# SAS/STAT® 13.1 User's Guide

# The QUANTREG Procedure

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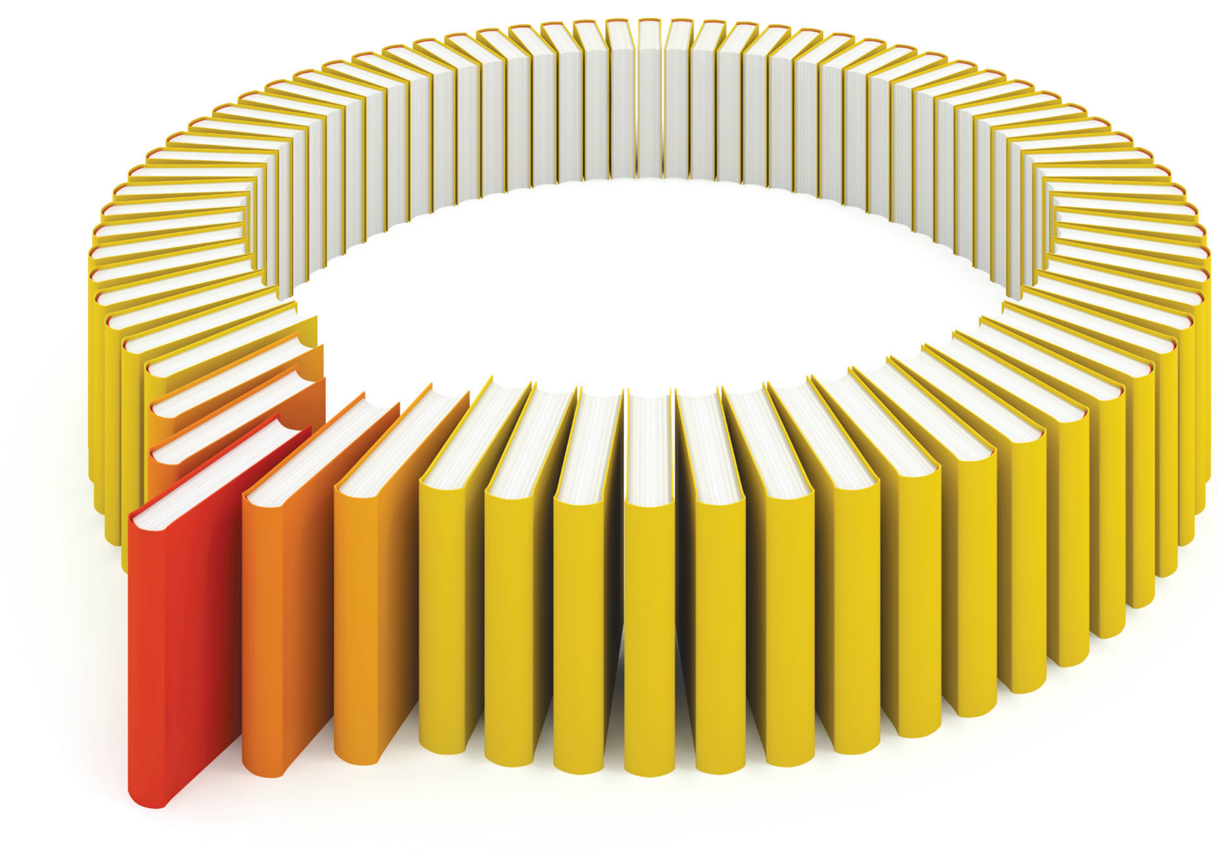


# Chapter 81
# The QUANTREG Procedure

**Contents**

# Overview: QUANTREG Procedure

The QUANTREG procedure uses quantile regression to model the effects of covariates on the conditional quantiles of a response variable.

Quantile regression was introduced by Koenker and Bassett (1978) as an extension of ordinary least squares (OLS) regression, which models the relationship between one or more covariates $X$ and the *conditional mean* of the response variable $Y$ given $X = x$. Quantile regression extends the OLS regression to model the *conditional quantiles* of the response variable, such as the median or the 90th percentile. Quantile regression is particularly useful when the rate of change in the conditional quantile, expressed by the regression coefficients, depends on the quantile.

**Figure 81.1** Trout Density in Streams

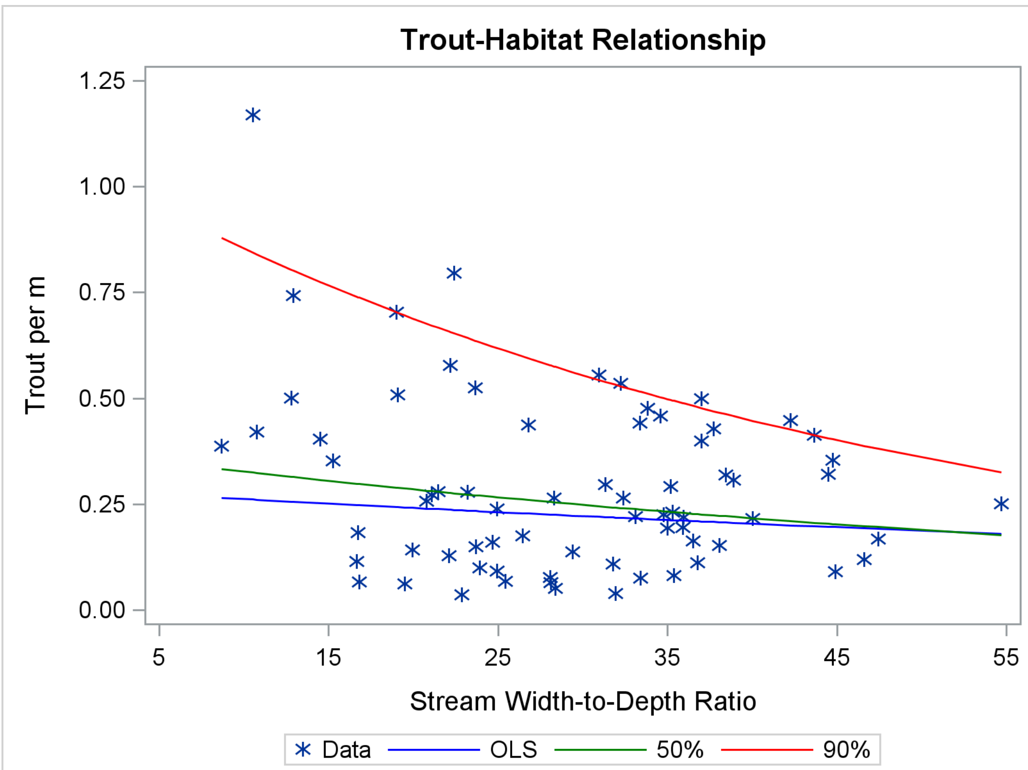
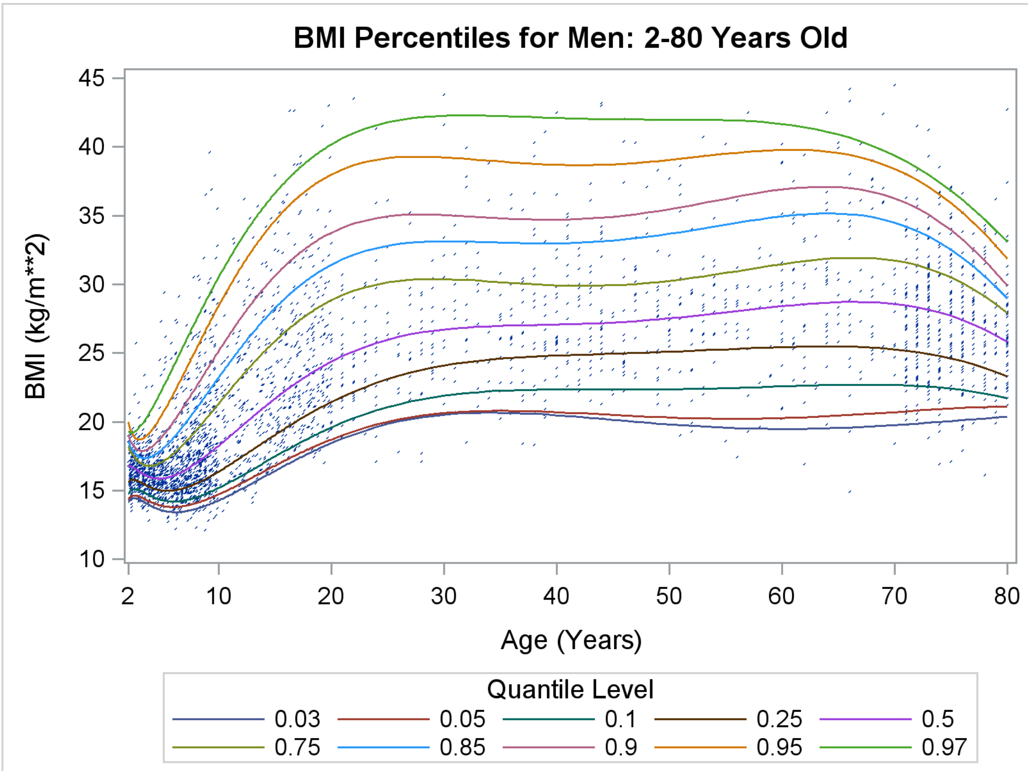



Figure 81.1 illustrates an ecological study in which modeling upper conditional quantiles reveals additional information. The points represent measurements of trout density and stream width-to-depth ratio that were taken at 13 streams over seven years.

As analyzed by Dunham, Cade, and Terrell (2002), both the ratio and the trout density depend on a number of unmeasured limiting factors that are related to the integrity of stream habitat. The interaction of these factors results in unequal variances for the conditional distributions of density given the ratio. When the ratio is the "active" limiting effect, changes in the upper conditional percentiles of density provide a better estimate of this effect than changes in the conditional mean.

The red and green curves represent the conditional 90th and 50th percentiles of density as determined by the QUANTREG procedure. The analysis was done by using a simple linear regression model for the logarithm of density. (The curves in Figure 81.1 were obtained by transforming the fitted lines back to the original scale. For more information, see the section "Analysis of Fish-Habitat Relationships" on page 6874.) The slope parameter for the 90th percentile has an estimated value of –0.0215 and is significant with a *p*-value less than 0.01. On the other hand, the slope parameter for the 50th percentile is not significantly different from 0. Similarly, the slope parameter for the mean, which is obtained with OLS regression, is not significantly different from 0.

**Figure 81.2** Percentiles for Body Mass Index




Quantile regression is especially useful when the data are heterogeneous in the sense that the tails and the central location of the conditional distributions vary differently with the covariates. An even more pronounced example of heterogeneity is shown in Figure 81.2, which plots the body mass index of 8,250 men versus their age.

Here, both upper (overweight) and lower (underweight) conditional quantiles are important because they provide the basis for developing growth charts and establishing health standards. The curves in Figure 81.2 were determined by using the QUANTREG procedure to perform polynomial quantile regression. For more information, see the section "Growth Charts for Body Mass Index" on page 6880. Clearly, the rate of change with age (as expressed by the regression coefficients), particularly for ages less than 20, is different for each conditional quantile.

Heterogeneous data occur in many fields, including biomedicine, econometrics, survival analysis, and ecology. Quantile regression, which includes median regression as a special case, provides a complete picture of the covariate effect when a set of percentiles is modeled. So it can capture important features of the data that might be missed by models that average over the conditional distribution.

Because it makes no distributional assumption about the error term in the model, quantile regression offers considerable model robustness. The assumption of normality, which is often made with OLS regression in order to compute conditional quantiles as offsets from the mean, forces a common set of regression coefficients for all the quantiles. Obviously, quantiles with common slopes would be inappropriate in the preceding examples.

Quantile regression is also flexible because it does not involve a link function that relates the variance and the mean of the response variable. Generalized linear models, which you can fit with the GENMOD procedure, require both a link function and a distributional assumption such as the normal or Poisson distribution. The goal of generalized linear models is inference about the regression parameters in the linear predictor for the mean of the population. In contrast, the goal of quantile regression is inference about regression coefficients for the conditional quantiles of a response variable that is usually assumed to be continuous.

Quantile regression also offers a degree of data robustness. Unlike OLS regression, quantile regression is robust to extreme points in the response direction (outliers). However, it is not robust to extreme points in the covariate space (leverage points). When both types of robustness are of concern, consider using the ROBUSTREG procedure (Chapter 84, "The ROBUSTREG Procedure.")

Unlike OLS regression, quantile regression is equivariant to monotone transformations of the response variable. For example, as illustrated in the trout example, the logarithm of the 90th conditional percentile of trout density is the 90th conditional percentile of the logarithm of density.

Quantile regression cannot be carried out simply by segmenting the unconditional distribution of the response variable and then obtaining least squares fits for the subsets. This approach leads to disastrous results when, for example, the data include outliers. In contrast, quantile regression uses *all* of the data for fitting quantiles, even the extreme quantiles.

## Features

The main features of the QUANTREG procedure are as follows:

- offers simplex, interior point, and smoothing algorithms for estimation
- provides sparsity, rank, and resampling methods for confidence intervals
- provides asymptotic and bootstrap methods for covariance and correlation matrices of the estimated parameters
- provides the Wald, likelihood ratio, and rank tests for the regression parameter estimates and the Wald test for heteroscedasticity
- provides outlier and leverage-point diagnostics
- enables parallel computing when multiple processors are available
- provides rowwise or columnwise output data sets with multiple quantiles
- provides regression quantile spline fits
- produces fit plots, diagnostic plots, and quantile process plots by using ODS Graphics

The next section provides notation and a formal definition for quantile regression.

## Quantile Regression

Quantile regression generalizes the concept of a univariate quantile to a conditional quantile given one or more covariates. Recall that a student's score on a test is at the $\tau$ quantile if his or her score is better than that of $100\tau\%$ of the students who took the test. The score is also said to be at the $100\tau$ percentile.

For a random variable $Y$ with probability distribution function

$$F(y) = \text{ Prob } (Y \leq y)$$

the $\tau$ quantile of $Y$ is defined as the inverse function

$$Q(\tau) = \text{ inf } \{y : F(y) \geq \tau\}$$

where the quantile level $\tau$ ranges between 0 and 1. In particular, the median is $Q(1/2)$.

For a random sample $\{y_1, \ldots, y_n\}$ of $Y$, it is well known that the sample median minimizes the sum of absolute deviations:

$$\text{median} = \arg\min_{\xi \in \mathbf{R}} \sum_{i=1}^{n} |y_i - \xi|$$

Likewise, the general $\tau$ sample quantile $\xi(\tau)$, which is the analog of $Q(\tau)$, is formulated as the minimizer

$$\xi(\tau) = \arg\min_{\xi \in \mathbf{R}} \sum_{i=1}^{n} \rho_\tau(y_i - \xi)$$

where $\rho_\tau(z) = z(\tau - I(z < 0))$, $0 < \tau < 1$, and where $I(\cdot)$ denotes the indicator function. The loss function $\rho_\tau$ assigns a weight of $\tau$ to positive residuals $y_i - \xi$ and a weight of $1 - \tau$ to negative residuals.

Using this loss function, the linear conditional quantile function extends the $\tau$ sample quantile $\xi(\tau)$ to the regression setting in the same way that the linear conditional mean function extends the sample mean. Recall that OLS regression estimates the linear conditional mean function $E(Y|X = x) = \mathbf{x}'\boldsymbol{\beta}$ by solving for

$$\hat{\boldsymbol{\beta}} = \arg\min_{\boldsymbol{\beta} \in \mathbf{R}^p} \sum_{i=1}^{n} (y_i - \mathbf{x}_i'\boldsymbol{\beta})^2$$

The estimated parameter $\hat{\boldsymbol{\beta}}$ minimizes the sum of squared residuals in the same way that the sample mean $\hat{\mu}$ minimizes the sum of squares:

$$\hat{\mu} = \arg\min_{\mu \in \mathbf{R}} \sum_{i=1}^{n} (y_i - \mu)^2$$

Likewise, quantile regression estimates the linear conditional quantile function, $Q_Y(\tau|X = x) = \mathbf{x}'\boldsymbol{\beta}(\tau)$, by solving the following equation for $\tau \in (0, 1)$:

$$\hat{\boldsymbol{\beta}}(\tau) = \arg\min_{\boldsymbol{\beta} \in \mathbf{R}^p} \sum_{i=1}^{n} \rho_\tau(y_i - \mathbf{x}_i'\boldsymbol{\beta})$$

The quantity $\hat{\boldsymbol{\beta}}(\tau)$ is called the $\tau$ *regression quantile*. The case $\tau = 0.5$ (which minimizes the sum of absolute residuals) corresponds to median regression (which is also known as $L_1$ regression).

The following set of regression quantiles is referred to as the *quantile process*:

$$\{\boldsymbol{\beta}(\tau) : \tau \in (0, 1)\}$$

The QUANTREG procedure computes the quantile function $Q_Y(\tau|X = x)$ and conducts statistical inference on the estimated parameters $\hat{\boldsymbol{\beta}}(\tau)$.

# Getting Started: QUANTREG Procedure

The following examples demonstrate how you can use the QUANTREG procedure to fit linear models for selected quantiles or for the entire quantile process. The first example explains the use of the procedure in a fish-habitat example, and the second example explains the use of the procedure to construct growth charts for body mass index.

## Analysis of Fish-Habitat Relationships

Quantile regression is used extensively in ecological studies (Cade and Noon 2003). Recently, Dunham, Cade, and Terrell (2002) applied quantile regression to analyze fish-habitat relationships for Lahontan cutthroat trout in 13 streams of the eastern Lahontan basin, which covers most of northern Nevada and parts of southern Oregon. The density of trout (number of trout per meter) was measured by sampling stream sites from 1993 to 1999. The width-to-depth ratio of the stream site was determined as a measure of stream habitat.

The goal of this study was to explore the relationship between the conditional quantiles of trout density and the width-to-depth ratio. The scatter plot of the data in Figure 81.1 indicates a nonlinear relationship, so it is reasonable to fit regression models for the conditional quantiles of the log of density. Because regression quantiles are equivariant under any monotonic (linear or nonlinear) transformation (Koenker and Hallock 2001), the exponential transformation converts the conditional quantiles to the original density scale.

The data set trout, which follows, includes the average numbers of Lahontan cutthroat trout per meter of stream (Density), the logarithm of Density (LnDensity), and the width-to-depth ratios (WDRatio) for 71 samples:

```
data trout;
   input Density WDRatio LnDensity @@;
   datalines;
 0.38732     8.6819    -0.94850    1.16956    10.5102     0.15662
 0.42025    10.7636    -0.86690    0.50059    12.7884    -0.69197
 0.74235    12.9266    -0.29793    0.40385    14.4884    -0.90672
 0.35245    15.2476    -1.04284    0.11499    16.6495    -2.16289
 0.18290    16.7188    -1.69881    0.06619    16.7859    -2.71523
 0.70330    19.0141    -0.35197    0.50845    19.0548    -0.67639

   ... more lines ...

 0.25125    54.6916    -1.38129
;
```

The following statements use the QUANTREG procedure to fit a simple linear model for the 50th and 90th percentiles of LnDensity:

```
ods graphics on;

proc quantreg data=trout alpha=0.1 ci=resampling;
   model LnDensity = WDRatio / quantile=0.5 0.9
                               CovB seed=1268;
   test WDRatio / wald lr;
run;
```

The MODEL statement specifies a simple linear regression model with LnDensity as the response variable $Y$ and WDRatio as the covariate $X$. The QUANTILE= option requests that the regression quantile function $Q(\tau|X = x) = \mathbf{x}'\boldsymbol{\beta}(\tau)$ be estimated by solving the following equation, where $\tau = (0.5, 0.9)$:

$$\hat{\boldsymbol{\beta}}(\tau) = \arg\min_{\boldsymbol{\beta}\in\mathbf{R}^2} \sum_{i=1}^{n} \rho_\tau(y_i - \mathbf{x}_i'\boldsymbol{\beta})$$

By default, the regression coefficients $\hat{\boldsymbol{\beta}}(\tau)$ are estimated by using the simplex algorithm, which is explained in the section "Simplex Algorithm" on page 6900. The ALPHA= option requests 90% confidence limits for the regression parameters, and the option CI=RESAMPLING specifies that the intervals be computed by using the Markov chain marginal bootstrap (MCMB) resampling method of He and Hu (2002). When you specify the CI=RESAMPLING option, the QUANTREG procedure also computes standard errors, $t$ values, and $p$-values of regression parameters by using the MCMB resampling method. The SEED= option specifies a seed for the resampling method. The COVB option requests covariance matrices for the estimated regression coefficients, and the TEST statement requests tests for the hypothesis that the slope parameter (the coefficient of WDRatio) is 0.

Figure 81.3 displays model information and summary statistics for the variables in the model. The summary statistics include the median and the standardized median absolute deviation (MAD), which are robust measures of univariate location and scale, respectively. For more information about the standardized MAD, see Huber (1981, p. 108).

**Figure 81.3** Model Fitting Information and Summary Statistics

The QUANTREG Procedure

Model Information

| | |
|---|---|
| Data Set | WORK.TROUT |
| Dependent Variable | LnDensity |
| Number of Independent Variables | 1 |
| Number of Observations | 71 |
| Optimization Algorithm | Simplex |
| Method for Confidence Limits | Resampling |

Summary Statistics

| Variable | Q1 | Median | Q3 | Mean | Standard Deviation | MAD |
|---|---|---|---|---|---|---|
| WDRatio | 22.0917 | 29.4083 | 35.9382 | 29.1752 | 9.9859 | 10.4970 |
| LnDensity | -2.0511 | -1.3813 | -0.8669 | -1.4973 | 0.7682 | 0.8214 |

Figure 81.4 and Figure 81.5 display the parameter estimates, standard errors, 95% confidence limits, *t* values, and *p*-values that are computed by the resampling method.

**Figure 81.4** Parameter Estimates at QUANTILE=0.5

Parameter Estimates

| Parameter | DF | Estimate | Standard Error | 90% Confidence Limits | | t Value | Pr > \|t\| |
|---|---|---|---|---|---|---|---|
| Intercept | 1 | -0.9811 | 0.3952 | -1.6400 | -0.3222 | -2.48 | 0.0155 |
| WDRatio | 1 | -0.0136 | 0.0123 | -0.0341 | 0.0068 | -1.11 | 0.2705 |

**Figure 81.5** Parameter Estimates at QUANTILE=0.9

Parameter Estimates

| Parameter | DF | Estimate | Standard Error | 90% Confidence Limits | | t Value | Pr > \|t\| |
|---|---|---|---|---|---|---|---|
| Intercept | 1 | 0.0576 | 0.2606 | -0.3769 | 0.4921 | 0.22 | 0.8257 |
| WDRatio | 1 | -0.0215 | 0.0075 | -0.0340 | -0.0091 | -2.88 | 0.0053 |

The 90th percentile of trout density can be predicted from the width-to-depth ratio as follows:

$$\hat{y}_{90} = \exp(0.0576 - 0.0215x)$$

This is the upper dashed curve that is plotted in Figure 81.1. The lower dashed curve for the median can be obtained in a similar fashion.

The covariance matrices for the estimated parameters are shown in Figure 81.6. The resampling method that is used for the confidence intervals is also used to compute these matrices.

