1}^{N} \left| \frac{\nu_{i} - \nu_{i}}{\nu_{i}} \right| / N$$

where $\nu_{i}$ is the forecasted volatility, and $N$ is the number of ex post simulation replicates. The results are summarized in Table 1.

Table 1. Volatility Forecasts

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<th>Period Est/For</th>
<th>GARCH</th>
<th>SAMPLE SD</th>
<th>EWMA</th>
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<td>6E/3F</td>
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<tr>
<td>12E/1F</td>
<td>0.00054</td>
<td>0.00067</td>
<td>0.00063</td>
</tr>
<tr>
<td>12E/3F</td>
<td>0.00121</td>
<td>0.00138</td>
<td>0.00170</td>
</tr>
</tbody>
</table>

| MAE 6E/1F      | 0.01169 | 0.01175   | 0.01308 |
| 6E/3F          | 0.02169 | 0.02508   | 0.02849 |
| 12E/1F         | 0.01731 | 0.01885   | 0.01968 |
| 12E/3F         | 0.02352 | 0.02750   | 0.03143 |

The GARCH volatility outperforms the sample and EWMA volatilities in this limited ex post simulation. The sample standard deviation shows better forecast performance than the EWMA volatility. Therefore, the EWMA method is not recommendable to forecast the monthly volatilities even though Taylor (1986) argues that the EWMA method is good for the next day forecast of the market volatility.

CONCLUSION

This paper explored the application of the family of GARCH models using the AUTOREG procedure. The option valuation model is also discussed using other SAS procedures. The AUTOREG procedure is shown to be very useful to model the financial applications. PROC AUTOREG is especially beneficial to forecast the conditional mean and variances of the time series data. For example, the option valuation can efficiently be made using the GARCH-type conditional variance predictions. The ex post simulation study indicates that the volatility of the GARCH model is superior to the currently used volatilities.

REFERENCES


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