INTRODUCTION

Monitoring processes or services for the purpose of quality improvement is a critical goal in most product and services oriented environments today. In the workplace it is customary for the customer to demand insight into the quality of the products or services they purchase. To provide this insight, the manufacturer or service provider must oftentimes exhibit quantitative evidence of the key characteristics of the processes or products. A key question upfront is that of the monitoring goals and the corresponding sampling frequency dictated by such goals. This quantitative evidence may be as fundamental as that reflected in a quality control chart on the mean, range, variance, or some other statistical entity all the way up to that of using multivariate statistics to study joint relationships of multiple variables in a process. To perform these tasks oftentimes dictates a need to collect and maintain considerable data loads. As an example, if one is tracking the viscosity of a chemical product, a priori decisions must be made in regard to such things as:

(a) What will be the criterion to identify an inadequate product (e.g., a suspect item might be one that falls outside a 3 standard deviation envelope --- as reflected in a quality control chart)?

(b) What will be a sufficient sampling frequency (i.e., monitoring rate) to gain adequate insight into the process behavior?

(c) Monitoring a process usually requires use of various statistical estimators of underlying process parameters, consequently, how does one ensure that the estimators meet project goals (e.g., in control charting, one should insist on some understanding of the accuracy of the sample mean and variance as estimators of the true process mean and variance)?

(d) Since numerous products and services are manufactured or administered in a temporal manner, when should a critical monitoring parameter estimate be updated (e.g., in control charting, when should one update the variance estimate of a process to ensure a more accurate control chart from its usage)?

Continuing with the example of monitoring viscosity, numerous critical decisions are required including: 1) a choice for what variance value to use (e.g., if it is estimated from sampled data, how much data should be utilized and when should it be updated to best reflect current acceptable viscosity characteristics?), 2) a choice of a decision logic that permits adequate warning with a minimum of false alarms, 3) identification of decision points in time to consider updating estimates of process characteristics, and 4) monitoring frequencies (i.e., is it sufficient to monitor viscosity hourly, daily, weekly, etc.) to meet preselected accuracy goals. This paper addresses these issues using statistical hypothesis testing methods. Specifically, using the concept of the "power" of a test and the actual decision logic in testing hypotheses concerning the mean of a process, relationships between appropriate sample sizes, detection of significant off-aim drifts, and associated confidence statements about all conclusions are attainable.
Unfortunately, these statistical concepts are not always straightforward for the layman (i.e., nonstatistician) and any insight that supports clarity is highly desirable (and critical) to the correct usage and interpretation of such tools. This paper provides a SAS/GRA I'll program that can be utilized to present these concepts compactly and with considerable clarity for the layman.

2 Approach

Emphasis is on monitoring quality via the means and variances. The differing cases discussed differ only in the number of processes (i.e., populations) being monitored.

2.1 Detecting Off-Aim Drifts in the Process Mean $\mu$

Consider monitoring a particular product characteristic with a desirable preselected "on-aim" (mean, $\mu$) value of $\mu_0$. Since variation is an inherent part of the typical manufacturing process, the product manufacturer must necessarily monitor (i.e., sample) the process periodically to maintain adequate insight into product characteristics. It is especially desirable to be able to answer the following question: What sample size, $n$, should be utilized to detect an off-aim drift (i.e., away from $\mu_0$) toward a value $\mu_1$ such that the probability is $1 - \beta$ of detecting when the difference $\delta = (\mu_0 - \mu_1)/s$ exceeds a preselected value (e.g., 1.5) with a confidence probability of $1 - \alpha$ that the collective conclusions hold true. Notice that the difference $\delta$ is the magnitude of off-aim drift in terms of standard deviation units $s$ (i.e., $s$ is the sample standard deviation) and, hence, is a unitless quantity. Consequently, since $\delta$ is unitless, the results are applicable to numerous monitoring tasks. Obviously, it seems reasonable to expect $n$, $s$, $\delta$, $\alpha$, and $\beta$ to be functionally related in some manner. From the use of statistical hypothesis testing logic, this connection is easily seen to be given by the following implicit functional relationship:

$$[F^{-1}(1 - \beta) + F^{-1}(1 - \alpha/2)]/\sqrt{n} = \delta$$  \hspace{1cm} (1)