**Figure 81.6** Covariance Matrices of the Estimated Parameters

**Estimated Covariance Matrix for Quantile Level = 0.5**

| | Intercept | WDRatio |
|---|---|---|
| Intercept | 0.156191 | -.004653 |
| WDRatio | -.004653 | 0.000151 |

**Estimated Covariance Matrix for Quantile Level = 0.9**

| | Intercept | WDRatio |
|---|---|---|
| Intercept | 0.067914 | -.001877 |
| WDRatio | -.001877 | 0.000056 |

The tests requested by the TEST statement are shown in Figure 81.7. Both the Wald test and the likelihood ratio test indicate that the coefficient of width-to-depth ratio is significantly different from 0 at the 90th percentile, but the difference is not significant at the median.

**Figure 81.7** Tests of Significance

**Test Results**

| Quantile Level | Test | Test Statistic | DF | Chi-Square | Pr > ChiSq |
|---|---|---|---|---|---|
| 0.5 | Wald | 1.2339 | 1 | 1.23 | 0.2666 |
| 0.5 | Likelihood Ratio | 1.1467 | 1 | 1.15 | 0.2842 |
| 0.9 | Wald | 8.3031 | 1 | 8.30 | 0.0040 |
| 0.9 | Likelihood Ratio | 9.0529 | 1 | 9.05 | 0.0026 |

In many quantile regression problems it is useful to examine how the estimated regression parameters for each covariate change as a function of $\tau$ in the interval $(0, 1)$. The following statements use the QUANTREG procedure to request the estimated quantile processes $\hat{\boldsymbol{\beta}}(\tau)$ for the slope and intercept parameters:

```
proc quantreg data=trout alpha=0.1 ci=resampling;
   model LnDensity = WDRatio / quantile=process seed=1268
                               plot=quantplot;
run;
```

The QUANTILE=PROCESS option requests an estimate of the quantile process for each regression parameter. The options ALPHA=0.1 and CI=RESAMPLING specify that 90% confidence bands for the quantile processes be computed by using the resampling method.

Figure 81.8 displays a portion of the objective function table for the entire quantile process. The objective function is evaluated at 77 values of $\tau$ in the interval $(0, 1)$. The table also provides predicted values of the conditional quantile function $Q(\tau)$ at the mean for WDRatio, which can be used to estimate the conditional density function.

**Figure 81.8** Objective Function

**Objective Function for Quantile Process**

| Label | Quantile Level | Objective Function | Predicted at Mean |
|---|---|---|---|
| t0 | 0.005634 | 0.7044 | -3.2582 |
| t1 | 0.020260 | 2.5331 | -3.0331 |
| t2 | 0.031348 | 3.7421 | -2.9376 |
| t3 | 0.046131 | 5.2538 | -2.7013 |
| . | . | . | . |
| . | . | . | . |
| . | . | . | . |
| t73 | 0.945705 | 4.1433 | -0.4361 |
| t74 | 0.966377 | 2.5858 | -0.4287 |
| t75 | 0.976060 | 1.8512 | -0.4082 |
| t76 | 0.994366 | 0.4356 | -0.4082 |

Figure 81.9 displays a portion of the table of the quantile processes for the estimated parameters and confidence limits.

**Figure 81.9** Objective Function

**Parameter Estimates for Quantile Process**

| Label | Quantile | Intercept | WDRatio |
|---|---|---|---|
| . | . | . | . |
| . | . | . | . |
| . | . | . | . |
| t57 | 0.765705 | -0.42205 | -0.01335 |
| lower90 | 0.765705 | -0.91952 | -0.02682 |
| upper90 | 0.765705 | 0.07541 | 0.00012 |
| t58 | 0.786206 | -0.32688 | -0.01592 |
| lower90 | 0.786206 | -0.80883 | -0.02895 |
| upper90 | 0.786206 | 0.15507 | -0.00289 |
| . | . | . | . |
| . | . | . | . |
| . | . | . | . |

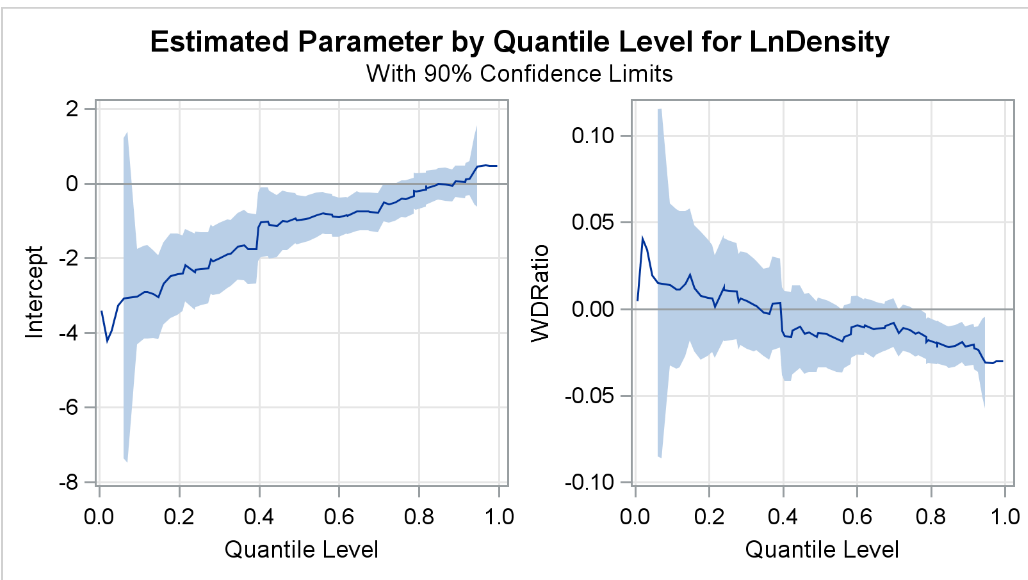
When ODS Graphics is enabled, the PLOT=QUANTPLOT option in the MODEL statement requests a plot of the estimated quantile processes.

For more information about enabling and disabling ODS Graphics, see the section "Enabling and Disabling ODS Graphics" on page 606 in Chapter 21, "Statistical Graphics Using ODS."

The left side of Figure 81.10 displays the process for the intercept, and the right side displays the process for the coefficient of WDRatio.

The process plot for WDRatio shows that the slope parameter changes from positive to negative as the quantile increases and that it changes sign with a sharp drop at the 40th percentile. The 90% confidence bands show that the relationship between LnDensity and WDRatio (expressed by the slope) is not significant below the 78th percentile. This situation can also be seen in Figure 81.9, which shows that 0 falls between the lower and upper confidence limits of the slope parameter for quantiles below 0.78. Since the confidence intervals for the extreme quantiles are not stable because of insufficient data, the confidence band is not displayed outside the interval (0.05, 0.95).

**Figure 81.10** Quantile Processes for Intercept and Slope



## Growth Charts for Body Mass Index

Body mass index (BMI) is defined as the ratio of weight (kg) to squared height ($m^2$) and is a widely used measure for categorizing individuals as overweight or underweight. The percentiles of BMI for specified ages are of particular interest. As age increases, these percentiles provide growth patterns of BMI not only for the majority of the population, but also for underweight or overweight extremes of the population. In addition, the percentiles of BMI for a specified age provide a reference for individuals at that age with respect to the population.

Smooth quantile curves have been widely used for reference charts in medical diagnosis to identify unusual subjects, whose measurements lie in the tails of the reference distribution. This example explains how to use the QUANTREG procedure to create growth charts for BMI.

A SAS data set named bmimen was created by merging and cleaning the 1999–2000 and 2001–2002 survey results for men that is published by the National Center for Health Statistics. This data set contains the variables Weight (kg), Height (m), BMI (kg/$m^2$), Age (year), and SeQN (respondent sequence number) for 8,250 men (Chen 2005).

The data set that is used in this example is a subset of the original data set of Chen (2005). It contains the two variables BMI and Age with 3,264 observations.

```
data bmimen;
   input BMI Age @@;
   SqrtAge = sqrt(Age);
   InveAge = 1/Age;
   LogBMI  = log(BMI);
   datalines;
18.6  2.0 17.1  2.0 19.0  2.0 16.8  2.0 19.0  2.1  15.5   2.1
16.7  2.1 16.1  2.1 18.0  2.1 17.8  2.1 18.3  2.1  16.9   2.1
15.9  2.1 20.6  2.1 16.7  2.1 15.4  2.1 15.9  2.1  17.7   2.1

   ... more lines ...

29.0 80.0 24.1 80.0 26.6 80.0 24.2 80.0 22.7 80.0  28.4  80.0
26.3 80.0 25.6 80.0 24.8 80.0 28.6 80.0 25.7 80.0  25.8  80.0
22.5 80.0 25.1 80.0 27.0 80.0 27.9 80.0 28.5 80.0  21.7  80.0
33.5 80.0 26.1 80.0 28.4 80.0 22.7 80.0 28.0 80.0  42.7  80.0
;
```

The logarithm of BMI is used as the response. (Although this does not improve the quantile regression fit, it helps with statistical inference.) A preliminary median regression is fitted with a parametric model, which involves six powers of Age.

The following statements invoke the QUANTREG procedure:

```
proc quantreg data=bmimen algorithm=interior(tolerance=1e-5) ci=resampling;
   model logbmi = inveage sqrtage age sqrtage*age
                  age*age age*age*age
                  / diagnostics cutoff=4.5 quantile=.5 seed=1268;
   id age bmi;
   test_age_cubic: test age*age*age / wald lr rankscore(tau);
run;
```

The MODEL statement provides the model, and the option QUANTILE=0.5 requests median regression. The ALGORITHM= option requests that the interior point algorithm be used to compute $\hat{\boldsymbol{\beta}}(\frac{1}{2})$. For more information about this algorithm, see the section "Interior Point Algorithm" on page 6901.

Figure 81.11 displays the estimated parameters, standard errors, 95% confidence intervals, $t$ values, and $p$-values that are computed by the resampling method, which is requested by the CI= option. All of the parameters are considered significant because the $p$-values are smaller than 0.001.

**Figure 81.11** Parameter Estimates with Median Regression: Men

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**Parameter Estimates**

| Parameter | DF | Estimate | Standard Error | 95% Confidence Limits | | t Value | Pr > \|t\| |
|---|---|---|---|---|---|---|---|
| Intercept | 1 | 7.8909 | 0.8168 | 6.2895 | 9.4924 | 9.66 | <.0001 |
| InveAge | 1 | -1.8354 | 0.4350 | -2.6884 | -0.9824 | -4.22 | <.0001 |
| SqrtAge | 1 | -5.1247 | 0.7135 | -6.5237 | -3.7257 | -7.18 | <.0001 |
| Age | 1 | 1.9759 | 0.2537 | 1.4785 | 2.4733 | 7.79 | <.0001 |
| SqrtAge*Age | 1 | -0.3347 | 0.0424 | -0.4179 | -0.2515 | -7.89 | <.0001 |
| Age*Age | 1 | 0.0227 | 0.0029 | 0.0170 | 0.0284 | 7.77 | <.0001 |
| Age*Age*Age | 1 | -0.0000 | 0.0000 | -0.0001 | -0.0000 | -7.40 | <.0001 |

The TEST statement requests Wald, likelihood ratio, and rank tests for the significance of the cubic term in Age. The test results, shown in Figure 81.12, indicate that this term is significant. Higher-order terms are not significant.

**Figure 81.12** Test of Significance for Cubic Term

**Test test_age_cubic Results**

| Test | Test Statistic | DF | Chi-Square | Pr > ChiSq |
|---|---|---|---|---|
| Wald | 54.7417 | 1 | 54.74 | <.0001 |
| Likelihood Ratio | 56.9473 | 1 | 56.95 | <.0001 |
| Rank_Tau | 42.5731 | 1 | 42.57 | <.0001 |

Median regression and, more generally, quantile regression are robust to extremes of the response variable. The DIAGNOSTICS option in the MODEL statement requests a diagnostic table of outliers, shown in Figure 81.13, which uses a cutoff value that is specified in the CUTOFF= option. The variables that are specified in the ID statement are included in the table.

With CUTOFF=4.5, 14 men are identified as outliers. All of these men have large positive standardized residuals, which indicates that they are overweight for their age. The cutoff value 4.5 is ad hoc. It corresponds to a probability less than 0.5E–5 if normality is assumed, but the standardized residuals for median regression usually do not meet this assumption.

In order to construct the chart shown in Figure 81.2, the same model that is used for median regression is used for other quantiles. The QUANTREG procedure can compute fitted values for multiple quantiles.

**Figure 81.13** Diagnostics with Median Regression

**Diagnostics**

| Obs | Age | BMI | Standardized Residual | Outlier |
|---|---|---|---|---|
| 1337 | 8.900000 | 36.500000 | 5.3575 | * |
| 1376 | 9.200000 | 39.600000 | 5.8723 | * |
| 1428 | 9.400000 | 36.900000 | 5.3036 | * |
| 1505 | 9.900000 | 35.500000 | 4.8862 | * |
| 1764 | 14.900000 | 46.800000 | 5.6403 | * |
| 1838 | 16.200000 | 50.400000 | 5.9138 | * |
| 1845 | 16.300000 | 42.600000 | 4.6683 | * |
| 1870 | 16.700000 | 42.600000 | 4.5930 | * |
| 1957 | 18.100000 | 49.900000 | 5.5053 | * |
| 2002 | 18.700000 | 52.700000 | 5.8106 | * |
| 2016 | 18.900000 | 48.400000 | 5.1603 | * |
| 2264 | 32.000000 | 55.600000 | 5.3085 | * |
| 2291 | 35.000000 | 60.900000 | 5.9406 | * |
| 2732 | 66.000000 | 14.900000 | -4.7849 | * |

The following statements request fitted values for 10 quantile levels that range from 0.03 to 0.97:

```
proc quantreg data=bmimen algorithm=interior(tolerance=1e-5) ci=none;
   model logbmi = inveage sqrtage age sqrtage*age
                  age*age age*age*age
                  / quantile=0.03,0.05,0.1,0.25,0.5,0.75,
                             0.85,0.90,0.95,0.97;
   output out=outp pred=p/columnwise;
run;

data outbmi;
   set outp;
   pbmi = exp(p);
run;

proc sgplot data=outbmi;
   title 'BMI Percentiles for Men: 2-80 Years Old';
   yaxis label='BMI (kg/m**2)' min=10 max=45 values=(10 15 20 25 30 35 40 45);
   xaxis label='Age (Years)' min=2 max=80 values=(2 10 20 30 40 50 60 70 80);

   scatter x=age y=bmi /markerattrs=(size=1);
   series  x=age y=pbmi/group=QUANTILE;
run;
```

The fitted values are stored in the OUTPUT data set outp. The COLUMNWISE option arranges these fitted values for all quantiles in the single variable p by groups of the quantiles. After the exponential transformation, both the fitted BMI values and the original BMI values are plotted against age to create the display shown in Figure 81.2.

The fitted quantile curves reveal important information. During the quick growth period (ages 2 to 20), the dispersion of BMI increases dramatically. It becomes stable during middle age, and then it contracts after age 60. This pattern suggests that effective population weight control should start in childhood.

Compared to the 97th percentile in reference growth charts that were published by the Centers for Disease Control and Prevention (CDC) in 2000 (Kuczmarski, Ogden, and Guo 2002), the 97th percentile for 10-year-old boys in Figure 81.2 is 6.4 BMI units higher (an increase of 27%). This can be interpreted as a warning of overweight or obesity. See Chen (2005) for a detailed analysis.

# Syntax: QUANTREG Procedure

The following statements are available in the QUANTREG procedure:

**PROC QUANTREG** < *options* > **;**
  **BY** *variables* **;**
  **CLASS** *variables* < / **TRUNCATE** > **;**
  **EFFECT** *name*=**SPLINE(***variables* < / *options* >**)** **;**
  **ESTIMATE** < '*label*' > *estimate-specification* < / *options* > **;**
  **ID** *variables* **;**
  **MODEL** *response* **=** < *effects* > < / *options* > **;**
  **OUTPUT** < **OUT=** *SAS-data-set* > < *options* > **;**
  **PERFORMANCE** < *options* > **;**
  **TEST** *effects* < / *options* > **;**
  **WEIGHT** *variable* **;**

The PROC QUANTREG statement invokes the QUANTREG procedure. The CLASS statement specifies which explanatory variables are treated as categorical. The ID statement names variables to identify observations in the outlier diagnostics tables. The MODEL statement is required and specifies the variables used in the regression. Main effects and interaction terms can be specified in the MODEL statement, as in the GLM procedure (Chapter 44, "The GLM Procedure.") The OUTPUT statement creates an output data set that contains predicted values, residuals, and estimated standard errors. The PERFORMANCE statement tunes the performance of PROC QUANTREG by using single or multiple processors available in the hardware. The TEST statement requests linear tests for the model parameters. The WEIGHT statement identifies a variable in the input data set whose values are used to weight the observations. Multiple OUTPUT and TEST statements are allowed in one invocation of PROC QUANTREG.

The EFFECT and ESTIMATE statements are also available in other procedures. Summary descriptions of functionality and syntax for these statements are provided in this chapter, and you can find full documentation about them in Chapter 19, "Shared Concepts and Topics."

## PROC QUANTREG Statement

**PROC QUANTREG** < *options* > **;**

The PROC QUANTREG statement invokes the QUANTREG procedure. Table 81.1 summarizes the *options* available in the PROC QUANTREG statement.

**Table 81.1** PROC QUANTREG Statement Options

| Option | Description |
|---|---|
| ALGORITHM= | Specifies an algorithm to estimate the regression parameters |
| CI= | Specifies a method to compute confidence intervals |
| DATA= | Specifies the input SAS data set |
| INEST= | Specifies an input SAS data set that contains initial estimates |
| NAMELEN= | Specifies the length of effect names |
| ORDER= | Specifies the order in which to sort classification variables |

**Table 81.1** *continued*

| Option | Description |
|---|---|
| OUTEST= | Specifies an output SAS data set containing the parameter estimates |
| PLOT | Specifies options that control details of the plots |

You can specify the following *options* in the PROC QUANTREG statement.

**ALGORITHM=***algorithm* **<(** *suboptions* **) >**

specifies an algorithm for estimating the regression parameters. Three algorithms are available: simplex (SIMPLEX), interior point (INTERIOR), and smoothing (SMOOTH).

The default algorithm depends on the number of observations ($n$) and the number of covariates ($p$) in the model estimation. See Table 81.2 for the relevant defaults.

**Table 81.2** The Default Estimation Algorithm

| | $p \leq 100$ | $p > 100$ |
|---|---|---|
| $n \leq 5000$ | Simplex | Smoothing |
| $n > 5000$ | Interior point | Smoothing |

Table 81.3 summarizes the *options* available for each of these methods.

**Table 81.3** Options for Estimation Algorithms

| *algorithm* | **Algorithm** | *suboption* |
|---|---|---|
| INTERIOR | Interior point | KAPPA=, MAXIT=, and TOLERANCE= |
| SIMPLEX | Simplex | MAXSTATIONARY= |
| SMOOTH | Smoothing | RRATIO= |

If you specify ALGORITHM=SIMPLEX, you can specify the following *suboption*:

**MAXSTATIONARY=***m* requests that the algorithm terminate if the objective function has not improved for *m* consecutive iterations. By default, *m* = 1000.

If you specify ALGORITHM=INTERIOR, you can specify the following *suboptions*:

**KAPPA=***value* specifies the step-length parameter for the interior point algorithm. This parameter should be between 0 and 1. The larger the parameter, the faster the algorithm. However, numeric instability can occur as the parameter approaches 1. By default, KAPPA=0.99995. For more information, see the section "Interior Point Algorithm" on page 6901.

**MAXIT=***m* sets the maximum number of iterations for the interior point algorithm. By default, MAXIT=1000.

**TOLERANCE=***value* specifies the tolerance for the convergence criterion of the interior point algorithm. By default, TOLERANCE=1E–8. The QUANTREG procedure uses the duality gap as the convergence criterion. For more information, see the section "Interior Point Algorithm" on page 6901.

If you specify ALGORITHM=INTERIOR, you can also use the PERFORMANCE statement to enable parallel computing when multiple processors are available in the hardware.

If you specify ALGORITHM=SMOOTH, you can specify the following *suboption*:

**RRATIO=***value* specifies the reduction ratio for the smoothing algorithm. This ratio is used to reduce the threshold of the smoothing algorithm. The *value* should be between 0 and 1. In theory, the smaller the reduction ratio, the faster the smoothing algorithm. However, in practice, the optimal ratio is quite dependent on the data. For more information, see the section "Smoothing Algorithm" on page 6905.

**ALPHA=***value*

sets the confidence level for the confidence intervals for regression parameters. The *value* must be between 0 and 1. The default is ALPHA=0.05, which corresponds to a 0.95 confidence interval.

**CI=NONE | RANK | SPARSITY< (BF | HS) >< /IID > | RESAMPLING< (NREP=***n***) >**

specifies a method for computing confidence intervals for regression parameters. When you specify CI=SPARSITY or CI=RESAMPLING, the QUANTREG procedure also computes standard errors, $t$ values, and $p$-values for regression parameters.

The following table summarizes these methods.

**Table 81.4** Options for Confidence Intervals

| Value of CI= | Method | Additional Options |
|---|---|---|
| NONE | No confidence intervals computed | |
| RANK | By inverting rank-score tests | |
| RESAMPLING | By resampling | NREP |
| SPARSITY | By estimating sparsity function | HS, BF, and IID |

By default, when there are fewer than 5,000 observations, fewer than 20 variables in the data set, and the algorithm is simplex, the QUANTREG procedure computes confidence intervals by using the inverted rank-score test method. Otherwise, the resampling method is used.

By default, confidence intervals are not computed for the quantile process, which is estimated when you specify the QUANTILE=PROCESS option in the MODEL statement. Confidence intervals for the quantile process are computed by using the sparsity or resampling methods when you specify CI=SPARSITY or CI=RESAMPLING, respectively. The rank method for confidence intervals is not available for quantile processes because it is computationally prohibitive.

When you specify the SPARSITY option, you have two suboptions for estimating the sparsity function. If you specify the IID suboption, the sparsity function is estimated by assuming that the errors in the linear model are independent and identically distributed (iid). By default, the sparsity function is estimated by assuming that the conditional quantile function is locally linear. For more information, see the section "Sparsity" on page 6907. For both methods, two bandwidth selection methods are available: You can specify the BF suboption for the Bofinger method or the the HS suboption for the Hall-Sheather method. By default, the Hall-Sheather method is used.

When you specify the RESAMPLING option, you can specify the NREP=*n* suboption for the number of repetitions. By default, NREP=200. The value of *n* must be greater than 50.

**DATA=***SAS-data-set*

specifies the input SAS data set to be used by the QUANTREG procedure. By default, the most recently created SAS data set is used.

**INEST=***SAS-data-set*

specifies an input SAS data set that contains initial estimates for all the parameters in the model. The interior point algorithm and the smoothing algorithm use these estimates as a start. For a detailed description of the contents of the INEST= data set, see the section “INEST= Data Set” on page 6914.

**NAMELEN=***n*

restricts the length of effect names in tables and output data sets to *n* characters, where *n* is a value between 20 and 200. By default, NAMELEN=20.