Arriving at this relationship amounts to that of finding the power, $1 - \beta$, in testing the statistical hypothesis $H_0: \mu = \mu_0$ versus the alternative $H_1: \mu = \mu_1$ with an overall confidence coefficient of $1 - \alpha$. The notation $F^{-1}$ in the above is the inverse of the cumulative distribution function of the $t$-distribution with $n - 1$ degrees of freedom (see any introductory statistics text, e.g., [2]). To the layman, such an expression lacks clarity; however, use of SAS/GRAPH provides the needed insight. Specifically, by fixing $\alpha = .05$ (a frequent choice which guarantees a confidence coefficient of $95$), letting the vertical axis represent choices of the sample size $n$, the horizontal axis represent candidate values of $\delta$, and by overlaying various curves corresponding to various choices of the detection probabilities (i.e., values of $1 - \beta$), one attains insight into these interrelationships by investigating the graph in Figure 1. This graph was generated by a few lines of SAS/GRAPH (see the program listing in Section 3) and can easily be altered to get various other related plots (e.g., one may wish to select differing values of confidence $1 - \alpha$ and/or differing probabilities of detection, $1 - \beta$).

2.2 Detecting Changes in the Process Variance $\sigma^2$

Since numerous products and services are manufactured or administered in a temporal manner, when should the variance of the critical monitoring parameter estimate be updated (e.g., in the simple task of control charting, when should one update the variance estimator to ensure a more accurate control chart from its usage)? Thus, a critical question regarding a process variance is: what sample size, $n$, should be utilized to detect a shift in the variance $\sigma^2$ (i.e., away from $\sigma^2_0$ toward a value $\sigma^2_1$) such that the probability is $1 - \beta$ of detecting when the ratio $\sigma^2_0/\sigma^2_1$ exceeds a preselected value (e.g., 3) with a confidence probability of $1 - \alpha$ that the collective conclusions hold true. The functional relationship between $n$, $\alpha$, $\beta$, and the ratio $K = \sigma^2_0/\sigma^2_1$ is given implicitly as follows:

\hspace{1cm}
As in the case for \( \mu \) (discussed above), arriving at this relationship consists of finding the power, \( 1 - \beta \), in testing \( H_0: \sigma^2 = \sigma_0^2 \) versus \( H_1: \sigma^2 = \sigma_1^2 \) with an overall confidence coefficient of \( 1 - \alpha \). Figure 2 provides considerable clarity regarding these interrelationships (the generation of this plot using SAS/GRAPH is discussed in Section 3 as well).

\[
K = \frac{F^{-1}(1 - \alpha/2)}{F^{-1}(\beta)}
\]  

(2)

2.3 DETECTING DIFFERENCES BETWEEN TWO PROCESS VARIANCES AND MEANS

At times, two geographically remote plants are tasked with manufacturing the same product for shipment to customers. A key issue is whether or not the two plants produce the same product (i.e., what are the conditions that lead to declaring a difference in the variances and means). First, for variances, the implicit relationship of interest is

\[
k = F^{-1}(1 - \alpha/2)/F^{-1}(\beta)
\]  

(3)

which is plotted in Figure 3 using the SAS/GRAPH program discussed in Section 3. It should be noted the function \( F^{-1} \) in equation (3) is the inverse of the cumulative \( F \)-distribution with \( n - 1 \) degrees of freedom for both, the numerator and denominator (see [2] for a discussion of the \( F \)-distribution). As before, this functional relationship results from the logistics that are the basis for testing \( H_0: \sigma^2 = \sigma_0^2 \) versus \( H_1: \sigma^2 = \sigma_1^2 \) (where \( k \neq 1 \)). Similarly, if \( \sigma_1^2 \) and \( \sigma_2^2 \) are acceptably the same, the following relationship establishes the implicit relationship between \( \alpha, \beta, n, \) and \( \delta = (\mu_1 - \mu_2)/s \). Figure 4 provides the clarification needed to see these relationships; again, the SAS/GRAPH program that generated this plot is discussed in Section 3.

\[
\sqrt{2[F_{12n-1}^{-1}(1 - \beta) + F_{12n-1}^{-1}(1 - \alpha/2)]/\sqrt{n}} = \delta
\]  

(4)

In this expression, \( F_{12n-1}^{-1} \) is the inverse of the cumulative \( t \)-distribution with \( 2n - 2 \) degrees of freedom [2].

2.4 DETECTING DIFFERENCES BETWEEN SEVERAL PROCESS MEANS

Generalizing from monitoring two process means to that of several means results in yet other relationships (see reference [1]). Again, these interrelationships are not
easily understood without resorting to graphical depictions like those given in the previous section. In particular, a graph can easily be generated for the situation whereby there are \( k \) means and the plot depicts the minimal sample size required to detect a maximum difference \( \Delta \) between a pair of the \( k \) means with a detection probability \( 1 - \beta \) and with overall confidence \( 1 - \alpha \) (to be discussed at conference time).