**ORDER=DATA | FORMATTED | FREQ | INTERNAL**

specifies the sort order for the levels of the classification variables (which are specified in the CLASS statement). This option applies to the levels for all classification variables, except when you use the (default) ORDER=FORMATTED option with numeric classification variables that have no explicit format. In that case, the levels of such variables are ordered by their internal value.

The ORDER= option can take the following values:

| Value of ORDER= | Levels Sorted By |
|---|---|
| **DATA** | Order of appearance in the input data set |
| **FORMATTED** | External formatted value, except for numeric variables with no explicit format, which are sorted by their unformatted (internal) value |
| **FREQ** | Descending frequency count; levels with the most observations come first in the order |
| **INTERNAL** | Unformatted value |

By default, ORDER=FORMATTED. For ORDER=FORMATTED and ORDER=INTERNAL, the sort order is machine-dependent. For more information about sort order, see the chapter on the SORT procedure in the *Base SAS Procedures Guide* and the discussion of BY-group processing in *SAS Language Reference: Concepts*.

**OUTEST=***SAS-data-set*

specifies an output SAS data set to contain the parameter estimates for all quantiles. See the section “OUTEST= Data Set” on page 6914 for a detailed description of the contents of the OUTEST= data set.

**PLOT | PLOTS< (***global-plot-options***) > < =***plot-request* **>**

**PLOT | PLOTS< (***global-plot-options***) > < =(***plot-request* **<** *. . . plot-request* **> ) >**

specifies options that control details of the plots. These plots fall into two categories: diagnostic plots and fit plots. You can also use the PLOT= option in the MODEL statement to request the quantile process plot for any effects that are specified in the model. If you do not specify the PLOTS= option, PROC QUANTREG produces the quantile fit plot by default when a single continuous variable is specified in the model.

When you specify only one *plot-request*, you can omit the parentheses around the plot request.

Here are some examples:

```
plots=ddplot
plots=(ddplot rdplot)
```

ODS Graphics must be enabled before plots can be requested. For example:

```
ods graphics on;

proc quantreg plots=fitplot;
   model y=x1;
run;

ods graphics off;
```

For more information about enabling and disabling ODS Graphics, see the section "Enabling and Disabling ODS Graphics" on page 606 in Chapter 21, "Statistical Graphics Using ODS."

You can specify the following *global-plot-options*, which apply to all plots that PROC QUANTREG generates:

**MAXPOINTS=NONE |** *number*

suppresses plots that have elements that require processing more than *number* points. The default is MAXPOINTS=5000. This cutoff is ignored if you specify MAXPOINTS=NONE.

**ONLY**

suppresses the default quantile fit plot. Only plots specifically requested are displayed.

You can specify the following *plot-requests*:

**ALL**

creates all appropriate plots.

**DDPLOT< (LABEL=ALL | LEVERAGE | NONE | OUTLIER) >**

creates a plot of robust distance against Mahalanobis distance. For more information about robust distance, see the section "Leverage Point and Outlier Detection" on page 6913. The LABEL= option specifies how the points on this plot are to be labeled, as summarized by Table 81.5.

**Table 81.5** Options for Label

| Value of LABEL= | Label Method |
|---|---|
| ALL | Label all points |
| LEVERAGE | Label leverage points |
| NONE | No labels |
| OUTLIERS | Label outliers |

By default, the QUANTREG procedure labels both outliers and leverage points.

If you specify ID variables in the ID statement, the values of the first ID variable are used as labels; otherwise, observation numbers are used as labels.

**FITPLOT< (NOLIMITS | SHOWLIMITS | NODATA) >**

creates a plot of fitted conditional quantiles against the single continuous variable that is specified in the model. This plot is produced only when the response is modeled as a function of a single continuous variable. Multiple lines or curves are drawn on this plot if you specify several quantiles with the QUANTILE= option in the MODEL statement. By default, confidence limits are added to the plot when a single quantile is requested, and the confidence limits are not shown on the plot when multiple quantiles are requested. The NOLIMITS option suppresses the display of the confidence limits. The SHOWLIMITS option adds the confidence limits when multiple quantiles are requested. The NODATA option suppresses the display of the observed data, which are superimposed on the plot by default.

**HISTOGRAM**

creates a histogram (based on the quantile regression estimates) for the standardized residuals. The histogram is superimposed with a normal density curve and a kernel density curve.

**NONE**

suppresses all plots.

**QQPLOT**

creates the normal quantile-quantile plot (based on the quantile regression estimates) for the standardized residuals.

**RDPLOT< (LABEL=ALL | LEVERAGE | NONE | OUTLIER) >**

creates the plot of standardized residual against robust distance. For more information about robust distance, see the section "Leverage Point and Outlier Detection" on page 6913.

The LABEL= option specifies a label method for points on this plot. These label methods are described in Table 81.5.

By default, the QUANTREG procedure labels both outliers and leverage points.

If you specify ID variables in the ID statement, the values of the first ID variable are used as labels; otherwise, observation numbers are used as labels.

**PP**

requests preprocessing to speed up the interior point algorithm or the smoothing algorithm. The preprocessing uses a subsampling algorithm (which assumes that the data set is evenly distributed) to iteratively reduce the original problem to a smaller one. Preprocessing should be used only for very large data sets, such as data sets with more than 100,000 observations. For more information, see Portnoy and Koenker (1997).

## BY Statement

**BY** *variables* **;**

You can specify a BY statement with PROC QUANTREG to obtain separate analyses of observations in groups that are defined by the BY variables. When a BY statement appears, the procedure expects the input data set to be sorted in order of the BY variables. If you specify more than one BY statement, only the last one specified is used.

If your input data set is not sorted in ascending order, use one of the following alternatives:

- Sort the data by using the SORT procedure with a similar BY statement.
- Specify the NOTSORTED or DESCENDING option in the BY statement for the QUANTREG procedure. The NOTSORTED option does not mean that the data are unsorted but rather that the data are arranged in groups (according to values of the BY variables) and that these groups are not necessarily in alphabetical or increasing numeric order.
- Create an index on the BY variables by using the DATASETS procedure (in Base SAS software).

For more information about BY-group processing, see the discussion in *SAS Language Reference: Concepts*. For more information about the DATASETS procedure, see the discussion in the *Base SAS Procedures Guide*.

## CLASS Statement

**CLASS** *variables* < / **TRUNCATE** > **;**

The CLASS statement names the classification variables to be used in the model. Typical classification variables are Treatment, Sex, Race, Group, and Replication. If you use the CLASS statement, it must appear before the MODEL statement.

Classification variables can be either character or numeric. By default, class levels are determined from the entire set of formatted values of the CLASS variables.

**NOTE:** Prior to SAS 9, class levels were determined by using no more than the first 16 characters of the formatted values. To revert to this previous behavior, you can use the TRUNCATE option in the CLASS statement.

In any case, you can use formats to group values into levels. See the discussion of the FORMAT procedure in the *Base SAS Procedures Guide* and the discussions of the FORMAT statement and SAS formats in *SAS Formats and Informats: Reference*. You can adjust the order of CLASS variable levels with the ORDER= option in the PROC QUANTREG statement. You can specify the following *option* in the CLASS statement after a slash (/):

**TRUNCATE**

specifies that class levels should be determined by using only up to the first 16 characters of the formatted values of CLASS variables. When formatted values are longer than 16 characters, you can use this option to revert to the levels as determined in releases prior to SAS 9.

## EFFECT Statement

**EFFECT** *name*=*effect-type* (*variables* < / *options*>) **;**

The EFFECT statement enables you to construct special collections of columns for design matrices. These collections are referred to as *constructed effects* to distinguish them from the usual model effects that are formed from continuous or classification variables, as discussed in the section "GLM Parameterization of Classification Variables and Effects" on page 387 in Chapter 19, "Shared Concepts and Topics."

You can specify the following *effect-types*:

**COLLECTION** is a collection effect that defines one or more variables as a single effect with multiple degrees of freedom. The variables in a collection are considered as a unit for estimation and inference.

**LAG** is a classification effect in which the level that is used for a given period corresponds to the level in the preceding period.

**MULTIMEMBER | MM** is a multimember classification effect whose levels are determined by one or more variables that appear in a CLASS statement.

**POLYNOMIAL | POLY** is a multivariate polynomial effect in the specified numeric variables.

**SPLINE** is a regression spline effect whose columns are univariate spline expansions of one or more variables. A spline expansion replaces the original variable with an expanded or larger set of new variables.

Table 81.6 summarizes the *options* available in the EFFECT statement.

**Table 81.6** EFFECT Statement Options

| Option | Description |
|---|---|
| **Collection Effects Options** | |
| DETAILS | Displays the constituents of the collection effect |
| **Lag Effects Options** | |
| DESIGNROLE= | Names a variable that controls to which lag design an observation is assigned |
| DETAILS | Displays the lag design of the lag effect |
| NLAG= | Specifies the number of periods in the lag |
| PERIOD= | Names the variable that defines the period |
| WITHIN= | Names the variable or variables that define the group within which each period is defined |
| **Multimember Effects Options** | |
| NOEFFECT | Specifies that observations with all missing levels for the multimember variables should have zero values in the corresponding design matrix columns |
| WEIGHT= | Specifies the weight variable for the contributions of each of the classification effects |
| **Polynomial Effects Options** | |
| DEGREE= | Specifies the degree of the polynomial |
| MDEGREE= | Specifies the maximum degree of any variable in a term of the polynomial |
| STANDARDIZE= | Specifies centering and scaling suboptions for the variables that define the polynomial |

**Table 81.6** *continued*

| Option | Description |
|---|---|
| **Spline Effects Options** | |
| BASIS= | Specifies the type of basis (B-spline basis or truncated power function basis) for the spline effect |
| DEGREE= | Specifies the degree of the spline effect |
| KNOTMETHOD= | Specifies how to construct the knots for the spline effect |

For more information about the syntax of these *effect-types* and how columns of constructed effects are computed, see the section "EFFECT Statement" on page 397 in Chapter 19, "Shared Concepts and Topics."

## ESTIMATE Statement

**ESTIMATE** < '*label*' > *estimate-specification* < **(***divisor***=***n***)** >
< **,** . . . < '*label*' > *estimate-specification* < **(***divisor***=***n***)** > >
< / *options* > **;**

The ESTIMATE statement provides a mechanism for obtaining custom hypothesis tests. Estimates are formed as linear estimable functions of the form $\mathbf{L}\boldsymbol{\beta}$. You can perform hypothesis tests for the estimable functions, construct confidence limits, and obtain specific nonlinear transformations.

Table 81.7 summarizes the *options* available in the ESTIMATE statement.

**Table 81.7** ESTIMATE Statement Options

| Option | Description |
|---|---|
| **Construction and Computation of Estimable Functions** | |
| DIVISOR= | Specifies a list of values to divide the coefficients |
| NOFILL | Suppresses the automatic fill-in of coefficients for higher-order effects |
| SINGULAR= | Tunes the estimability checking difference |
| **Degrees of Freedom and *p*-values** | |
| ADJUST= | Determines the method for multiple comparison adjustment of estimates |
| ALPHA=$\alpha$ | Determines the confidence level $(1-\alpha)$ |
| LOWER | Performs one-sided, lower-tailed inference |
| STEPDOWN | Adjusts multiplicity-corrected *p*-values further in a step-down fashion |
| TESTVALUE= | Specifies values under the null hypothesis for tests |
| UPPER | Performs one-sided, upper-tailed inference |

**Table 81.7** *continued*

| Option | Description |
|---|---|
| **Statistical Output** | |
| CL | Constructs confidence limits |
| CORR | Displays the correlation matrix of estimates |
| COV | Displays the covariance matrix of estimates |
| E | Prints the **L** matrix |
| JOINT | Produces a joint *F* or chi-square test for the estimable functions |
| SEED= | Specifies the seed for computations that depend on random numbers |

For details about the syntax of the ESTIMATE statement, see the section "ESTIMATE Statement" on page 444 in Chapter 19, "Shared Concepts and Topics."

## ID Statement

**ID** *variables* **;**

When the diagnostics table is requested by the DIAGNOSTICS option in the MODEL statement, the variables listed in the ID statement are displayed in addition to the observation number. These values are useful for identifying observations. If the ID statement is omitted, only the observation number is displayed.

## MODEL Statement

**<** *label***: > MODEL** *response* **=** < *effects* > < / *options* > **;**

You can specify main effects and interaction terms in the MODEL statement, as you can in the GLM procedure (Chapter 44, "The GLM Procedure.") Classification variables in the MODEL statement must also be specified in the CLASS statement.

The optional *label*, which must be a valid SAS name, is used to label output from the matching MODEL statement.

### Options

Table 81.8 summarizes the *options* available in the MODEL statement.

**Table 81.8** MODEL Statement Options

| Option | Description |
|---|---|
| CORRB | Produces the estimated correlation matrix |
| COVB | Produces the estimated covariance matrix |
| CUTOFF= | Specifies the multiplier of the cutoff value for outlier detection |
| DIAGNOSTICS | Requests the outlier diagnostics |
| ITPRINT | Displays the iteration history |
| LEVERAGE | Requests an analysis of leverage points |

**Table 81.8** *continued*

| Option | Description |
|---|---|
| NODIAG | Suppresses the computation for outlier diagnostics |
| NOINT | Specifies no-intercept regression |
| NOSUMMARY | Suppresses the computation for summary statistics |
| PLOT= | Requests plots |
| QUANTILE= | Specifies the quantile levels |
| SCALE= | Specifies the scale value used to compute the standardized residuals |
| SEED= | Specifies the seed for the random number generator |
| SINGULAR= | Specifies the tolerance for testing singularity |

You can specify the following *options* for the model fit.

**CORRB**

produces the estimated correlation matrix of the parameter estimates. When the resampling method is used to compute the confidence intervals, the QUANTREG procedure computes the bootstrap correlation. When the sparsity method is used to compute the confidence intervals, PROC QUANTREG bases its computation of the asymptotic correlation on an estimator of the sparsity function. The rank method for confidence intervals does not provide a correlation estimate.

**COVB**

produces the estimated covariance matrix of the parameter estimates. When the resampling method is used to compute the confidence intervals, the QUANTREG procedure computes the bootstrap covariance. When the sparsity method is used to compute the confidence intervals, PROC QUANTREG bases its computation of the asymptotic covariance on an estimator of the sparsity function. The rank method for confidence intervals does not provide a covariance estimate.

**CUTOFF=***value*

specifies the multiplier of the cutoff value for outlier detection. By default, CUTOFF=3.

**DIAGNOSTICS< (ALL) >**

requests the outlier diagnostics. By default, only observations that are identified as outliers or leverage points are displayed. To request that all observations be displayed, specify the ALL option.

**ITPRINT**

displays the iteration history of the interior point algorithm or the smoothing algorithm.

**LEVERAGE< (CUTOFF=***value* **| CUTOFFALPHA=***value* **| H=***n***) >**

requests an analysis of leverage points for the continuous covariates. The results are added to the diagnostics table, which you can request with the DIAGNOSTICS option in the MODEL statement. You can specify the cutoff value for leverage-point detection with the CUTOFF= option. The default cutoff value is $\sqrt{\chi^2_{p;1-\alpha}}$, where $\alpha$ can be specified with the CUTOFFALPHA= option. By default, $\alpha = 0.025$. You can use the H= option to specify the number of points to be minimized for the MCD algorithm used for the leverage-point analysis. By default, H=$[(3n + p + 1)/4]$, where $n$ is the number of observations and $p$ is the number of independent variables. The LEVERAGE option is ignored if the model includes classification variables as covariates.

**NODIAG**

suppresses the computation for outlier diagnostics. If you specify the NODIAG option, the diagnostics summary table is not provided.

**NOINT**

specifies no intercept regression.

**NOSUMMARY**

suppresses the computation of summary statistics. If you specify the NOSUMMARY option, the summary statistics table is not provided.

**PLOT=***plot-option*

**PLOTS=(***plot-option***)**

You can use the PLOTS= option in the MODEL statement together with ODS Graphics to request the quantile process plot in addition to all that plots that you request in the PLOT= option in the PROC QUANTREG statement. If you specify the same *plot-option* in both the PROC QUANTREG statement and the MODEL statement, the *plot-option* in the PROC QUANTREG statement overwrites the *plot-option* in the MODEL statement .

You can specify the following *plot-option* only in the MODEL statement:

**QUANTPLOT< (EFFECTS) < / < NOLIMITS > < EXTENDCI > < UNPACK > < OLS > > >**

plots the regression quantile process. The estimated coefficient of each specified covariate effect is plotted as a function of the quantile. If you do not specify a covariate effect, quantile processes are plotted for all covariate effects in the MODEL statement. You can use the NOLIMITS option to suppress confidence bands for the quantile processes. By default, confidence bands are plotted, and process plots are displayed in panels, each of which can hold up to four plots. By default, the confidence limits are plotted for quantiles in the range between 0.05 and 0.95. You can use the EXTENDCI option to plot the confidence limits even for quantiles outside this range. You can use the UNPACK option to create individual process plots. For an individual process plot, you can superimpose the ordinary least squares estimate by specifying the OLS option.

ODS Graphics must be enabled before you request plots.

For more information about enabling and disabling ODS Graphics, see the section "Enabling and Disabling ODS Graphics" on page 606 in Chapter 21, "Statistical Graphics Using ODS."

**QUANTILE=***number-list* **| PROCESS**

specifies the quantile levels for the quantile regression. You can specify any number of quantile levels in $(0, 1)$. You can also compute the entire quantile process by specifying the PROCESS option. Only the simplex algorithm is available for computing the quantile process.

If you do not specify the QUANTILE= option, the QUANTREG procedure fits a median regression, which corresponds to QUANTILE=0.5.

**SCALE=***number*

specifies the scale value to use to compute the standardized residuals. By default, the scale is computed as the corrected median of absolute residuals. See the section "Leverage Point and Outlier Detection" on page 6913 for details.

**SEED=***number*
specifies the seed for the random number generator used to compute the MCMB confidence intervals. This seed is also used to randomly select the subgroups for preprocessing when you specify the PP option in the PROC QUANTREG statement. If you do not specify a seed, or if you specify a value less than or equal to 0, the seed is generated from reading the time of day from the computer clock.

By default or if you specify SEED=0, the QUANTREG procedure generates a seed between one and one billion.

**SINGULAR=***value*
sets the tolerance for testing singularity of the information matrix and the crossproducts matrix for the initial least squares estimates. Approximately, the test requires that a pivot be at least this value times the original diagonal value. By default, SINGULAR=1E–12.

## OUTPUT Statement

**OUTPUT** < **OUT=***SAS-data-set*> *keyword***=***name* < ... *keyword***=***name*> < / **COLUMNWISE** > **;**

The OUTPUT statement creates a SAS data set to contain statistics that are calculated after PROC QUANTREG fits models for all specified quantiles that are specified in the QUANTILE= option in the MODEL statement. At least one specification of the form *keyword=name* is required.

All variables in the original data set are included in the new data set, along with the variables that are created from options in the OUTPUT statement. These new variables contain fitted values and estimated quantiles. If you want to create a SAS data set in a permanent library, you must specify a two-level name. For more information about permanent libraries and SAS data sets, see *SAS Language Reference: Concepts*.

If you specify multiple quantiles in the MODEL statement, the COLUMNWISE option arranges the created OUTPUT data set in columnwise form. This arrangement repeats the input data for each quantile. By default, the OUTPUT data set is created in rowwise form. For each appropriate keyword specified in the OUTPUT statement, one variable for each specified quantile is generated. These variables appear in the sorted order of the specified quantiles.

The following specifications can appear in the OUTPUT statement:

**OUT**=*SAS-data-set* specifies the new data set. By default, PROC QUANTREG uses the DATA*n* convention to name the new data set.

*keyword=name* specifies the statistics to include in the output data set and gives names to the new variables. For each desired statistic, specify a *keyword* from the following list of *keywords*, an equal sign, and the name of a variable to contain the statistic.

You can specify the following *keywords*:

**LEVERAGE** specifies a variable to indicate leverage points. To include this variable in the OUTPUT data set, you must specify the LEVERAGE option in the MODEL statement. See the section "Leverage Point and Outlier Detection" on page 6913 for how to define LEVERAGE.

**MAHADIST | MD** names a variable to contain the Mahalanobis distance. To include this variable in the OUTPUT data set, you must specify the LEVERAGE option in the MODEL statement.

**OUTLIER** specifies a variable to indicate outliers. See the section "Leverage Point and Outlier Detection" on page 6913 for how to define OUTLIER.

**PREDICTED | P** names a variable to contain the estimated response.

**QUANTILE | Q** names a variable to contain the quantile for which the quantile regression is fitted. If you specify the COLUMNWISE option, this variable is created by default. If multiple quantiles are specified in the MODEL statement and the COLUMNWISE option is not specified, this variable is not created.

**RESIDUAL | RES** names a variable to contain the residuals (unstandardized):

$$y_i - \mathbf{x}_i'\hat{\boldsymbol{\beta}}$$

**ROBDIST | RD** names a variable to contain the robust MCD distance. To include this variable in the OUTPUT data set, you must specify the LEVERAGE option in the MODEL statement.

**SPLINE | SP** names a variable to contain the estimated spline effect, which includes all spline effects in the model and their interactions.

**SRESIDUAL | SR** names a variable to contain the standardized residuals:

$$\frac{y_i - \mathbf{x}_i'\hat{\boldsymbol{\beta}}}{\hat{\sigma}}$$

See the section "Leverage Point and Outlier Detection" on page 6913 for how to compute $\sigma$.

**STDP** names a variable to contain the estimates of the standard errors of the estimated response.

## PERFORMANCE Statement

You can use the PERFORMANCE statement to change default options that affect the performance of PROC QUANTREG and to request tables that show the performance options in effect and the timing details.

**PERFORMANCE** < *options* > **;**

You can specify the following *options*:

**CPUCOUNT=***number* **| ACTUAL**

specifies the number of processors to use in the computation of the interior point algorithm. CPUCOUNT=ACTUAL sets CPUCOUNT to be the number of physical processors available, which this can be less than the physical number of CPUs if the SAS process has been restricted by system administration tools. You can specify any integer from 1 to 1024 for *number*. Setting CPUCOUNT= to a *number* greater than the actual number of available CPUs might result in reduced performance. If CPUCOUNT=1, then NOTHREADS is in effect, and PROC QUANTREG uses singly threaded code. This option overrides the SAS system option CPUCOUNT=.

**DETAILS**

creates the PerfSettings table that shows the performance settings in effect and the "Timing" table that provides a broad timing breakdown of the PROC QUANTREG step.

**THREADS**

enables multithreaded computation for the interior point algorithm. If you do not specify the ALGORITHM=INTERIOR option in the PROC QUANTREG statement, then PROC QUANTREG ignores this option and uses singly threaded code. This option overrides the SAS system option THREADS | NOTHREADS.

**NOTHREADS**

disables multithreaded computation for the interior point algorithm. This option overrides the SAS system option THREADS | NOTHREADS.

## TEST Statement

< *label*: > **TEST** *effects* < / *options* > ;

In quantile regression analysis, you might be interested in testing whether a covariate effect is statistically significant for a given quantile. In other situations, you might be interested in testing whether the coefficients of a covariate are the same across a set of quantiles. You can use the TEST Statement to perform these tests.

You can submit multiple TEST statements, provided that they appear after the MODEL statement. The optional *label*, which must be a valid SAS name, is used to identify output from the corresponding TEST statement. For more information about these tests, see the section "Linear Test" on page 6911.

### Testing Effects of Covariates

You can use TEST statement to obtain a test for the canonical linear hypothesis concerning the parameters of the tested effects,

$$\beta_j = 0, \quad j = i_1, \ldots, i_q$$

where $q$ is the total number of parameters of the tested effects. The tested *effects* can be any set of effects in the MODEL statement. You can specify three types of tests (Wald, likelihood ratio, and rank methods) for testing effects of covariates by using the following *options* in the TEST statement after a slash (/):

**WALD**

requests Wald tests.

**LR**

requests likelihood ratio tests.

**RANKSCORE < (NORMAL | WILCOXON | SIGN | TAU) >**

requests rank tests. The NORMAL, WILCOXON, and SIGN functions are implemented and suitable for iid error models, and the TAU score function is implemented and appropriate for non-iid error models. By default, the TAU score function is used. See Koenker (2005) for more information about the score functions.

### Testing for Heteroscedasticity

You can test whether there is any difference among the estimated coefficients across quantiles if several quantiles are specified in the MODEL statement. The test for such heteroscedasticity can be requested by the option QINTERACT after a slash (/) in the TEST statement. See Example 81.5.

## WEIGHT Statement

**WEIGHT** *variable* **;**

The WEIGHT statement specifies a weight *variable* in the input data set.

To request weighted quantile regression, place the weights in a *variable*. The values of the WEIGHT *variable* can be nonintegral and are not truncated. Observations with nonpositive or missing values for the weight variable do not contribute to the fit of the model. For more information about weighted quantile regression, see the section "Details: QUANTREG Procedure" on page 6899.

# Details: QUANTREG Procedure

## Quantile Regression as an Optimization Problem

The model for linear quantile regression is

$$\mathbf{y} = \mathbf{A}'\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

where $\mathbf{y} = (y_1, \ldots, y_n)'$ is the $(n \times 1)$ vector of responses, $\mathbf{A}' = (\mathbf{x}_1, \ldots, \mathbf{x}_n)'$ is the $(n \times p)$ regressor matrix, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)'$ is the $(p \times 1)$ vector of unknown parameters, and $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_n)'$ is the $(n \times 1)$ vector of unknown errors.

$L_1$ regression, also known as median regression, is a natural extension of the sample median when the response is conditioned on the covariates. In $L_1$ regression, the least absolute residuals estimate $\hat{\boldsymbol{\beta}}_{LAR}$, referred to as the $L_1$-*norm estimate*, is obtained as the solution of the following minimization problem:

$$\min_{\boldsymbol{\beta} \in \mathbf{R}^p} \sum_{i=1}^{n} |y_i - \mathbf{x}_i'\boldsymbol{\beta}|$$

More generally, for quantile regression Koenker and Bassett (1978) defined the $\tau$ *regression quantile*, $0 < \tau < 1$, as any solution to the following minimization problem:

$$\min_{\boldsymbol{\beta} \in \mathbf{R}^p} \left[ \sum_{i \in \{i : y_i \geq \mathbf{x}_i'\boldsymbol{\beta}\}} \tau |y_i - \mathbf{x}_i'\boldsymbol{\beta}| + \sum_{i \in \{i : y_i < \mathbf{x}_i'\boldsymbol{\beta}\}} (1 - \tau) |y_i - \mathbf{x}_i'\boldsymbol{\beta}| \right]$$

The solution is denoted as $\hat{\boldsymbol{\beta}}(\tau)$, and the $L_1$-norm estimate corresponds to $\hat{\boldsymbol{\beta}}(1/2)$. The $\tau$ regression quantile is an extension of the $\tau$ sample quantile $\hat{\xi}(\tau)$, which can be formulated as the solution of

$$\min_{\xi \in \mathbf{R}} \left[ \sum_{i \in \{i: y_i \geq \xi\}} \tau |y_i - \xi| + \sum_{i \in \{i: y_i < \xi\}} (1-\tau)|y_i - \xi| \right]$$

If you specify weights $w_i, i = 1, \ldots, n$, with the WEIGHT statement, weighted quantile regression is carried out by solving

$$\min_{\boldsymbol{\beta}_w \in \mathbf{R}^p} \left[ \sum_{i \in \{i: y_i \geq \mathbf{x}_i' \boldsymbol{\beta}_w\}} w_i \tau |y_i - \mathbf{x}_i' \boldsymbol{\beta}_w| + \sum_{i \in \{i: y_i < \mathbf{x}_i' \boldsymbol{\beta}_w\}} w_i (1-\tau)|y_i - \mathbf{x}_i' \boldsymbol{\beta}_w| \right]$$

Weighted regression quantiles $\boldsymbol{\beta}_w$ can be used for L-estimation (Koenker and Zhao 1994).

## Optimization Algorithms

The optimization problem for median regression has been formulated and solved as a linear programming (LP) problem since the 1950s. Variations of the simplex algorithm, especially the method of Barrodale and Roberts (1973), have been widely used to solve this problem. The simplex algorithm is computationally demanding in large statistical applications, and in theory the number of iterations can increase exponentially with the sample size. This algorithm is often useful with data that contain no more than tens of thousands of observations.

Several alternatives have been developed to handle $L_1$ regression for larger data sets. The interior point approach of Karmarkar (1984) solves a sequence of quadratic problems in which the relevant interior of the constraint set is approximated by an ellipsoid. The worst-case performance of the interior point algorithm has been proved to be better than the worst-case performance of the simplex algorithm. More important, experience has shown that the interior point algorithm is advantageous for larger problems.

Like $L_1$ regression, general quantile regression fits nicely into the standard primal-dual formulations of linear programming.

In addition to the interior point method, various heuristic approaches are available for computing $L_1$-type solutions. Among these, the finite smoothing algorithm of Madsen and Nielsen (1993) is the most useful. It approximates the $L_1$-type objective function with a smoothing function, so that the Newton-Raphson algorithm can be used iteratively to obtain a solution after a finite number of iterations. The smoothing algorithm extends naturally to general quantile regression.

The QUANTREG procedure implements the simplex, interior point, and smoothing algorithms. The remainder of this section describes these algorithms in more detail.

### Simplex Algorithm

Let $\boldsymbol{\mu} = [\mathbf{y} - \mathbf{A}'\boldsymbol{\beta}]_+$, $\boldsymbol{\nu} = [\mathbf{A}'\boldsymbol{\beta} - \mathbf{y}]_+$, $\boldsymbol{\phi} = [\boldsymbol{\beta}]_+$, and $\boldsymbol{\varphi} = [-\boldsymbol{\beta}]_+$, where $[z]_+$ is the nonnegative part of $z$.

Let $D_{LAR}(\boldsymbol{\beta}) = \sum_{i=1}^n |y_i - \mathbf{x}_i'\boldsymbol{\beta}|$. For the $L_1$ problem, the simplex approach solves $\min_{\boldsymbol{\beta}} D_{LAR}(\boldsymbol{\beta})$ by reformulating it as the constrained minimization problem

$$\min_{\boldsymbol{\beta}} \{\mathbf{e}'\boldsymbol{\mu} + \mathbf{e}'\boldsymbol{\nu} | \mathbf{y} = \mathbf{A}'\boldsymbol{\beta} + \boldsymbol{\mu} - \boldsymbol{\nu}, \{\boldsymbol{\mu}, \boldsymbol{\nu}\} \in \mathbf{R}_+^n\}$$

where $\mathbf{e}$ denotes an $(n \times 1)$ vector of ones.

Let $\mathbf{B} = [\mathbf{A}' \ -\mathbf{A}' \ \mathbf{I} \ -\mathbf{I}]$, $\boldsymbol{\theta} = (\boldsymbol{\phi}' \ \boldsymbol{\varphi}' \ \boldsymbol{\mu}' \ \boldsymbol{\nu}')'$, and $\mathbf{d} = (\mathbf{0}' \ \mathbf{0}' \ \mathbf{e}' \ \mathbf{e}')'$, where $\mathbf{0}' = (0\ 0\ \ldots\ 0)_p$. The reformulation presents a standard LP problem:

$$\begin{array}{ll} (P) & \min_{\boldsymbol{\theta}} \mathbf{d}'\boldsymbol{\theta} \\ \text{subject to} & \mathbf{B}\boldsymbol{\theta} = \mathbf{y} \\ & \boldsymbol{\theta} \geq \mathbf{0} \end{array}$$

This problem has the following dual formulation:

$$\begin{array}{ll} (D) & \max_{z} \mathbf{y}'\mathbf{z} \\ \text{subject to} & \mathbf{B}'\mathbf{z} \leq \mathbf{d} \end{array}$$

This formulation can be simplified as

$$\max_{z} \mathbf{y}'\mathbf{z}; \quad \text{subject to } \mathbf{A}\mathbf{z} = \mathbf{0}, \mathbf{z} \in [-1, 1]^n$$

By setting $\boldsymbol{\eta} = \frac{1}{2} + \frac{1}{2}\mathbf{e}$, $\mathbf{b} = \frac{1}{2}\mathbf{A}\mathbf{e}$, the problem becomes

$$\max_{\boldsymbol{\eta}} \mathbf{y}'\boldsymbol{\eta}; \quad \text{subject to } \mathbf{A}\boldsymbol{\eta} = \mathbf{b}, \boldsymbol{\eta} \in [0, 1]^n$$

For quantile regression, the minimization problem is $\min_{\boldsymbol{\beta}} \sum \rho_\tau(y_i - \mathbf{x}_i'\boldsymbol{\beta})$, and a similar set of steps leads to the dual formulation

$$\max_{z} \mathbf{y}'\mathbf{z}; \quad \text{subject to } \mathbf{A}\mathbf{z} = (1 - \tau)\mathbf{A}\mathbf{e}, \mathbf{z} \in [0, 1]^n$$

The QUANTREG procedure solves this LP problem by using the simplex algorithm of Barrodale and Roberts (1973). This algorithm exploits the special structure of the coefficient matrix $\mathbf{B}$ by solving the primary LP problem $(P)$ in two stages: The first stage chooses the columns in $\mathbf{A}'$ or $-\mathbf{A}'$ as pivotal columns. The second stage interchanges the columns in $\mathbf{I}$ or –$\mathbf{I}$ as basis or nonbasis columns, respectively. The algorithm obtains an optimal solution by executing these two stages interactively. Moreover, because of the special structure of $\mathbf{B}$, only the main data matrix $\mathbf{A}$ is stored in the current memory.

Although this special version of the simplex algorithm was introduced for median regression, it extends naturally to quantile regression for any given quantile and even to the entire quantile process (Koenker and d'Orey 1994). It greatly reduces the computing time that is required by the general simplex algorithm, and it is suitable for data sets with fewer than 5,000 observations and 50 variables.

## Interior Point Algorithm

There are many variations of interior point algorithms. The QUANTREG procedure uses the primal-dual predictor-corrector algorithm that was implemented by Lustig, Marsten, and Shanno (1992). Roos, Terlaky, and Vial (1997) provide more information about this particular algorithm. The following brief introduction of this algorithm uses the notation in the first reference.

To be consistent with the conventional LP setting, let $\mathbf{c} = -\mathbf{y}$, let $\mathbf{b} = (1 - \tau)\mathbf{A}\mathbf{e}$, and let $u$ be the general upper bound. The linear program to be solved is

$$\min\{\mathbf{c}'\mathbf{z}\}$$

subject to $\mathbf{A}\mathbf{z} = \mathbf{b}$

$$\mathbf{0} \leq \mathbf{z} \leq \mathbf{u}$$

To simplify the computation, this is treated as the *primal* problem. The problem has *n* variables. The index *i* denotes a variable number, and *k* denotes an iteration number. If *k* is used as a subscript or superscript, it denotes "of iteration *k*."

Let $\mathbf{v}$ be the primal slack so that $\mathbf{z} + \mathbf{v} = \mathbf{u}$. Associate dual variables $w$ with these constraints. The interior point algorithm solves the system of equations to satisfy the Karush-Kuhn-Tucker (KKT) conditions for optimality:

$$\mathbf{A}\mathbf{z} = \mathbf{b}$$
$$\mathbf{z} + \mathbf{v} = \mathbf{u}$$
$$\mathbf{A}'\mathbf{t} + \mathbf{s} - \mathbf{w} = \mathbf{c}$$
$$\mathbf{ZSe} = \mathbf{0}$$
$$\mathbf{VWe} = \mathbf{0}$$
$$\mathbf{z}, \mathbf{s}, \mathbf{v}, \mathbf{w} \geq \mathbf{0}$$

where

$\mathbf{W} = \mathrm{diag}(\mathbf{w})$ (that is, $W_{i,j} = w_i$ if $i = j$; $W_{i,j} = 0$ otherwise)

$\mathbf{V} = \mathrm{diag}(\mathbf{v}), \mathbf{Z} = \mathrm{diag}(\mathbf{z}), \mathbf{S} = \mathrm{diag}(\mathbf{s})$

These are the conditions for feasibility, with the addition of *complementarity* conditions $\mathbf{ZSe} = \mathbf{0}$ and $\mathbf{VWe} = \mathbf{0}$. $\mathbf{c}'\mathbf{z} = \mathbf{b}'\mathbf{t} - \mathbf{u}'\mathbf{w}$ must occur at the optimum. Complementarity forces the optimal objectives of the primal and dual to be equal, $\mathbf{c}'\mathbf{z}_{opt} = \mathbf{b}'\mathbf{t}_{opt} - \mathbf{u}'\mathbf{w}_{opt}$, because

$$0 = \mathbf{v}'_{opt}\mathbf{w}_{opt} = (\mathbf{u} - \mathbf{z}_{opt})'\mathbf{w}_{opt} = \mathbf{u}'\mathbf{w}_{opt} - \mathbf{z}'_{opt}\mathbf{w}_{opt}$$
$$0 = \mathbf{z}'_{opt}\mathbf{s}_{opt} = \mathbf{s}'_{opt}\mathbf{z}_{opt} = (\mathbf{c} - \mathbf{A}'\mathbf{t}_{opt} + \mathbf{w}_{opt})'\mathbf{z}_{opt}$$
$$= \mathbf{c}'\mathbf{z}_{opt} - \mathbf{t}'_{opt}(\mathbf{A}\mathbf{z}_{opt}) + \mathbf{w}'_{opt}\mathbf{z}_{opt} = \mathbf{c}'\mathbf{z}_{opt} - \mathbf{b}'\mathbf{t}_{opt} + \mathbf{u}'\mathbf{w}_{opt}$$

Therefore

$$0 = \mathbf{c}'\mathbf{z}_{opt} - \mathbf{b}'\mathbf{t}_{opt} + \mathbf{u}'\mathbf{w}_{opt}$$

The *duality gap*, $\mathbf{c}'\mathbf{z} - \mathbf{b}'\mathbf{t} + \mathbf{u}'\mathbf{w}$, measures the convergence of the algorithm. You can specify a tolerance for this convergence criterion in the TOLERANCE= option in the PROC QUANTREG statement.

Before the optimum is reached, it is possible for a solution $(\mathbf{z}, \mathbf{t}, \mathbf{s}, \mathbf{v}, \mathbf{w})$ to violate the KKT conditions in one of several ways:

- Primal bound constraints can be broken: $\delta_b = \mathbf{u} - \mathbf{z} - \mathbf{v} \neq \mathbf{0}$.
- Primal constraints can be broken: $\delta_c = \mathbf{b} - \mathbf{A}\mathbf{z} \neq \mathbf{0}$.
- Dual constraints can be broken: $\delta_d s = \mathbf{c} - \mathbf{A}'\mathbf{t} - \mathbf{s} + \mathbf{w} \neq \mathbf{0}$.

- Complementarity conditions are unsatisfied: $\mathbf{z}'\mathrm{s} \neq 0$ and $\mathbf{v}'\mathbf{w} \neq 0$.

The interior point algorithm works by using Newton's method to find a direction $(\boldsymbol{\Delta} z^k, \boldsymbol{\Delta} t^k, \boldsymbol{\Delta} s^k, \boldsymbol{\Delta} v^k, \boldsymbol{\Delta} w^k)$ to move from the current solution $(\mathbf{z}^k, \mathbf{t}^k, \mathrm{s}^k, \mathbf{v}^k, \mathbf{w}^k)$ toward a better solution:

$$(\mathbf{z}^{k+1}, \mathbf{t}^{k+1}, \mathrm{s}^{k+1}, \mathbf{v}^{k+1}, \mathbf{w}^{k+1}) = (\mathbf{z}^k, \mathbf{t}^k, \mathrm{s}^k, \mathbf{v}^k, \mathbf{w}^k) + \kappa(\boldsymbol{\Delta} z^k, \boldsymbol{\Delta} t^k, \boldsymbol{\Delta} s^k, \boldsymbol{\Delta} v^k, \boldsymbol{\Delta} w^k)$$

$\kappa$ is the *step length* and is assigned a value as large as possible, but not so large that a $\mathbf{z}_i^{k+1}$ or $\mathrm{s}_i^{k+1}$ is "too close" to 0. You can control the step length in the KAPPA= option in the PROC QUANTREG statement.

The QUANTREG procedure implements a predictor-corrector variant of the primal-dual interior point algorithm. First, Newton's method is used to find a direction $(\boldsymbol{\Delta} z^k_{aff}, \boldsymbol{\Delta} t^k_{aff}, \boldsymbol{\Delta} s^k_{aff}, \boldsymbol{\Delta} v^k_{aff}, \boldsymbol{\Delta} w^k_{aff})$ in which to move. This is known as the *affine* step.

In iteration *k*, the affine step system that must be solved is

$$\begin{aligned}
&\boldsymbol{\Delta} z_{aff} + \boldsymbol{\Delta} v_{aff} = \boldsymbol{\delta}_b \\
&\mathbf{A}\boldsymbol{\Delta} z_{aff} = \boldsymbol{\delta}_c \\
&\mathbf{A}'\boldsymbol{\Delta} t_{aff} + \boldsymbol{\Delta} s_{aff} - \boldsymbol{\Delta} w_{aff} = \boldsymbol{\delta}_d \\
&\mathbf{S}\boldsymbol{\Delta} z_{aff} + \mathbf{Z}\boldsymbol{\Delta} s_{aff} = -\mathbf{ZSe} \\
&\mathbf{V}\boldsymbol{\Delta} w_{aff} + \mathbf{W}\boldsymbol{\Delta} z_{aff} = -\mathbf{VWe}
\end{aligned}$$

Therefore, the following computations are involved in solving the affine step, where $\kappa$ is the *step length* as before:

$$\begin{aligned}
&\boldsymbol{\Theta} = \mathbf{SZ}^{-1} + \mathbf{WV}^{-1} \\
&\boldsymbol{\rho} = \boldsymbol{\Theta}^{-1}(\boldsymbol{\delta}_d + (\mathbf{S} - \mathbf{W})\mathrm{e} - \mathbf{V}^{-1}\mathbf{W}\boldsymbol{\delta}_b) \\
&\boldsymbol{\Delta} t_{aff} = (\mathbf{A}\boldsymbol{\Theta}^{-1}\mathbf{A}')^{-1}(\boldsymbol{\delta}_c + \mathbf{A}\boldsymbol{\rho}) \\
&\boldsymbol{\Delta} z_{aff} = \boldsymbol{\Theta}^{-1}\mathbf{A}'\boldsymbol{\Delta} t_{aff} - \boldsymbol{\rho} \\
&\boldsymbol{\Delta} v_{aff} = \boldsymbol{\delta}_b - \boldsymbol{\Delta} z_{aff} \\
&\boldsymbol{\Delta} w_{aff} = -\mathbf{We} - \mathbf{V}^{-1}\mathbf{W}\boldsymbol{\Delta} z_{aff} \\
&\boldsymbol{\Delta} s_{aff} = -\mathbf{Se} - \mathbf{Z}^{-1}\mathbf{S}\boldsymbol{\Delta} z_{aff} \\
&(\mathbf{z}_{aff}, \mathbf{t}_{aff}, \mathrm{s}_{aff}, \mathbf{v}_{aff}, \mathbf{w}_{aff}) = (\mathbf{z}, \mathbf{t}, \mathrm{s}, \mathbf{v}, \mathbf{w}) + \kappa(\boldsymbol{\Delta} z_{aff}, \boldsymbol{\Delta} t_{aff}, \boldsymbol{\Delta} s_{aff}, \boldsymbol{\Delta} v_{aff}, \boldsymbol{\Delta} w_{aff})
\end{aligned}$$

The success of the affine step is gauged by calculating the complementarity of $\mathbf{z}'\mathrm{s}$ and $\mathbf{v}'\mathbf{w}$ at $(\mathbf{z}^k_{aff}, \mathbf{t}^k_{aff}, \mathrm{s}^k_{aff}, \mathbf{v}^k_{aff}, \mathbf{w}^k_{aff})$ and comparing it with the complementarity at the starting point $(\mathbf{z}^k, \mathbf{t}^k, \mathrm{s}^k, \mathbf{v}^k, \mathbf{w}^k)$. If the affine step was successful in reducing the complementarity by a substantial amount, the need for centering is not great. Therefore, a value close to 0 is assigned to $\sigma$ in the following second linear system, which is used to determine a centering vector.

The following linear system is solved to determine a centering vector $(\mathbf{\Delta} z_c, \mathbf{\Delta} t_c, \mathbf{\Delta} s_c, \mathbf{\Delta} v_c, \mathbf{\Delta} w_c)$ from $(\mathbf{z}_{aff}, \mathbf{t}_{aff}, \mathbf{s}_{aff}, \mathbf{v}_{aff}, \mathbf{w}_{aff})$:

$$
\begin{aligned}
&\mathbf{\Delta} z_c + \mathbf{\Delta} v_c = \mathbf{0} \\
&\mathbf{A}\mathbf{\Delta} z_c = \mathbf{0} \\
&\mathbf{A}'\mathbf{\Delta} t_c + \mathbf{\Delta} s_c - \mathbf{\Delta} w_c = \mathbf{0} \\
&\mathbf{S}\mathbf{\Delta} z_c + \mathbf{Z}\mathbf{\Delta} s_c = -\mathbf{Z}_{aff}\mathbf{S}_{aff}\mathbf{e} + \sigma\boldsymbol{\mu}\mathbf{e} \\
&\mathbf{V}\mathbf{\Delta} w_c + \mathbf{W}\mathbf{\Delta} v_c = -\mathbf{V}_{aff}\mathbf{W}_{aff}\mathbf{e} + \sigma\boldsymbol{\mu}\mathbf{e} \\
&\text{where } \zeta_{start} = \mathbf{z}'\mathbf{s} + \mathbf{v}'\mathbf{w}, \text{ complementarity at the start of the iteration} \\
&\zeta_{aff} = \mathbf{z}'_{aff}\mathbf{s}_{aff} + \mathbf{v}'_{aff}\mathbf{w}_{aff}, \text{ the affine complementarity} \\
&\mu = \zeta_{aff}/2n, \text{ the average complementarity} \\
&\sigma = (\zeta_{aff}/\zeta_{start})^3
\end{aligned}
$$

However, if the affine step was unsuccessful, then centering is deemed beneficial, and a value close to 1.0 is assigned to $\sigma$. In other words, the value of $\sigma$ is adaptively altered depending on the progress made toward the optimum.