Sample Size versus Detection Probability (two means)

\[
F_{n-k}^{-1}(1 - \beta) + F_{n-k}^{-1}(1 - \alpha/2) - 1 = \delta / \sqrt{\lambda^T(X^T X)^{-1}} \lambda
\]

which is easily implemented in SAS/GRAPH (to be discussed at conference time). It should be noted that in equation (5), \( F_{n-k}^{-1} \) is the inverse of the cumulative distribution function of the \( t \)-distribution with \( n - k \) degrees of freedom [1].

3 Discussion of SAS/GRAPH Program

The program listing given below is the specific program used to generate the plot in Figure 1. The generation of the other plots given in Figures 2 through 5 result from minor modifications of this listing; these modifications are detailed in the discussion following the listing. It should be noted that this particular program generates 4 overlaid plots in each case considered in this paper; the user may wish to only plot one or possibly change the values of \( \alpha \) and/or \( \beta \), or alter the output in some other way. As can be seen, the program is small and modifications to generate differing outputs than given herein should be straightforward.

1. /* OPTIONS DEV=VGA; /*
2. LEGEND1 LABEL=NONE VALUE=(COLOR=GREEK
   '1-b=.95' '1-b=.85' '1-b=.75'
   '1-b=.65') ACROSS=1 FRAME
   POSITION=(INSIDE MIDDLE) MODE=PROTECT
   OFFSET=(5,0);
3. SYMBOL1 C=BLUE I=JOIN V=NONE;
4. SYMBOL2 C=GREEN I=JOIN V=NONE;
5. SYMBOL3 C=YELLOW I=JOIN V=NONE;
6. SYMBOL4 C=RED I=JOIN V=NONE;
7. AXIS1 LABEL=(F=GREEK 'd') ORDER=(.25
   TO 2 BY .25);
8. OPTIONS VSIZE=4.5 IN HSIZE=1.5 IN;
9. DATA MADE;
10. ALPHA=.05;
11. BETA1=.05;
12. BETA2=.15;
13. BETA3=.25;
14. BETA4=.35;
15. PI=1-BETA1;P2=1-BETA2;P3=1-BETA3;P4=1-BETA4;
16. ARRAY D(4);ARRAY H(4) P1-P4;
17. DO N=5 TO 51;
18. DO l=1 TO 4;
19. D(I)=TINV(P(I),N-1)+TINV(1-ALPHA/2,N-
   1))/N**.5);
To obtain the plot in Figure 2, one needs to change program line 19 to:
\[ D(I) = (\text{CINV}(I-\alpha/2, N-I)) / (\text{CINV}(1-P(I), N-I)) \]
and the OVERLAY portion of program line 25 to:
\[ \text{OVERLAY HAXIS=AXIS3 VAXIS=0 TO 50 BY 10 VREF=10 20 30 40 50 HREF=.5 .75 1 1.25 1.5 1.75 LEGEND=LEGEND1;} \]

For the plot in Figure 3, line 19 becomes:
\[ D(I) = (\text{FINV}(1-\alpha/2, N-1, N-I)) / (\text{FINV}(1-P(I), N-1, N-I)) \]
and the OVERLAY portion of line 25 is changed to:
\[ \text{OVERLAY HAXIS=AXIS4 VAXIS=0 TO 50 BY 10 VREF=10 20 30 40 50 HREF=3 4 5 6 7 8 LEGEND=LEGEND1;} \]

Finally, for the plot in Figure 4, line 19 becomes:
\[ D(I) = (\text{TINV}(P(I)/2, N-2)) + (\text{TINV}(1-\alpha/2, 2*N-2)) / (\text{N}/2**.5); \]
and the OVERLAY portion of line 25 is changed to:
\[ \text{OVERLAY HAXIS=AXIS2 VAXIS=0 TO 50 BY 10 VREF=10 20 30 40 50 HREF=.75 1 1.25 1.5 1.75 2 2.25 LEGEND=LEGEND1.} \]

Note that the use of the HAXIS statement was necessary to get the plot to take on a certain desirable look. In particular, this part of the plot is determined by the range of values the user wishes to see for \( \delta \) (in the case of tests on means) or \( k \) (in the case of tests on the variances). Further discussion regarding the flexibility and options in the above SAS/GRAPH program will be taken up at conference time.

4 Discussion/Conclusions
The availability of SAS/GRAPH with options like the overlay capability, as well as its many other flexibilities, can greatly aid the layman (i.e., the nonstatistician). This paper demonstrates how invaluable SAS/GRAPH can be in establishing easily readable graphics generated from somewhat complicated statistical, functional relationships. Several key relationships are included that are of value in gaining insight to support critical decisions related to sample sizes, probabilities of detecting pre-selected off-aim process drifts, magnitudes of the drift to be sensed, along with confidence coefficients for joint results.

5 References