Therefore, the following computations are involved in solving the centering step:

$$
\begin{aligned}
&\boldsymbol{\rho} = \mathbf{\Theta}^{-1}(\sigma\mu(\mathbf{Z}^{-1} - \mathbf{V}^{-1})\mathbf{e} - \mathbf{Z}^{-1}\mathbf{Z}_{aff}\mathbf{S}_{aff}\mathbf{e} + \mathbf{V}^{-1}\mathbf{V}_{aff}\mathbf{W}_{aff}\mathbf{e}) \\
&\mathbf{\Delta} t_c = (\mathbf{A}\mathbf{\Theta}^{-1}\mathbf{A}')^{-1}\mathbf{A}\boldsymbol{\rho} \\
&\mathbf{\Delta} z_c = \mathbf{\Theta}^{-1}\mathbf{A}'\mathbf{\Delta} t_c - \boldsymbol{\rho} \\
&\mathbf{\Delta} v_c = -\mathbf{\Delta} z_c \\
&\mathbf{\Delta} w_c = \sigma\mu\mathbf{V}^{-1}\mathbf{e} - \mathbf{V}^{-1}\mathbf{V}_{aff}\mathbf{W}_{aff}\mathbf{e} - \mathbf{V}^{-1}\mathbf{W}_{aff}\mathbf{\Delta} v_c \\
&\mathbf{\Delta} s_c = \sigma\mu\mathbf{Z}^{-1}\mathbf{e} - \mathbf{Z}^{-1}\mathbf{Z}_{aff}\mathbf{S}_{aff}\mathbf{e} - \mathbf{Z}^{-1}\mathbf{S}_{aff}\mathbf{\Delta} z_c
\end{aligned}
$$

Then

$$
\begin{aligned}
&(\mathbf{\Delta} z, \mathbf{\Delta} t, \mathbf{\Delta} s, \mathbf{\Delta} v, \mathbf{\Delta} w) = (\mathbf{\Delta} z_{aff}, \mathbf{\Delta} t_{aff}, \mathbf{\Delta} s_{aff}, \mathbf{\Delta} v_{aff}, \mathbf{\Delta} w_{aff}) + (\mathbf{\Delta} z_c, \mathbf{\Delta} t_c, \mathbf{\Delta} s_c, \mathbf{\Delta} v_c, \mathbf{\Delta} w_c) \\
&(\mathbf{z}^{k+1}, \mathbf{t}^{k+1}, \mathbf{s}^{k+1}, \mathbf{v}^{k+1}, \mathbf{w}^{k+1}) = (\mathbf{z}^k, \mathbf{t}^k, \mathbf{s}^k, \mathbf{v}^k, \mathbf{w}^k) + \kappa(\mathbf{\Delta} z, \mathbf{\Delta} t, \mathbf{\Delta} s, \mathbf{\Delta} v, \mathbf{\Delta} w)
\end{aligned}
$$

where, as before, $\kappa$ is the step length, which is assigned a value as large as possible but not so large that a $\mathbf{z}_i^{k+1}$, $\mathbf{s}_i^{k+1}$, $\mathbf{v}_i^{k+1}$, or $\mathbf{w}_i^{k+1}$ is "too close" to 0.

Although the predictor-corrector variant entails solving two linear systems instead of one, fewer iterations are usually required to reach the optimum. The additional overhead of the second linear system is small because the matrix $(\mathbf{A}\mathbf{\Theta}^{-1}\mathbf{A}')$ has already been factorized in order to solve the first linear system.

You can specify the starting point in the INEST= option in the PROC QUANTREG statement. By default, the starting point is set to be the least squares estimate.

### Smoothing Algorithm

To minimize the sum of the absolute residuals $D_{LAR}(\boldsymbol{\beta})$, the smoothing algorithm approximates the nondifferentiable function $D_{LAR}$ by the following smooth function(which is referred to as the Huber function),

$$D_\gamma(\boldsymbol{\beta}) = \sum_{i=1}^{n} H_\gamma(r_i(\boldsymbol{\beta}))$$

where

$$H_\gamma(t) = \begin{cases} t^2/(2\gamma) & \text{if } |t| \le \gamma \\ |t| - \gamma/2 & \text{if } |t| > \gamma \end{cases}$$

Here $r_i(\boldsymbol{\beta}) = y_i - \mathbf{x}_i'\boldsymbol{\beta}$, and the threshold $\gamma$ is a positive real number. The function $D_\gamma$ is continuously differentiable, and a minimizer $\beta_\gamma$ of $D_\gamma$ is close to a minimizer $\hat{\boldsymbol{\beta}}_{LAR}$ of $D_{LAR}(\boldsymbol{\beta})$ when $\gamma$ is close to 0.

The advantage of the smoothing algorithm as described in Madsen and Nielsen (1993) is that the $L_1$ solution $\hat{\boldsymbol{\beta}}_{LAR}$ can be detected when $\gamma > 0$ is small. In other words, it is not necessary to let $\gamma$ converge to 0 in order to find a minimizer of $D_{LAR}(\boldsymbol{\beta})$. The algorithm terminates before going through the entire sequence of values of $\gamma$ that are generated by the algorithm. Convergence is indicated by no change of the status of residuals $r_i(\boldsymbol{\beta})$ as $\gamma$ goes through this sequence.

The smoothing algorithm extends naturally from $L_1$ regression to general quantile regression (Chen 2007). The function

$$D_{\rho_\tau}(\boldsymbol{\beta}) = \sum_{i=1}^{n} \rho_\tau(y_i - \mathbf{x}_i'\boldsymbol{\beta})$$

can be approximated by the smooth function

$$D_{\gamma,\tau}(\boldsymbol{\beta}) = \sum_{i=1}^{n} H_{\gamma,\tau}(r_i(\boldsymbol{\beta}))$$

where

$$H_{\gamma,\tau}(t) = \begin{cases} t(\tau-1) - \frac{1}{2}(\tau-1)^2\gamma & \text{if } t \le (\tau-1)\gamma \\ \frac{t^2}{2\gamma} & \text{if } (\tau-1)\gamma \le t \le \tau\gamma \\ t\tau - \frac{1}{2}\tau^2\gamma & \text{if } t \ge \tau\gamma \end{cases}$$

The function $H_{\gamma,\tau}$ is determined by whether $r_i(\boldsymbol{\beta}) \le (\tau-1)\gamma$, $r_i(\boldsymbol{\beta}) \ge \tau\gamma$, or $(\tau-1)\gamma \le r_i(\boldsymbol{\beta}) \le \tau\gamma$. These inequalities divide $\mathbf{R}^p$ into subregions that are separated by the parallel hyperplanes $r_i(\boldsymbol{\beta}) = (\tau-1)\gamma$ and $r_i(\boldsymbol{\beta}) = \tau\gamma$. The set of all such hyperplanes is denoted by $B_{\gamma,\tau}$:

$$B_{\gamma,\tau} = \{\boldsymbol{\beta} \in \mathbf{R}^p | \exists i : r_i(\boldsymbol{\beta}) = (\tau-1)\gamma \text{ or } r_i(\boldsymbol{\beta}) = \tau\gamma\}$$

Define the sign vector $\mathbf{s}_\gamma(\boldsymbol{\beta}) = (s_1(\boldsymbol{\beta}), \ldots, s_n(\boldsymbol{\beta}))'$ as

$$s_i = s_i(\boldsymbol{\beta}) = \begin{cases} -1 & \text{if } r_i(\boldsymbol{\beta}) \le (\tau-1)\gamma \\ 0 & \text{if } (\tau-1)\gamma \le r_i(\boldsymbol{\beta}) \le \tau\gamma \\ 1 & \text{if } r_i(\boldsymbol{\beta}) \ge \tau\gamma \end{cases}$$

and introduce

$$w_i = w_i(\boldsymbol{\beta}) = 1 - s_i^2(\boldsymbol{\beta})$$

Therefore,

$$H_{\gamma,\tau}(r_i(\boldsymbol{\beta})) = \frac{1}{2\gamma} w_i r_i^2(\boldsymbol{\beta})$$
$$+s_i[\frac{1}{2}r_i(\boldsymbol{\beta}) + \frac{1}{4}(1-2\tau)\gamma + s_i(r_i(\boldsymbol{\beta})(\tau - \frac{1}{2}) - \frac{1}{4}(1-2\tau+2\tau^2)\gamma)]$$

This equation yields

$$D_{\gamma,\tau}(\boldsymbol{\beta}) = \frac{1}{2\gamma}\mathbf{r}'\mathbf{W}_{\gamma,\tau}\mathbf{r} + \mathbf{v}'(s)\mathbf{r} + c(s)$$

where $\mathbf{W}_{\gamma,\tau}$ is the diagonal $n \times n$ matrix with diagonal elements $w_i(\boldsymbol{\beta})$, $\mathbf{v}'(s) = (s_1((2\tau-1)s_1+1)/2, \ldots, s_n((2\tau-1)s_n+1)/2)$, $c(s) = \sum[\frac{1}{4}(1-2\tau)\gamma s_i - \frac{1}{4}s_i^2(1-2\tau+2\tau^2)\gamma]$, and $r(\boldsymbol{\beta}) = (r_1(\boldsymbol{\beta}), \ldots, r_n(\boldsymbol{\beta}))'$.

The gradient of $D_{\gamma,\tau}$ is given by

$$D^{(1)}_{\gamma,\tau}(\boldsymbol{\beta}) = -\mathbf{A}[\frac{1}{\gamma}\mathbf{W}_{\gamma,\tau}(\boldsymbol{\beta})r(\boldsymbol{\beta}) + v(s)]$$

For $\boldsymbol{\beta} \in \mathbf{R}^p \backslash B_{\gamma,\tau}$ the Hessian exists and is given by

$$D^{(2)}_{\gamma,\tau}(\boldsymbol{\beta}) = \frac{1}{\gamma}\mathbf{A}\mathbf{W}_{\gamma,\tau}(\boldsymbol{\beta})\mathbf{A}'$$

The gradient is a continuous function in $\mathbf{R}^p$, whereas the Hessian is piecewise constant.

Following Madsen and Nielsen (1993), the vector s is referred to as a *$\gamma$-feasible sign vector* if there exists $\boldsymbol{\beta} \in \mathbf{R}^p \backslash B_{\gamma,\tau}$ with $\mathrm{s}_\gamma(\boldsymbol{\beta}) = \mathrm{s}$. If s is $\gamma$-feasible, then $Q_s$ is defined as the quadratic function $Q_s(\boldsymbol{\alpha})$ that is derived from $D_{\gamma,\tau}(\boldsymbol{\beta})$ by substituting s for $\mathrm{s}_\gamma$. Thus, for any $\boldsymbol{\beta}$ with $\mathrm{s}_\gamma = \mathrm{s}$,

$$Q_s(\boldsymbol{\alpha}) = \frac{1}{2}(\boldsymbol{\alpha} - \boldsymbol{\beta})' D^{(2)}_{\gamma,\tau}(\boldsymbol{\beta})(\boldsymbol{\alpha} - \boldsymbol{\beta}) + D^{(1)}_{\gamma,\tau}(\boldsymbol{\beta})(\boldsymbol{\alpha} - \boldsymbol{\beta}) + D_{\gamma,\tau}(\boldsymbol{\beta})$$

In the domain $C_s = \{\alpha | \mathrm{s}_\gamma(\alpha) = \mathrm{s}\}$,

$$D_{\gamma,\tau}(\alpha) = Q_s(\boldsymbol{\alpha})$$

For each $\gamma > 0$ and $\theta \in \mathbf{R}^p$, there can be one or several corresponding quadratics, $Q_s$. If $\theta \notin B_{\gamma,\tau}$, then $Q_s$ is characterized by $\theta$ and $\gamma$. However, for $\theta \in B_{\gamma,\tau}$, the quadratic is not unique. Therefore, the following *reference* determines the quadratic:

$$(\gamma, \theta, \mathrm{s})$$

Again following Madsen and Nielsen (1993), let $(\gamma, \theta, \mathrm{s})$ be a *feasible reference* if s is a $\gamma$-feasible sign vector, where $\theta \in C_s$, and let $(\gamma, \theta, \mathrm{s})$ be a *solution reference* if s is feasible and $\theta$ minimizes $D_{\gamma,\tau}$.

The smoothing algorithm for minimizing $D_{\rho_\tau}$ is based on minimizing $D_{\gamma,\tau}$ for a set of decreasing $\gamma$. For each new value of $\gamma$, information from the previous solution is used. Finally, when $\gamma$ is small enough, a solution can be found by the following modified Newton-Raphson algorithm as stated by Madsen and Nielsen (1993):

1. Find an initial solution reference $(\gamma, \boldsymbol{\beta}_{\gamma}, \mathrm{s})$.
2. Repeat the following substeps until $\gamma = 0$.
   a) Decrease $\gamma$.
   b) Find a solution reference $(\gamma, \boldsymbol{\beta}_{\gamma}, \mathrm{s})$.

   $\boldsymbol{\beta}_0$ is the solution.

By default, the initial solution reference is found by letting $\boldsymbol{\beta}_{\gamma}$ be the least squares solution. Alternatively, you can specify the initial solution reference with the INEST= option in the PROC QUANTREG statement. Then $\gamma$ and s are chosen according to these initial values.

There are several approaches for determining a decreasing sequence of values of $\gamma$. The QUANTREG procedure uses a strategy by Madsen and Nielsen (1993). The computation that is uses is not significant compared to the Newton-Raphson step. You can control the ratio of consecutive decreasing values of $\gamma$ by specifying the RRATIO= suboption in the ALGORITHM= option in the PROC QUANTREG statement. By default,

$$\text{RRATIO} = \begin{cases} 0.1 & \text{if } n \geq 10,000 \text{ and } p \leq 20 \\ 0.9 & \text{if } \frac{p}{n} \geq 0.1 \text{ or } \{n \leq 5,000 \text{ and } p \geq 300\} \\ 0.5 & \text{otherwise} \end{cases}$$

For the $L_1$ and quantile regression, it turns out that the smoothing algorithm is very efficient and competitive, especially for a *fat* data set—namely, when $\frac{p}{n} > 0.05$ and $\mathbf{AA}'$ is dense. See Chen (2007) for a complete smoothing algorithm and details.

## Confidence Interval

The QUANTREG procedure provides three methods to compute confidence intervals for the regression quantile parameter $\boldsymbol{\beta}(\tau)$: sparsity, rank, and resampling. The sparsity method is the most direct and the fastest, but it involves estimation of the sparsity function, which is not robust for data that are not independently and identically distributed. To deal with this problem, the QUANTREG procedure uses a local estimate of the sparsity function to compute a Huber sandwich estimate. The rank method, which computes confidence intervals by inverting the rank score test, does not suffer from this problem. However, the rank method uses the simplex algorithm and is computationally expensive with large data sets. The resampling method, which uses the bootstrap approach, addresses these problems, but at a computation cost.

Based on these properties, the QUANTREG uses a combination of the resampling and rank methods as the default. For data sets that have more than either 5,000 observations or more than 20 variables, the QUANTREG procedure uses the MCMB resampling method; otherwise it uses the rank method. You can request a particular method by using the CI= option in the PROC QUANTREG statement.

### Sparsity

Consider the linear model

$$y_i = \mathbf{x}_i'\boldsymbol{\beta} + \epsilon_i$$

Assume that $\{\epsilon_i\}$, $i = 1, \ldots, n$, are iid with a distribution $F$ and a density $f = F'$, where $f(F^{-1}(\tau)) > 0$ in a neighborhood of $\tau$. Under some mild conditions,

$$\sqrt{n}(\hat{\boldsymbol{\beta}}(\tau) - \boldsymbol{\beta}(\tau)) \rightarrow N(0, \omega^2(\tau, F)\boldsymbol{\Omega}^{-1})$$

where $\omega^2(\tau, F) = \tau(1-\tau)/f^2(F^{-1}(\tau))$ and $\boldsymbol{\Omega} = \lim_{n\rightarrow\infty} n^{-1} \sum \mathbf{x}_i \mathbf{x}_i'$ (Koenker and Bassett 1982b).

This asymptotic distribution for the regression quantile $\hat{\boldsymbol{\beta}}(\tau)$ can be used to construct confidence intervals. However, the reciprocal of the density function,

$$s(\tau) = [f(F^{-1}(\tau))]^{-1}$$

which is called the *sparsity function*, must first be estimated.

Because

$$s(t) = \frac{d}{dt} F^{-1}(t)$$

$s(t)$ can be estimated by the difference quotient of the empirical quantile function—that is,

$$\hat{s}_n(t) = [\hat{F}_n^{-1}(t + h_n) - \hat{F}_n^{-1}(t - h_n)]/2h_n$$

where $\hat{F}_n$ is an estimate of $F^{-1}$ and $h_n$ is a bandwidth that tends to 0 as $n \rightarrow \infty$.

The QUANTREG procedure provides two bandwidth methods. The Bofinger bandwidth

$$h_n = n^{-1/5}(\frac{4.5 s^2(t)}{(s^{(2)}(t))^2})^{1/5}$$

is an optimizer of mean squared error for standard density estimation. The Hall-Sheather bandwidth

$$h_n = n^{-1/3} z_\alpha^{2/3}(\frac{1.5 s(t)}{s^{(2)}(t)})^{1/3}$$

is based on Edgeworth expansions for studentized quantiles, where $s^{(2)}(t)$ is the second derivative of $s(t)$ and $z_\alpha$ satisfies $\Phi(z_\alpha) = 1 - \alpha/2$ for the construction of $1 - \alpha$ confidence intervals. The following quantity is not sensitive to $f$ and can be estimated by assuming $f$ is Gaussian:

$$\frac{s(t)}{s^{(2)}(t)} = \frac{f^2}{2(f^{(1)}/f)^2 + [(f^{(1)}/f)^2 - f^{(2)}/f]}$$

$\hat{F}^{-1}$ can be estimated in either of the following ways:

- by the empirical quantile function of the residuals from the quantile regression fit,

  $$\hat{F}^{-1}(t) = r_{(i)}, \quad \text{for } t \in [(i-1)/n, i/n),$$

- by the empirical quantile function of regression proposed by Bassett and Koenker (1982),

  $$\hat{F}^{-1}(t) = \bar{\mathbf{x}}'\hat{\boldsymbol{\beta}}(t)$$

The QUANTREG procedure interpolates the first empirical quantile function and produces the piecewise linear version:

$$\hat{F}^{-1}(t) = \begin{cases} r_{(1)} & \text{if } t \in [0, 1/2n) \\ \lambda r_{(i+1)} + (1-\lambda) r_{(i)} & \text{if } t \in [(2i-1)/2n, (2i+1)/2n) \\ r_{(n)} & \text{if } t \in [(2n-1), 1] \end{cases}$$

$\hat{F}^{-1}$ is set to a constant if $t \pm h_n$ falls outside [0, 1].

This estimator of the sparsity function is sensitive to the iid assumption. Alternately, Koenker and Machado (1999) consider the non-iid case. By assuming local linearity of the conditional quantile function $Q(\tau|x)$ in $x$, they propose a local estimator of the density function by using the difference quotient. A Huber sandwich estimate of the covariance and standard error is computed and used to construct the confidence intervals. One difficulty with this method is the selection of the bandwidth when using the difference quotient. With a small sample size, either the Bofinger or the Hall-Sheather bandwidth tends to be too large to assure local linearity of the conditional quantile function. The QUANTREG procedure uses a heuristic bandwidth selection in these cases.

By default, the QUANTREG procedure computes non-iid confidence intervals. You can request iid confidence intervals by specifying the IID option in the PROC QUANTREG statement.

## Inversion of Rank Tests

The classical theory of rank tests can be extended to test the hypothesis $H_0$: $\beta_2 = \eta$ in the linear regression model $\mathbf{y} = \mathbf{X}_1\beta_1 + \mathbf{X}_2\beta_2 + \boldsymbol{\epsilon}$. Here, $(\mathbf{X}_1, \mathbf{X}_2) = \mathbf{A}'$. See Gutenbrunner and Jureckova (1992) for more details. By inverting this test, confidence intervals can be computed for the regression quantiles that correspond to $\beta_2$.

The rank score function $\hat{a}_n(t) = (\hat{a}_{n1}(t), \ldots, \hat{a}_{nn}(t))$ can be obtained by solving the dual problem:

$$\max_a \{(\mathbf{y} - \mathbf{X}_2\boldsymbol{\eta})'a | \mathbf{X}_1'a = (1-t)\mathbf{X}_1'\mathbf{e},\ a \in [0,1]^n\}$$

For a fixed quantile $\tau$, integrating $\hat{a}_{ni}(t)$ with respect to the $\tau$-quantile score function

$$\varphi_\tau(t) = \tau - I(t < \tau)$$

yields the $\tau$-quantile scores

$$\hat{b}_{ni} = -\int_0^1 \varphi_\tau(t) d\hat{a}_{ni}(t) = \hat{a}_{ni}(\tau) - (1-\tau)$$

Under the null hypothesis $H_0$: $\beta_2 = \eta$,

$$S_n(\eta) = n^{-1/2}\mathbf{X}_2'\hat{b}_n(\eta) \to N(0, \tau(1-\tau)\boldsymbol{\Omega}_n)$$

for large $n$, where $\boldsymbol{\Omega}_n = n^{-1}\mathbf{X}_2'(\mathbf{I} - \mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1')\mathbf{X}_2$.

Let

$$T_n(\eta) = \frac{1}{\sqrt{\tau(1-\tau)}} S_n(\eta)\boldsymbol{\Omega}_n^{-1/2}$$

Then $T_n(\hat{\beta}_2(\tau)) = 0$ from the constraint $\mathbf{A}\hat{a} = (1-\tau)\mathbf{A}\mathbf{e}$ in the full model. In order to obtain confidence intervals for $\beta_2$, a critical value can be specified for $T_n$. The dual vector $\hat{a}_n(\eta)$ is a piecewise constant in $\eta$, and $\eta$ can be altered without compromising the optimality of $\hat{a}_n(\eta)$ as long as the signs of the residuals in the primal quantile regression problem do not change. When $\eta$ gets to such a boundary, the solution does change. But it can be restored by taking one simplex pivot. The process can continue in this way until $T_n(\eta)$ exceeds the specified critical value. Because $T_n(\eta)$ is piecewise constant, interpolation can be used to obtain the desired level of confidence interval (Koenker and d'Orey 1994).

### Resampling

The bootstrap can be implemented to compute confidence intervals for regression quantile estimates. As in other regression applications, both the residual bootstrap and the *xy*-pair bootstrap can be used. The former assumes iid random errors and resamples from the residuals, whereas the latter resamples *xy* pairs and accommodates some forms of heteroscedasticity. Koenker (1994) considered a more interesting resampling mechanism, resampling directly from the full regression quantile process, which he called the Heqf bootstrap.

In contrast with these bootstrap methods, Parzen, Wei, and Ying (1994) observed that the following estimating equation for the $\tau$ regression quantile is a pivotal quantity for the $\tau$ quantile regression parameter $\beta_\tau$:

$$S(\boldsymbol{\beta}) = n^{-1/2} \sum_{i=1}^{n} \mathbf{x}_i (\tau - I(y_i \leq \mathbf{x}_i' \boldsymbol{\beta}))$$

In other words, the distribution of $\mathbf{S}(\boldsymbol{\beta})$ can be generated exactly by a random vector $\mathbf{U}$, which is a weighted sum of independent, re-centered Bernoulli variables. They further showed that for large $n$, the distribution of $\hat{\boldsymbol{\beta}}(\tau) - \beta_\tau$ can be approximated by the conditional distribution of $\hat{\boldsymbol{\beta}}_U - \hat{\boldsymbol{\beta}}_n(\tau)$, where $\hat{\boldsymbol{\beta}}_U$ solves an augmented quantile regression problem by using $n + 1$ observations that have $\mathbf{x}_{n+1} = -n^{-1/2} u / \tau$ and $y_{n+1}$ sufficiently large for a given realization of $u$. By exploiting the asymptotically pivotal role of the quantile regression "gradient condition," this approach also achieves some robustness to certain heteroscedasticity.

Although the bootstrap method by Parzen, Wei, and Ying (1994) is much simpler, it is too time-consuming for relatively large data sets, especially for high-dimensional data sets. The QUANTREG procedure implements a new, general resampling method developed by He and Hu (2002), which is called the Markov chain marginal bootstrap (MCMB). For quantile regression, the MCMB method has the advantage that it solves $p$ one-dimensional equations instead of solving $p$-dimensional equations, as the previous bootstrap methods do. This greatly improves the feasibility of the resampling method in computing confidence intervals for regression quantiles.

## Covariance-Correlation

You can specify the COVB and CORRB options in the MODEL statement to request covariance and correlation matrices for the estimated parameters.

The QUANTREG procedure provides two methods for computing the covariance and correlation matrices of the estimated parameters: an asymptotic method and a bootstrap method. Bootstrap covariance and correlation matrices are computed when resampling confidence intervals are computed. Asymptotic covariance and correlation matrices are computed when asymptotic confidence intervals are computed. The rank method for confidence intervals does not provide a covariance-correlation estimate.

### Asymptotic Covariance-Correlation

This method corresponds to the sparsity method for the confidence intervals. For the sparsity function in the computation of the asymptotic covariance and correlation, the QUANTREG procedure provides both iid and non-iid estimates. By default, the QUANTREG procedure computes non-iid estimates.

### Bootstrap Covariance-Correlation

This method corresponds to the resampling method for the confidence intervals. The Markov chain marginal bootstrap (MCMB) method is used.

## Linear Test

Consider the linear model

$$y_i = \mathbf{x}_{1i}'\boldsymbol{\beta}_1 + \mathbf{x}_{2i}'\boldsymbol{\beta}_2 + \epsilon_i$$

where $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ are $p$- and $q$-dimensional unknown parameters and $\{\epsilon_i\}$, $i = 1, \ldots, n$, are errors with unknown density function $f_i$. Let $\mathbf{x}_i' = (\mathbf{x}_{1i}', \mathbf{x}_{2i}')$, and let $\hat{\boldsymbol{\beta}}_1(\tau)$ and $\hat{\boldsymbol{\beta}}_2(\tau)$ be the parameter estimates for $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$, respectively at the $\tau$ quantile. The covariance matrix $\boldsymbol{\Omega}$ for the parameter estimates is partitioned correspondingly as $\boldsymbol{\Omega}_{ij}$ with $i = 1, 2$; $j = 1, 2$; and $\boldsymbol{\Omega}^{22} = (\boldsymbol{\Omega}_{22} - \boldsymbol{\Omega}_{21}\boldsymbol{\Omega}_{11}^{-1}\boldsymbol{\Omega}_{12})^{-1}$.

### Testing Effects of Covariates

Three tests are available in the QUANTREG procedure for the linear null hypothesis $H_0 : \beta_2 = 0$ at the $\tau$ quantile:

- The Wald test statistic, which is based on the estimated coefficients for the unrestricted model, is given by

  $$T_W(\tau) = \hat{\boldsymbol{\beta}}_2'(\tau)\hat{\boldsymbol{\Sigma}}(\tau)^{-1}\hat{\boldsymbol{\beta}}_2(\tau)$$

  where $\hat{\boldsymbol{\Sigma}}(\tau)$ is an estimator of the covariance of $\hat{\boldsymbol{\beta}}_2(\tau)$. The QUANTREG procedure provides two estimators for the covariance, as described in the previous section. The estimator that is based on the asymptotic covariance is

  $$\hat{\boldsymbol{\Sigma}}(\tau) = \frac{1}{n}\hat{\omega}(\tau)^2\boldsymbol{\Omega}^{22}$$

  where $\hat{\omega}(\tau) = \sqrt{\tau(1-\tau)}\hat{s}(\tau)$ and $\hat{s}(\tau)$ is the estimated sparsity function. The estimator that is based on the bootstrap covariance is the empirical covariance of the MCMB samples.

- The likelihood ratio test is based on the difference between the objective function values in the restricted and unrestricted models. Let $D_0(\tau) = \sum \rho_\tau(y_i - \mathbf{x}_i\hat{\boldsymbol{\beta}}(\tau))$, and let $D_1(\tau) = \sum \rho_\tau(y_i - \mathbf{x}_{1i}\hat{\boldsymbol{\beta}}_1(\tau))$. Set

  $$T_{\text{LR}}(\tau) = 2(\tau(1-\tau)\hat{s}(\tau))^{-1}(D_1(\tau) - D_0(\tau))$$

  where $\hat{s}(\tau)$ is the estimated sparsity function.

- The rank test statistic is given by

  $$T_R(\tau) = \mathbf{S}_n' \mathbf{M}_n^{-1} \mathbf{S}_n / A^2(\varphi)$$

  where

  $$\mathbf{S}_n = n^{-1/2}(\mathbf{X}_2 - \hat{\mathbf{X}}_2)'\hat{\mathbf{b}}_n$$
  $$\mathbf{\Psi} = \mathrm{diag}(f_i(Q_{y_i}(\tau|\mathbf{x}_{1i}, \mathbf{x}_{2i})))$$
  $$\hat{\mathbf{X}}_2 = \mathbf{X}_1(\mathbf{X}_1'\mathbf{\Psi}\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{X}_2$$
  $$\mathbf{M}_n = (\mathbf{X}_2 - \hat{\mathbf{X}}_2)(\mathbf{X}_2 - \hat{\mathbf{X}}_2)'/n$$
  $$\hat{\mathbf{b}}_{ni} = \int_0^1 \hat{\mathbf{a}}_{ni}(t)d\varphi(t)$$
  $$\hat{\mathbf{a}}(t) = \max_a \{\mathbf{y}'\mathbf{a}|\mathbf{X}_1'\mathbf{a} = (1-t)\mathbf{X}_1'\mathbf{e},\ \mathbf{a} \in [0,1]^n\}$$
  $$A^2(\varphi) = \int_0^1 (\varphi(t) - \bar{\varphi}(t))^2 dt$$
  $$\bar{\varphi}(t) = \int_0^1 \varphi(t)dt$$

  and $\varphi(t)$ is one of the following score functions:

  - Wilcoxon scores: $\phi(t) = t - 1/2$
  - normal scores: $\phi(t) = \Phi^{-1}(t)$, where $\Phi$ is the normal distribution function
  - sign scores: $\phi(t) = 1/2\mathrm{sign}(t - 1/2)$
  - tau scores: $\phi_\tau(t) = \tau - I(t < \tau)$.

  The rank test statistic $T_R(\tau)$, unlike Wald tests or likelihood ratio tests, requires no estimation of the nuisance parameter $f_i$ under iid error models (Gutenbrunner et al. 1993).

Koenker and Machado (1999) prove that the three test statistics ($T_W(\tau)$, $T_{\mathrm{LR}}(\tau)$, and $T_R(\tau)$) are asymptotically equivalent and that their distributions converge to $\chi_q^2$ under the null hypothesis, where $q$ is the dimension of $\beta_2$.

## Testing for Heteroscedasticity

After you obtain the parameter estimates for several quantiles specified in the MODEL statement, you can test whether there are significant differences for the estimates for the same covariates across the quantiles. For example, if you want to test whether the parameters $\beta_2$ are the same across quantiles, the null hypothesis $H_0$ can be written as $\beta_2(\tau_1) = ... = \beta_2(\tau_k)$, where $\tau_j$, $j = 1, ..., k$, are the quantiles specified in the MODEL statement. See Koenker and Bassett (1982a) for details.

## Leverage Point and Outlier Detection

The QUANTREG procedure uses robust multivariate location and scale estimates for leverage-point detection.

Mahalanobis distance is defined as

$$\mathrm{MD}(\mathbf{x}_i) = [(\mathbf{x}_i - \bar{\mathbf{x}})'\bar{\mathbf{C}}(\mathbf{A})^{-1}(\mathbf{x}_i - \bar{\mathbf{x}})]^{1/2}$$

where $\bar{\mathbf{x}} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i$ and $\bar{\mathbf{C}}(\mathbf{A}) = \frac{1}{n-1}\sum_{i=1}^{n}(\mathbf{x}_i - \bar{\mathbf{x}})'(\mathbf{x}_i - \bar{\mathbf{x}})$ are the empirical multivariate location and scale, respectively. Here, $\mathbf{x}_i = (\mathbf{x}_{i1}, \ldots, \mathbf{x}_{i(p-1)})'$ does not include the intercept variable. The relationship between the Mahalanobis distance $\mathrm{MD}(\mathbf{x}_i)$ and the matrix $\mathbf{H} = (h_{ij}) = \mathbf{A}'(\mathbf{A}\mathbf{A}')^{-1}\mathbf{A}$ is

$$h_{ii} = \frac{1}{n-1}\mathrm{MD}_i^2 + \frac{1}{n}$$

Robust distance is defined as

$$\mathrm{RD}(\mathbf{x}_i) = [(\mathbf{x}_i - \mathbf{T}(\mathbf{A}))'\mathbf{C}(\mathbf{A})^{-1}(\mathbf{x}_i - \mathbf{T}(\mathbf{A}))]^{1/2}$$

where $\mathbf{T}(\mathbf{A})$ and $\mathbf{C}(\mathbf{A})$ are robust multivariate location and scale estimates that are computed according to the minimum covariance determinant (MCD) method of Rousseeuw and Van Driessen (1999).

These distances are used to detect leverage points. You can use the LEVERAGE and DIAGNOSTICS options in the MODEL statement to request leverage-point and outlier diagnostics, respectively. Two new variables, Leverage and Outlier, respectively, are created and saved in an output data set that is specified in the OUTPUT statement.

Let $C(p) = \sqrt{\chi^2_{p;1-\alpha}}$ be the cutoff value. The variable LEVERAGE is defined as

$$\text{LEVERAGE} = \begin{cases} 0 & \text{if } \mathrm{RD}(\mathbf{x}_i) \le C(p) \\ 1 & \text{otherwise} \end{cases}$$

You can specify a cutoff value in the LEVERAGE option in the MODEL statement.

Residuals $r_i, i = 1, \ldots, n$, that are based on quantile regression estimates are used to detect vertical outliers. The variable OUTLIER is defined as

$$\text{OUTLIER} = \begin{cases} 0 & \text{if } |r_i| \le k\sigma \\ 1 & \text{otherwise} \end{cases}$$

You can specify the multiplier $k$ of the cutoff value in the CUTOFF= option in the MODEL statement. You can specify the scale $\sigma$ in the SCALE= option in the MODEL statement. By default, $k = 3$ and the scale $\sigma$ is computed as the corrected median of the absolute residuals:

$\sigma = \mathrm{median}\{|r_i|/\beta_0, i = 1, \ldots, n\}$

where $\beta_0 = \Phi^{-1}(0.75)$ is an adjustment constant for consistency when the normal distribution is used.

An ODS table called DIAGNOSTICS contains the Leverage and Outlier variables.

## INEST= Data Set

The INEST= data set specifies initial estimates for all the parameters in the model. The INEST= data set must contain the intercept variable (named Intercept) and all independent variables in the MODEL statement.

If BY processing is used, the INEST= data set should also include the BY variables, and there must be at least one observation for each BY group. If there is more than one observation in one BY group, the first one read is used for that BY group.

If the INEST= data set also contains the _TYPE_ variable, only observations with the _TYPE_ value `'PARMS'` are used as starting values.

You can specify starting values for the interior point algorithm or the smoothing algorithm in the INEST= data set. The INEST= data set has the same structure as the OUTEST= data set, but it is not required to have all the variables or observations that appear in the OUTEST= data set. One simple use of the INEST= option is passing the previous OUTEST= data set directly to the next model as an INEST= data set, assuming that the two models have the same parameterization. If you specify more than one quantile in the MODEL statement, the same initial values are used for all quantiles.

## OUTEST= Data Set

The OUTEST= data set contains parameter estimates for the specified model with all quantiles. A set of observations is created for each quantile specified. You can also specify a label in the MODEL statement to distinguish between the estimates for different models that are used by the QUANTREG procedure.

If the QUANTREG procedure does not produce valid solutions, the parameter estimates are set to missing in the OUTEST data set.

If this data set is created, it contains all the variables that are specified in the MODEL statement and the BY statement. Each observation consists of parameter values for a specified quantile, and the dependent variable has the value –1.

The following variables are also added to the data set:

_MODEL_
: a character variable of length 8 that contains the label of the MODEL statement, if present. Otherwise, the variable's value is blank.

_ALGORITHM_
: a character variable of length 8 that contains the name of the algorithm that is used for computing the parameter estimates, either SIMPLEX, INTERIOR, or SMOOTH.

_TYPE_
: a character variable of length 8 that contains the type of the observation. This variable is fixed as PARMS to indicate that the observation includes parameter estimates.

_STATUS_
: a character variable of length 12 that contains the status of model fitting (either NORMAL, NOUNIQUE, or NOVALID).

Intercept
: a numeric variable that contains the intercept parameter estimates.

_QUANTILE_
: a numeric variable that contains the specified quantile levels.

Any specified BY variables are also added to the OUTEST= data set.

## Computational Resources

The various algorithms need different amounts of memory for working space. Let $p$ be the number of parameters that are estimated, $n$ be the number of observations that are used in the model estimation, and $s$ be the size (in bytes) of the double data type.

For the simplex algorithm, the minimum working space (in bytes) that is needed is

$$(2np + 6n + 10p)s$$

For the interior point algorithm, the minimum working space (in bytes) that is needed is

$$(np + p^2 + 13n + 4p)s$$

For the smoothing algorithm, the minimum working space (in bytes) that is needed is

$$(np + p^2 + 6n + 4p)s$$

For the last two algorithms, if you want to use preprocessing, the following additional amount of working space (in bytes) is needed:

$$(np + 6n + 2p)s$$

If sufficient space is available, the input data set is kept in memory. Otherwise, the input data set is reread as necessary, and the execution time of the procedure increases substantially.

## ODS Table Names

The QUANTREG procedure assigns a name to each table it creates. You can specify these names when you use the Output Delivery System (ODS) to select tables and create output data sets. These names are listed in the Table 81.9.

**Table 81.9** ODS Tables Produced in PROC QUANTREG

| ODS Table Name | Description | Statement | Option |
|---|---|---|---|
| ClassLevels | Classification variable levels | CLASS | Default |
| CorrB | Parameter estimate correlation matrix | MODEL | CORRB |
| CovB | Parameter estimate covariance matrix | MODEL | COVB |
| Diagnostics | Outlier diagnostics | MODEL | DIAGNOSTICS |
| DiagSummary | Summary of the outlier diagnostics | MODEL | DIAGNOSTICS |
| IPIterHistory | Iteration history (interior point) | MODEL | ITPRINT |
| ModelInfo | Model information | MODEL | Default |
| NObs | Number of observations | PROC | Default |
| ObjFunction | Objective function | MODEL | Default |
| ParameterEstimates | Parameter estimates | MODEL | Default |
| ParmInfo | Parameter indices | MODEL | Default |
| PerfSettings | Performance settings | PERFORMANCE | DETAILS |
| ProcessEst | Quantile process estimates | MODEL | QUANTILE= |
| ProcessObj | Objective function for quantile process | MODEL | QUANTILE= |

**Table 81.9** (continued)

| ODS Table Name | Description | Statement | Option |
|---|---|---|---|
| SMIterHistory | Iteration history (smoothing) | MODEL | ITPRINT |
| SummaryStatistics | Summary statistics for model variables | MODEL | Default |
| Tests | Results for tests | TEST | Default |
| ScalableTiming | Timing details | PERFORMANCE | DETAILS |

## ODS Graphics

Statistical procedures use ODS Graphics to create graphs as part of their output. ODS Graphics is described in detail in Chapter 21, "Statistical Graphics Using ODS."

Before you create graphs, ODS Graphics must be enabled (for example, by specifying the ODS GRAPHICS ON statement). For more information about enabling and disabling ODS Graphics, see the section "Enabling and Disabling ODS Graphics" on page 606 in Chapter 21, "Statistical Graphics Using ODS."

The overall appearance of graphs is controlled by ODS styles. Styles and other aspects of using ODS Graphics are discussed in the section "A Primer on ODS Statistical Graphics" on page 605 in Chapter 21, "Statistical Graphics Using ODS."

For a single quantile, two plots are particularly useful in revealing outliers and leverage points: a scatter plot of the standardized residuals for the specified quantile against the robust distances and a scatter plot of the robust distances against the classical Mahalanobis distances. You can request these two plots by using the PLOT=RDPLOT and PLOT=DDPLOT options, respectively.

You can also request a normal quantile-quantile plot and a histogram of the standardized residuals for the specified quantile by using the PLOT=QQPLOT and PLOT=HISTOGRAM options, respectively.

You can request a plot of fitted conditional quantiles by the single continuous variable that is specified in the model by using the PLOT=FITPLOT option.

All these plots can be requested by specifying corresponding plot options in either the PROC QUANTREG statement or the MODEL statement. If you specify same plot options in both statements, options in the PROC QUANTREG statement override options in the MODEL statement.

You can specify the PLOT=QUANTPLOT option only in the MODEL statement to request a quantile process plot with confidence bands.

The plot options in the PROC QUANTREG statement and the MODEL statement are summarized in Table 81.10. See the PLOT= option in the PROC QUANTREG statement and the PLOT= option in the MODEL statement for details.

**Table 81.10** Options for Plots

| Keyword | Plot |
|---|---|
| ALL | All appropriate plots |
| DDPLOT | Robust distance versus Mahalanobis distance |
| FITPLOT | Conditional quantile fit versus independent variable |
| HISTOGRAM | Histogram of standardized robust residuals |
| NONE | No plot |
| QUANTPLOT | Scatter plot of regression quantile |
| QQPLOT | Q-Q plot of standardized robust residuals |
| RDPLOT | Standardized robust residual versus robust distance |

The following subsections provide information about these graphs.

## ODS Graph Names

The QUANTREG procedure assigns a name to each graph it creates. You can use these names to refer to the graphs when you use ODS. The names along with the required statements and options are listed in Table 81.11.

**Table 81.11** Graphs Produced by PROC QUANTREG

| ODS Graph Name | Plot Description | Statement | Option |
|---|---|---|---|
| DDPlot | Robust distance versus Mahalanobis distance | PROC MODEL | DDPLOT |
| FitPlot | Quantile fit versus independent variable | PROC MODEL | FITPLOT |
| Histogram | Histogram of standardized robust residuals | PROC MODEL | HISTOGRAM |
| QQPlot | Q-Q plot of standardized robust residuals | PROC MODEL | QQPLOT |
| QuantPanel | Panel of quantile plots with confidence limits | MODEL | QUANTPLOT |
| QuantPlot | Scatter plot for regression quantiles with confidence limits | MODEL | QUANTPLOT UNPACK |
| RDPlot | Standardized robust residual versus robust distance | PROC MODEL | RDPLOT |

## Fit Plot

When the model has a single independent continuous variable (with or without the intercept), the QUANTREG procedure automatically creates a plot of fitted conditional quantiles against this independent variable for one or more quantiles that are specified in the MODEL statement.

The following example reuses the trout data set in the section "Analysis of Fish-Habitat Relationships" on page 6874 to show the fit plot for one or several quantiles:

```
ods graphics on;

proc quantreg data=trout ci=resampling;
   model LnDensity = WDRatio / quantile=0.9 seed=1268;
run;

proc quantreg data=trout ci=resampling;
   model LnDensity = WDRatio / quantile=0.5 0.75 0.9 seed=1268;
run;
```

For a single quantile, the confidence limits for the fitted conditional quantiles are also plotted if you specify the CI=RESAMPLING or CI=SPARSITY option. (See Figure 81.14.) For multiple quantiles, confidence limits are not plotted by default. (See Figure 81.15.) You can add the confidence limits on the plot by specifying the option PLOT=FITPLOT(SHOWLIMITS).

The QUANTREG procedure also provides fit plots for quantile regression splines and polynomials if they are based on a single continuous variable. (See Example 81.4 and Example 81.5 for some examples.)

**Figure 81.14** Fit Plot with Confidence Limits

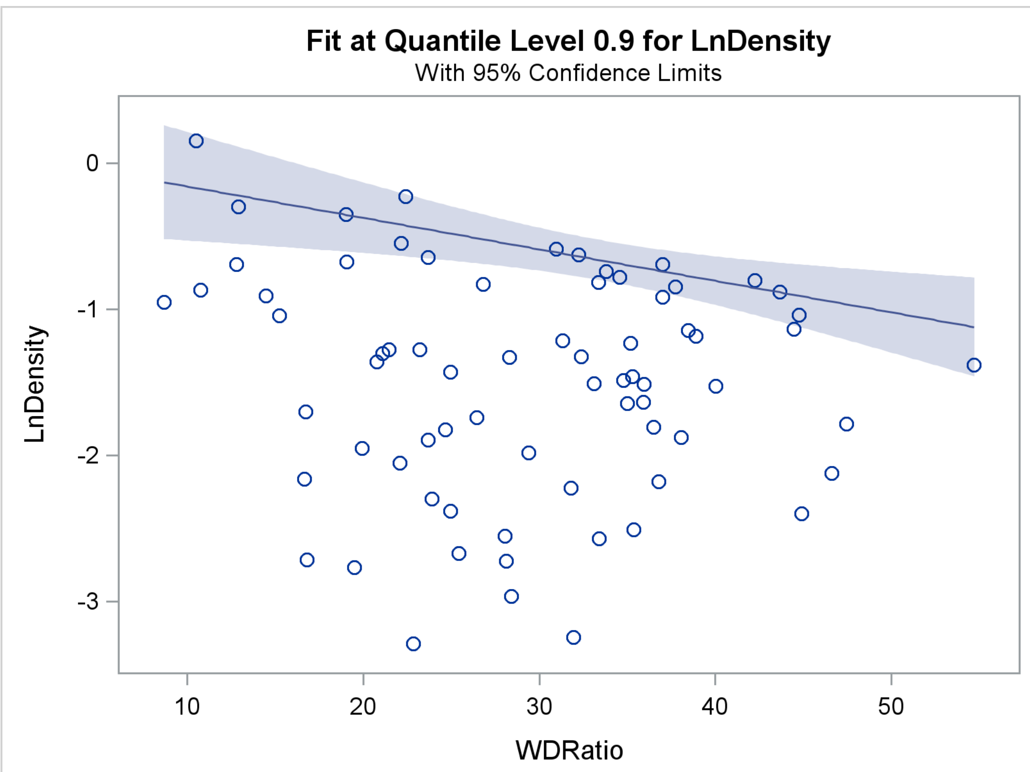
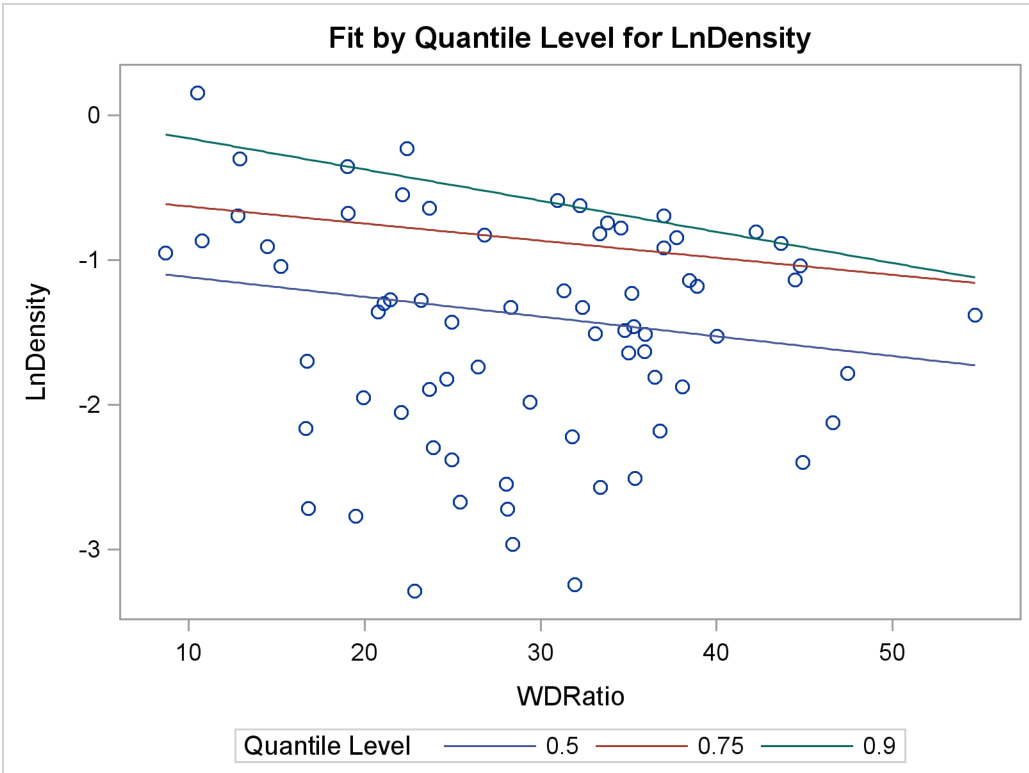


**Figure 81.15** Fit Plot for Multiple Quantiles




## Quantile Process Plot

A quantile process plot is a scatter plot of an estimated regression parameter against a quantile. You can request this plot by specifying the PLOT=QUANTPLOT option in the MODEL statement when multiple regression quantiles are computed or when the entire quantile process is computed. Quantile process plots are often used to check model variations at different quantiles, which is usually called model heterogeneity.

By default, panels are used to hold multiple process plots (up to four in each panel). You can use the UNPACK option to request individual process plots. Figure 81.10 in the section "Analysis of Fish-Habitat Relationships" on page 6874 shows a panel that includes two quantile process plots. Output 81.2.9 in Example 81.2 shows a single quantile process plot. Example 81.3 demonstrates more quantile process plots and their usage.

## Distance-Distance Plot

The distance-distance plot (DDPLOT) is mainly used for leverage-point diagnostics. It is a scatter plot of the robust distances against the classical Mahalanobis distances for the continuous independent variables. For more information about the robust distance, see the section "Leverage Point and Outlier Detection" on page 6913. If is a classification variable is specified in the model, this plot is not created.

You can use the PLOT=DDPLOT option to request this plot. The following statements use the growth data set in Example 81.2 to create a single plot, which is shown in Output 81.2.4 in Example 81.2:

```
ods graphics on;

proc quantreg data=growth ci=resampling plot=ddplot;
   model GDP = lgdp2 mse2 fse2 fhe2 mhe2 lexp2
               lintr2 gedy2 Iy2 gcony2 lblakp2 pol2 ttrad2
               / quantile=.5 diagnostics leverage(cutoff=8) seed=1268;
   id Country;
run;
```

The reference lines represent the cutoff values. The diagonal line is also drawn to show the distribution of the distances. By default, all outliers and leverage points are labeled with observation numbers. To change the default, you can use the LABEL= option as described in Table 81.5.

## Residual-Distance Plot

The residual-distance plot (RDPLOT) is used for both outlier and leverage-point diagnostics. It is a scatter plot of the standardized residuals against the robust distances. For more information about the robust distance, see the section "Leverage Point and Outlier Detection" on page 6913. If a classification variable is specified in the model, this plot is not created.

You can use the PLOT=RDPLOT option to request this plot. The following statements use the growth data set in Example 81.2 to create a single plot, which is shown in Output 81.2.3 in Example 81.2:

```
ods graphics on;

proc quantreg data=growth ci=resampling plot=rdplot;
   model GDP = lgdp2 mse2 fse2 fhe2 mhe2 lexp2
               lintr2 gedy2 Iy2 gcony2 lblakp2 pol2 ttrad2
               / quantile=.5 diagnostics leverage(cutoff=8) seed=1268;
   id Country;
run;
```

The reference lines represent the cutoff values. By default, all outliers and leverage points are labeled with observation numbers. To change the default, you can use the LABEL= option as described in Table 81.5.

If you specify ID variables instead of observation numbers in the ID statement, the values of the first ID variable are used as labels.

## Histogram and Q-Q Plot

PROC QUANTREG produces a histogram and a Q-Q plot for the standardized residuals. The histogram is superimposed with a normal density curve and a kernel density curve. Using the growth data set in Example 81.2, the following statements create the plot that is shown in Output 81.2.5 in Example 81.2:

```
ods graphics on;

proc quantreg data=growth ci=resampling plot=histogram;
   model GDP = lgdp2 mse2 fse2 fhe2 mhe2 lexp2
               lintr2 gedy2 Iy2 gcony2 lblakp2 pol2 ttrad2
               / quantile=.5 diagnostics leverage(cutoff=8) seed=1268;
   id Country;
run;
```

# Examples: QUANTREG Procedure

## Example 81.1: Comparison of Algorithms

This example illustrates and compares the three algorithms for regression estimation available in the QUANTREG procedure. The simplex algorithm is the default because of its stability. Although this algorithm is slower than the interior point and smoothing algorithms for large data sets, the difference is not as significant for data sets with fewer than 5,000 observations and 50 variables. The simplex algorithm can also compute the entire quantile process, which is shown in Example 81.2.

The following statements generate 1,000 random observations. The first 950 observations are from a linear model, and the last 50 observations are significantly biased in the *y*-direction. In other words, 5% of the observations are contaminated with outliers.

```
data a (drop=i);
   do i=1 to 1000;
      x1=rannor(1234);
      x2=rannor(1234);
      e=rannor(1234);
      if i > 950 then y=100 + 10*e;
      else y=10 + 5*x1 + 3*x2 + 0.5 * e;
      output;
   end;
run;
```

The following statements invoke the QUANTREG procedure to fit a median regression model with the default simplex algorithm. They produce the results that are shown in Output 81.1.1 through Output 81.1.3.

```
proc quantreg data=a;
   model y = x1 x2;
run;
```

Output 81.1.1 displays model information and summary statistics for variables in the model. It indicates that the simplex algorithm is used to compute the optimal solution and that the rank method is used to compute confidence intervals of the parameters.

By default, the QUANTREG procedure fits a median regression model. This is indicated by the quantile value 0.5 in Output 81.1.2, which also displays the objective function value and the predicted value of the response at the means of the covariates.

Output 81.1.3 displays parameter estimates and confidence limits. These estimates are reasonable, which indicates that median regression is robust to the 50 outliers.

**Output 81.1.1** Model Fit Information and Summary Statistics from the Simplex Algorithm

BMI Percentiles for Men: 2-80 Years Old

The QUANTREG Procedure

Model Information

| | |
|---|---|
| Data Set | WORK.A |
| Dependent Variable | y |
| Number of Independent Variables | 2 |
| Number of Observations | 1000 |
| Optimization Algorithm | Simplex |
| Method for Confidence Limits | Inv_Rank |

Summary Statistics

| Variable | Q1 | Median | Q3 | Mean | Standard Deviation | MAD |
|---|---|---|---|---|---|---|
| x1 | -0.6546 | 0.0230 | 0.7099 | 0.0222 | 0.9933 | 1.0085 |
| x2 | -0.7891 | -0.0747 | 0.6839 | -0.0401 | 1.0394 | 1.0857 |
| y | 6.1045 | 10.6936 | 14.9569 | 14.4864 | 20.4087 | 6.5696 |

**Output 81.1.2** Quantile and Objective Function from the Simplex Algorithm

Quantile Level and Objective Function

| | |
|---|---|
| Quantile Level | 0.5 |
| Objective Function | 2441.1927 |
| Predicted Value at Mean | 10.0259 |

**Output 81.1.3** Parameter Estimates from the Simplex Algorithm

Parameter Estimates

| Parameter | DF | Estimate | 95% Confidence Limits | |
|---|---|---|---|---|
| Intercept | 1 | 10.0364 | 9.9959 | 10.0756 |
| x1 | 1 | 5.0106 | 4.9602 | 5.0388 |
| x2 | 1 | 3.0294 | 2.9944 | 3.0630 |

The following statements refit the model by using the interior point algorithm:

```
proc quantreg algorithm=interior(tolerance=1e-6)
              ci=none data=a;
   model y = x1 x2 / itprint nosummary;
run;
```

The TOLERANCE= option specifies the stopping criterion for convergence of the interior point algorithm, which is controlled by the duality gap. Although the default criterion is 1E–8, the value 1E–6 is often sufficient. The ITPRINT option requests the iteration history for the algorithm. The option CI=NONE suppresses the computation of confidence limits, and the option NOSUMMARY suppresses the table of summary statistics.

Output 81.1.4 displays model fit information.

**Output 81.1.4** Model Fit Information from the Interior Point Algorithm

**BMI Percentiles for Men: 2-80 Years Old**

**The QUANTREG Procedure**

**Model Information**

| | |
|---|---|
| Data Set | WORK.A |
| Dependent Variable | y |
| Number of Independent Variables | 2 |
| Number of Observations | 1000 |
| Optimization Algorithm | Interior |

Output 81.1.5 displays the iteration history of the interior point algorithm. Note that the duality gap is less than 1E–6 in the final iteration. The table also provides the number of iterations, the number of corrections, the primal step length, the dual step length, and the objective function value at each iteration.

**Output 81.1.5** Iteration History for the Interior Point Algorithm

**Iteration History of Interior Point Algorithm**

| Duality Gap | Iter | Correction | Primal Step | Dual Step | Objective Function |
|---|---|---|---|---|---|
| 2623 | 1 | 1 | 0.3113 | 0.4910 | 3303.4688 |
| 3215 | 2 | 2 | 0.0427 | 1.0000 | 2461.3774 |
| 1127 | 3 | 3 | 0.9882 | 0.3653 | 2451.1337 |
| 760.88658 | 4 | 4 | 0.3381 | 1.0000 | 2442.8104 |
| 77.10290 | 5 | 5 | 1.0000 | 0.8916 | 2441.2627 |
| 8.43666 | 6 | 6 | 0.9370 | 0.8381 | 2441.2085 |
| 1.82868 | 7 | 7 | 0.8375 | 0.7674 | 2441.1985 |
| 0.40584 | 8 | 8 | 0.6980 | 0.8636 | 2441.1948 |
| 0.09550 | 9 | 9 | 0.9438 | 0.5955 | 2441.1930 |
| 0.00665 | 10 | 10 | 0.9818 | 0.9304 | 2441.1927 |
| 0.0002248 | 11 | 11 | 0.9179 | 0.9994 | 2441.1927 |
| 5.44651E-8 | 12 | 12 | 1.0000 | 1.0000 | 2441.1927 |

Output 81.1.6 displays the parameter estimates that are obtained by using the interior point algorithm. These estimates are identical to those obtained by using the simplex algorithm.

**Output 81.1.6** Parameter Estimates from the Interior Point Algorithm

**Parameter Estimates**

| Parameter | DF | Estimate |
|---|---|---|
| Intercept | 1 | 10.0364 |
| x1 | 1 | 5.0106 |
| x2 | 1 | 3.0294 |

The following statements refit the model by using the smoothing algorithm. They produce the results that are shown in Output 81.1.7 through Output 81.1.9.

```
proc quantreg algorithm=smooth(rratio=.5) ci=none data=a;
   model y = x1 x2 / itprint nosummary;
run;
```

The RRATIO= option controls the reduction speed of the threshold. Output 81.1.7 displays the model fit information.

**Output 81.1.7** Model Fit Information from the Smoothing Algorithm

**BMI Percentiles for Men: 2-80 Years Old**

**The QUANTREG Procedure**

**Model Information**

| | |
|---|---|
| Data Set | WORK.A |
| Dependent Variable | y |
| Number of Independent Variables | 2 |
| Number of Observations | 1000 |
| Optimization Algorithm | Smooth |

Output 81.1.8 displays the iteration history of the smoothing algorithm. The threshold controls the convergence. Note that the thresholds decrease by a factor of at least 0.5, which is the value specified in the RRATIO= option. The table also provides the number of iterations, the number of factorizations, the number of full updates, the number of partial updates, and the objective function value in each iteration. For details concerning the smoothing algorithm, see Chen (2007).

**Output 81.1.8** Iteration History for the Smoothing Algorithm

**Iteration History of Smoothing Algorithm**

| Threshold | Iter | Refac | Full Update | Partial Update | Objective Function |
|---|---|---|---|---|---|
| 227.24557 | 1 | 1 | 1000 | 0 | 4267.0988 |
| 116.94090 | 15 | 4 | 1480 | 2420 | 3631.9653 |
| 1.44064 | 17 | 4 | 1480 | 2583 | 2441.4719 |
| 0.72032 | 20 | 5 | 1980 | 2598 | 2441.3315 |
| 0.36016 | 22 | 6 | 2248 | 2607 | 2441.2369 |
| 0.18008 | 24 | 7 | 2376 | 2608 | 2441.2056 |
| 0.09004 | 26 | 8 | 2446 | 2613 | 2441.1997 |
| 0.04502 | 28 | 9 | 2481 | 2617 | 2441.1971 |
| 0.02251 | 30 | 10 | 2497 | 2618 | 2441.1956 |
| 0.01126 | 32 | 11 | 2505 | 2620 | 2441.1946 |
| 0.00563 | 34 | 12 | 2510 | 2621 | 2441.1933 |
| 0.00281 | 35 | 13 | 2514 | 2621 | 2441.1930 |
| 0.0000846 | 36 | 14 | 2517 | 2621 | 2441.1927 |
| 1E-12 | 37 | 14 | 2517 | 2621 | 2441.1927 |

Output 81.1.9 displays the parameter estimates that are obtained by using the smoothing algorithm. These estimates are identical to those obtained by using the simplex and interior point algorithms. All three algorithms should have the same parameter estimates unless the problem does not have a unique solution.

**Output 81.1.9** Parameter Estimates from the Smoothing Algorithm

**Parameter Estimates**

| Parameter | DF | Estimate |
|---|---|---|
| Intercept | 1 | 10.0364 |
| x1 | 1 | 5.0106 |
| x2 | 1 | 3.0294 |

The interior point algorithm and the smoothing algorithm offer better performance than the simplex algorithm for large data sets. For more information about choosing an appropriate algorithm on the basis of data set size. see Chen (2004). All three algorithms should have the same parameter estimates, unless the optimization problem has multiple solutions.

## Example 81.2: Quantile Regression for Econometric Growth Data

This example uses a SAS data set named Growth, which contains economic growth rates for countries during two time periods: 1965–1975 and 1975–1985. The data come from a study by Barro and Lee (1994) and have also been analyzed by Koenker and Machado (1999).

There are 161 observations and 15 variables in the data set. The variables, which are listed in the following table, include the national growth rates (GDP) for the two periods, 13 covariates, and a name variable

(Country) for identifying the countries in one of the two periods.

| Variable | Description |
|---|---|
| Country | Country's name and period |
| GDP | Annual change per capita in gross domestic product (GDP) |
| lgdp2 | Initial per capita GDP |
| mse2 | Male secondary education |
| fse2 | Female secondary education |
| fhe2 | Female higher education |
| mhe2 | Male higher education |
| lexp2 | Life expectancy |
| lintr2 | Human capital |
| gedy2 | Education/GDP |
| Iy2 | Investment/GDP |
| gcony2 | Public consumption/GDP |
| lblakp2 | Black market premium |
| pol2 | Political instability |
| ttrad2 | Growth rate terms trade |

The goal is to study the effect of the covariates on GDP. The following statements request median regression for a preliminary exploration. They produce the results that are in Output 81.2.1 through Output 81.2.6.

```
data growth;
   length Country$ 22;
   input Country GDP lgdp2 mse2 fse2 fhe2 mhe2 lexp2 lintr2 gedy2
         Iy2 gcony2 lblakp2 pol2 ttrad2 @@;
   datalines;
Algeria75              .0415 7.330 .1320 .0670 .0050 .0220 3.880 .1138 .0382
                       .1898 .0601 .3823 .0833 .1001
Algeria85              .0244 7.745 .2760 .0740 .0070 .0370 3.978 -.107 .0437
                       .3057 .0850 .9386 .0000 .0657
Argentina75            .0187 8.220 .7850 .6200 .0740 .1660 4.181 .4060 .0221
                       .1505 .0596 .1924 .3575 -.011
Argentina85            -.014 8.407 .9360 .9020 .1320 .2030 4.211 .1914 .0243
                       .1467 .0314 .3085 .7010 -.052
Australia75            .0259 9.101 2.541 2.353 .0880 .2070 4.263 6.937 .0348
                       .3272 .0257 .0000 .0080 -.016

   ... more lines ...

Zambia75               .0120 6.989 .3760 .1190 .0130 .0420 3.757 .4388 .0339
                       .3688 .2513 .3945 .0000 -.032
Zambia85               -.046 7.109 .4200 .2740 .0110 .0270 3.854 .8812 .0477
                       .1632 .2637 .6467 .0000 -.033
Zimbabwe75             .0320 6.860 .1450 .0170 .0080 .0450 3.833 .7156 .0337
                       .2276 .0246 .1997 .0000 -.040
Zimbabwe85             -.011 7.180 .2200 .0650 .0060 .0400 3.944 .9296 .0520
                       .1559 .0518 .7862 .7161 -.024
;
```

```
ods graphics on;

proc quantreg data=growth ci=resampling
                plots=(rdplot ddplot reshistogram);
   model GDP = lgdp2 mse2 fse2 fhe2 mhe2 lexp2
               lintr2 gedy2 Iy2 gcony2 lblakp2 pol2 ttrad2
               / quantile=.5 diagnostics leverage(cutoff=8) seed=1268;
   id Country;
   test_lgdp2: test lgdp2 / lr wald;
run;
```

The QUANTREG procedure uses the default simplex algorithm to estimate the parameters and uses the MCMB resampling method to compute confidence limits.

Output 81.2.1 displays model information and summary statistics for the variables in the model. Six summary statistics are computed, including the median and the median absolute deviation (MAD), which are robust measures of univariate location and scale, respectively. For the variable lintr2 (human capital), both the mean and standard deviation are much larger than the corresponding robust measures (median and MAD), indicating that this variable might have outliers.

**Output 81.2.1** Model Information and Summary Statistics

BMI Percentiles for Men: 2-80 Years Old

The QUANTREG Procedure

Model Information

| | |
|---|---|
| Data Set | WORK.GROWTH |
| Dependent Variable | GDP |
| Number of Independent Variables | 13 |
| Number of Observations | 161 |
| Optimization Algorithm | Simplex |
| Method for Confidence Limits | Resampling |

Summary Statistics

| Variable | Q1 | Median | Q3 | Mean | Standard Deviation | MAD |
|---|---|---|---|---|---|---|
| lgdp2 | 6.9890 | 7.7450 | 8.6080 | 7.7905 | 0.9543 | 1.1579 |
| mse2 | 0.3160 | 0.7230 | 1.2675 | 0.9666 | 0.8574 | 0.6835 |
| fse2 | 0.1270 | 0.4230 | 0.9835 | 0.7117 | 0.8331 | 0.5011 |
| fhe2 | 0.0110 | 0.0350 | 0.0890 | 0.0792 | 0.1216 | 0.0400 |
| mhe2 | 0.0400 | 0.1060 | 0.2060 | 0.1584 | 0.1752 | 0.1127 |
| lexp2 | 3.8670 | 4.0640 | 4.2430 | 4.0440 | 0.2028 | 0.2728 |
| lintr2 | 0.00160 | 0.5604 | 1.8805 | 1.4625 | 2.5491 | 1.0058 |
| gedy2 | 0.0248 | 0.0343 | 0.0466 | 0.0360 | 0.0141 | 0.0151 |
| Iy2 | 0.1396 | 0.1955 | 0.2671 | 0.2010 | 0.0877 | 0.0981 |
| gcony2 | 0.0480 | 0.0767 | 0.1276 | 0.0914 | 0.0617 | 0.0566 |
| lblakp2 | 0 | 0.0696 | 0.2407 | 0.1916 | 0.3070 | 0.1032 |
| pol2 | 0 | 0.0500 | 0.2429 | 0.1683 | 0.2409 | 0.0741 |
| ttrad2 | -0.0240 | -0.0100 | 0.00730 | -0.00570 | 0.0375 | 0.0239 |
| GDP | 0.00290 | 0.0196 | 0.0351 | 0.0191 | 0.0248 | 0.0237 |

Output 81.2.2 displays the parameter estimates and 95% confidence limits that are computed with the rank method.

**Output 81.2.2** Parameter Estimates

**Parameter Estimates**

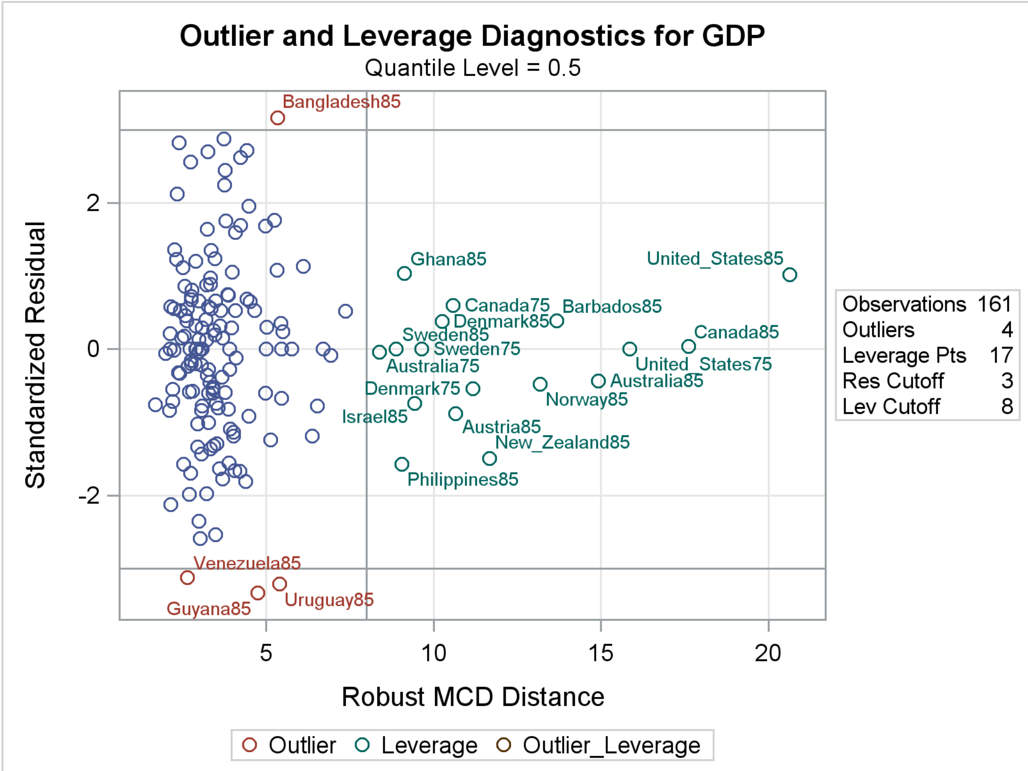
| Parameter | DF | Estimate | Standard Error | 95% Confidence Limits | | t Value | Pr > \|t\| |
|---|---|---|---|---|---|---|---|
| Intercept | 1 | -0.0488 | 0.0733 | -0.1937 | 0.0961 | -0.67 | 0.5065 |
| lgdp2 | 1 | -0.0269 | 0.0041 | -0.0350 | -0.0188 | -6.58 | <.0001 |
| mse2 | 1 | 0.0110 | 0.0080 | -0.0048 | 0.0269 | 1.38 | 0.1710 |
| fse2 | 1 | -0.0011 | 0.0088 | -0.0185 | 0.0162 | -0.13 | 0.8960 |
| fhe2 | 1 | 0.0148 | 0.0321 | -0.0485 | 0.0782 | 0.46 | 0.6441 |
| mhe2 | 1 | 0.0043 | 0.0268 | -0.0487 | 0.0573 | 0.16 | 0.8735 |
| lexp2 | 1 | 0.0683 | 0.0229 | 0.0232 | 0.1135 | 2.99 | 0.0033 |
| lintr2 | 1 | -0.0022 | 0.0015 | -0.0052 | 0.0008 | -1.44 | 0.1513 |
| gedy2 | 1 | -0.0508 | 0.1654 | -0.3777 | 0.2760 | -0.31 | 0.7589 |
| Iy2 | 1 | 0.0723 | 0.0248 | 0.0233 | 0.1213 | 2.92 | 0.0041 |
| gcony2 | 1 | -0.0935 | 0.0382 | -0.1690 | -0.0181 | -2.45 | 0.0154 |
| lblakp2 | 1 | -0.0269 | 0.0084 | -0.0435 | -0.0104 | -3.22 | 0.0016 |
| pol2 | 1 | -0.0301 | 0.0093 | -0.0485 | -0.0117 | -3.23 | 0.0015 |
| ttrad2 | 1 | 0.1613 | 0.0740 | 0.0149 | 0.3076 | 2.18 | 0.0310 |

Diagnostics for the median regression fit, which are requested in the PLOTS= option, are displayed in Output 81.2.3 and Output 81.2.4. Output 81.2.3 plots the standardized residuals from median regression against the robust MCD distance. This display is used to diagnose both vertical outliers and horizontal leverage points. Output 81.2.4 plots the robust MCD distance against the Mahalanobis distance. This display is used to diagnose leverage points.

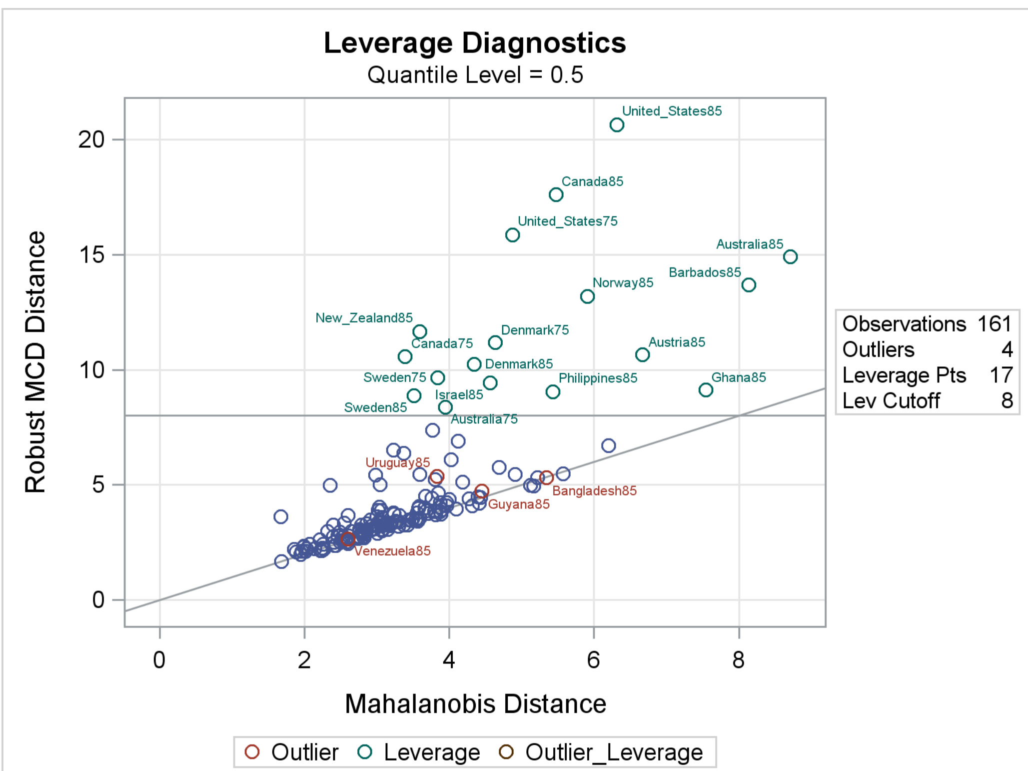
The cutoff value 8, which is specified in the LEVERAGE option, is close to the maximum of the Mahalanobis distance. Eighteen points are diagnosed as high leverage points, and almost all are countries with high human capital, which is the major contributor to the high leverage as observed from the summary statistics. Four points are diagnosed as outliers by using the default cutoff value of 3. However, these are not extreme outliers.

A histogram of the standardized residuals and two fitted density curves are displayed in Output 81.2.5. This output shows that median regression fits the data well.

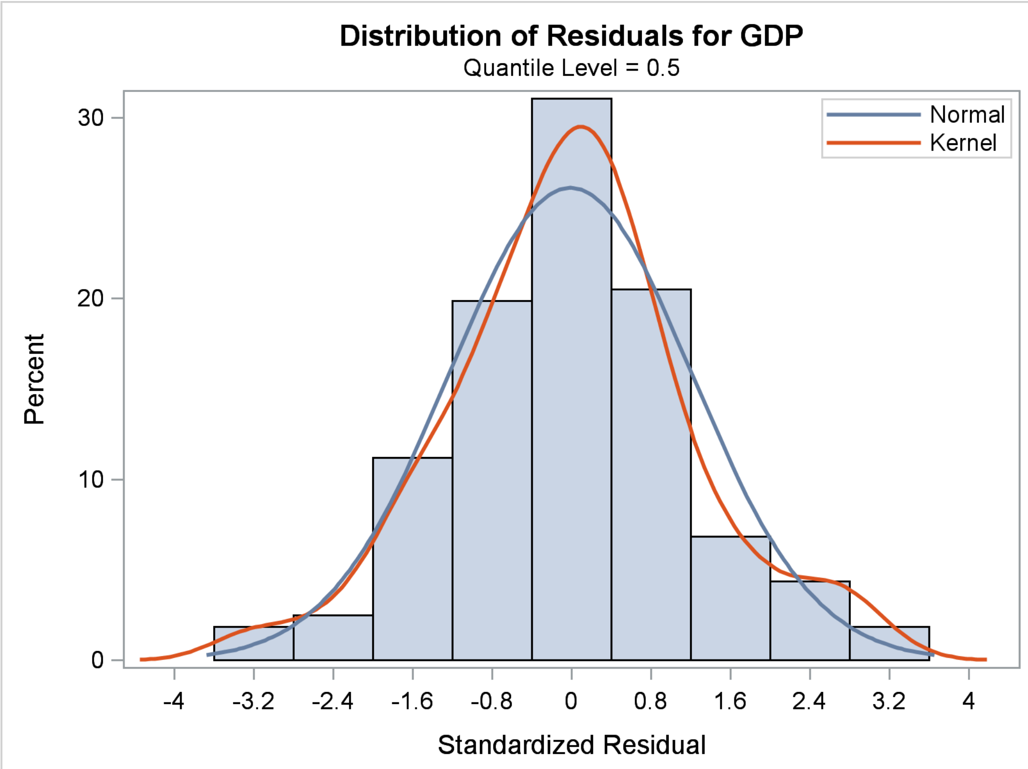
**Output 81.2.3** Plot of Residual versus Robust Distance




**Output 81.2.4** Plot of Robust Distance versus Mahalanobis Distance



**Output 81.2.5** Histogram for Residuals




Tests of significance for the initial per-capita GDP (LGDP2) are shown in Output 81.2.6.

**Output 81.2.6** Tests for Regression Coefficient

**Test test_lgdp2 Results**

| Test | Test Statistic | DF | Chi-Square | Pr > ChiSq |
|---|---|---|---|---|
| Wald | 43.2684 | 1 | 43.27 | <.0001 |
| Likelihood Ratio | 36.3047 | 1 | 36.30 | <.0001 |

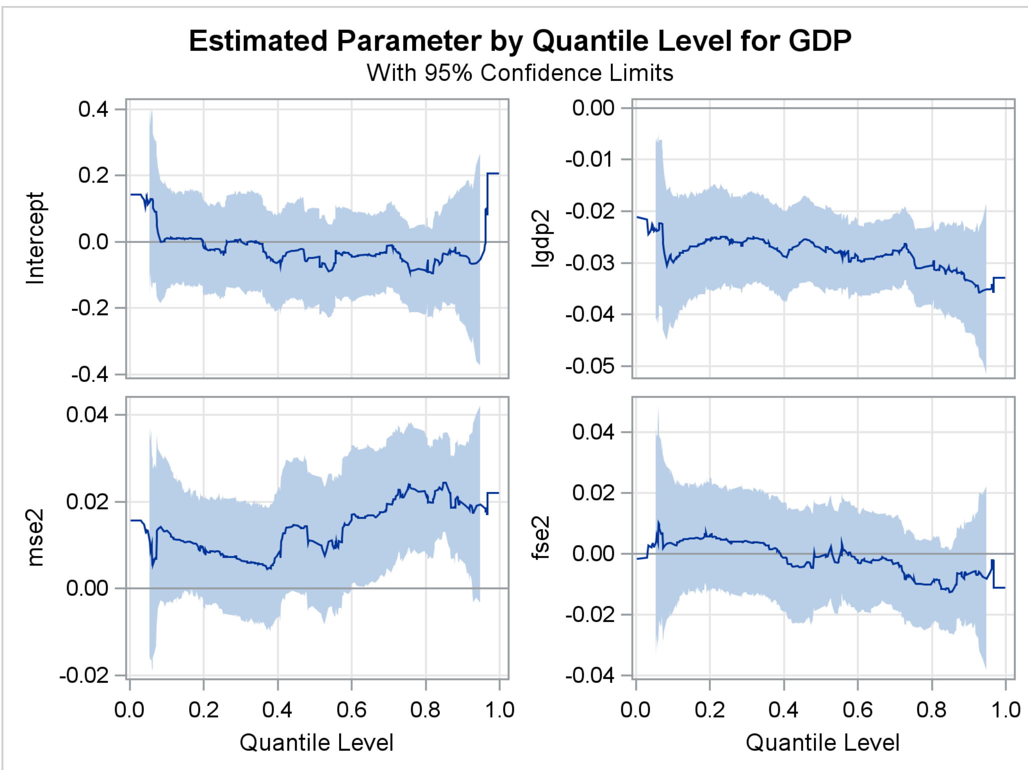
The QUANTREG procedure computes entire quantile processes for covariates when you specify QUANTILE=PROCESS in the MODEL statement, as follows:

```
proc quantreg data=growth ci=resampling;
   model GDP = lgdp2 mse2 fse2 fhe2 mhe2 lexp2 lintr2
               gedy2 Iy2 gcony2 lblakp2 pol2 ttrad2
               / quantile=process plot=quantplot seed=1268;
run;
```

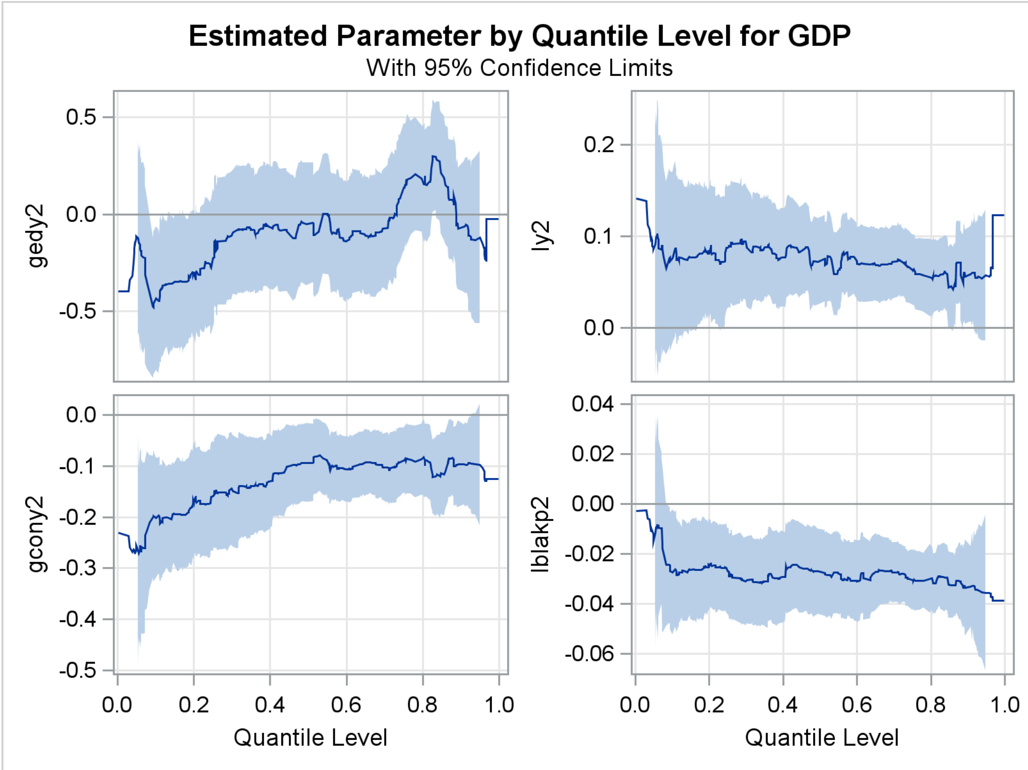
Confidence limits for quantile processes can be computed by using the sparsity or resampling methods. But they cannot be computed by using the rank method, because the computation would be prohibitively expensive.

A total of 14 quantile process plots are produced. Output 81.2.7 and Output 81.2.8 display two panels of eight selected process plots. The 95% confidence bands are shaded.

**Output 81.2.7** Quantile Processes with 95% Confidence Bands



**Output 81.2.8** Quantile Processes with 95% Confidence Bands

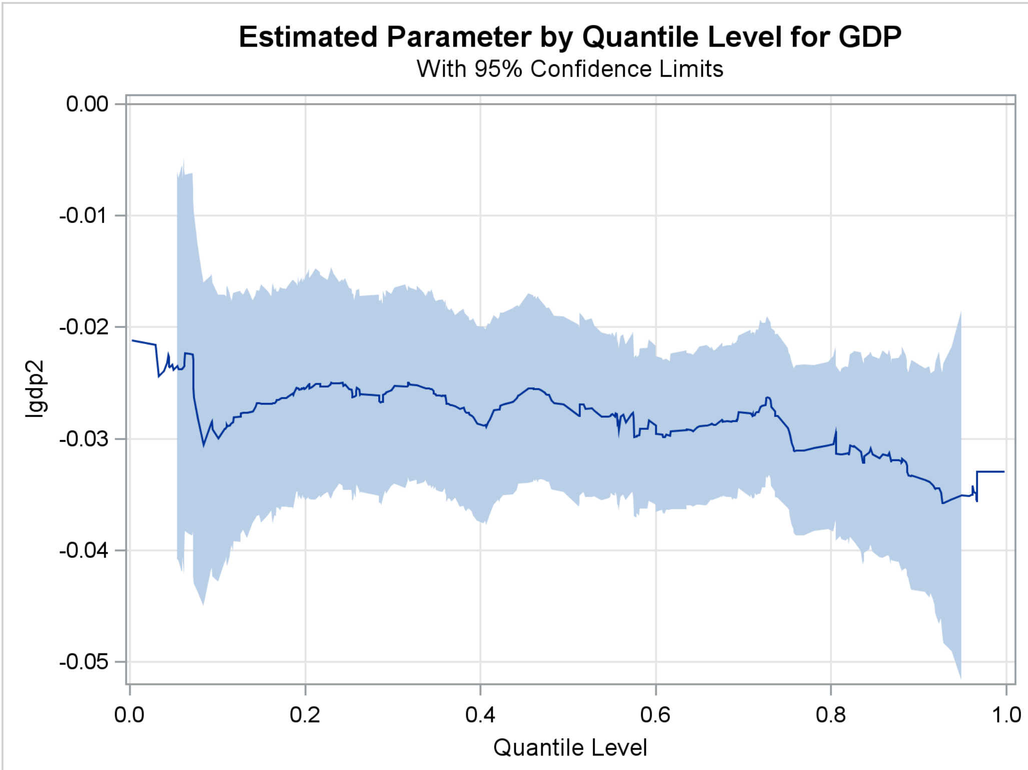



As pointed out by Koenker and Machado (1999), previous studies of the Barro growth data have focused on the effect of the initial per-capita GDP on the growth of this variable (annual change in per-capita GDP). The following statements request a single process plot for this effect:

```
proc quantreg data=growth ci=resampling;
   model GDP = lgdp2 mse2 fse2 fhe2 mhe2 lexp2 lintr2
               gedy2 Iy2 gcony2 lblakp2 pol2 ttrad2
               / quantile=process plot=quantplot(lgdp2) seed=1268;
run;
```

The plot is shown in Output 81.2.9.

**Output 81.2.9** Quantile Process Plot for LGDP2




The confidence bands here are computed by using the MCMB resampling method. In contrast, Koenker and Machado (1999) used the rank method to compute confidence limits for a few selected points. Output 81.2.9 suggests that the effect of the initial level of GDP is relatively constant over the entire distribution, with a slightly stronger effect in the upper tail.

The effects of other covariates are quite varied. An interesting covariate is public consumption divided by GDP (gcony2) (first plot in second panel), which has a constant effect over the upper half of the distribution and a larger effect in the lower tail. For an analysis of the effects of the other covariates, see Koenker and Machado (1999).

## Example 81.3: Quantile Regression Analysis of Birth-Weight Data

This example is patterned after a quantile regression analysis of covariates associated with birth weight that was carried out by Koenker and Hallock (2001). Their study uses a subset of the June 1997 Detailed Natality Data, which was published by the National Center for Health Statistics. The study demonstrates that conditional quantile functions provide more complete information about the covariate effects than ordinary least squares regression provides.

This example is based on Koenker and Hallock (2001); Abreveya (2001); it uses data for live, singleton births to mothers in the United States who were recorded as black or white, and who were between the ages of 18 and 45. For convenience, this example uses 50,000 observations, which are randomly selected from the qualified observations. Observations that have missing data for any of the variables are deleted. The data are available in the data set Sashelp.BWeight. The following step displays in Output 81.3.1 the variables in the data set:

```
proc contents varnum data=sashelp.bweight;
   ods select position;
run;
```

**Output 81.3.1** Sashelp.BWeight Data Set

BMI Percentiles for Men: 2-80 Years Old

The CONTENTS Procedure

Variables in Creation Order

| # | Variable | Type | Len | Label |
|---|---|---|---|---|
| 1 | Weight | Num | 8 | Infant Birth Weight |
| 2 | Black | Num | 8 | Black Mother |
| 3 | Married | Num | 8 | Married Mother |
| 4 | Boy | Num | 8 | Baby Boy |
| 5 | MomAge | Num | 8 | Mother's Age |
| 6 | MomSmoke | Num | 8 | Smoking Mother |
| 7 | CigsPerDay | Num | 8 | Cigarettes Per Day |
| 8 | MomWtGain | Num | 8 | Mother's Pregnancy Weight Gain |
| 9 | Visit | Num | 8 | Prenatal Visit |
| 10 | MomEdLevel | Num | 8 | Mother's Education Level |

The following step creates descriptive labels for the values of the classification variables Visit and MomEdLevel:

```
proc format;
   value vfmt 0 = 'No Visit'       1 = 'Second Trimester'
              2 = 'Last Trimester' 3 = 'First Trimester';
   value efmt 0 = 'High School'    1 = 'Some College'
              2 = 'College'        3 = 'Less Than High School';
run;
```

There are four levels of maternal education. When you specify the ORDER=INTERNAL option, PROC QUANTREG treats the highest unformatted value (3, which represents that the mother's education level is less than high school) as a reference level. The regression coefficients of other levels measure the effect relative to this level. Likewise, there are four levels of prenatal medical care of the mother, and a first visit in the first trimester serves as the reference level.

The following statements fit a regression model for 19 quantiles of birth weight, which are evenly spaced in the interval $(0, 1)$. The model includes linear and quadratic effects for the age of the mother and for weight gain during pregnancy.

```
ods graphics on;

proc quantreg ci=sparsity/iid algorithm=interior(tolerance=5.e-4)
   data=sashelp.bweight order=internal;
```

```
      class Visit MomEdLevel;
      model Weight = Black Married Boy Visit MomEdLevel MomSmoke
      CigsPerDay MomAge MomAge*MomAge
      MomWtGain MomWtGain*MomWtGain /
      quantile= 0.05 to 0.95 by 0.05
      plot=quantplot;
      format Visit vfmt. MomEdLevel efmt.;
   run;
```

Output 81.3.2 displays the model information and summary statistics for the variables in the model.

**Output 81.3.2** Model Information and Summary Statistics

**BMI Percentiles for Men: 2-80 Years Old**

**The QUANTREG Procedure**

**Model Information**

| | | |
|---|---|---|
| Data Set | SASHELP.BWEIGHT | Infant Birth Weight |
| Dependent Variable | Weight | Infant Birth Weight |
| Number of Independent Variables | 9 | |
| Number of Continuous Independent Variables | 7 | |
| Number of Class Independent Variables | 2 | |
| Number of Observations | 50000 | |
| Optimization Algorithm | Interior | |
| Method for Confidence Limits | Sparsity | |

**Summary Statistics**

| Variable | Q1 | Median | Q3 | Mean | Standard Deviation | MAD |
|---|---|---|---|---|---|---|
| Black | 0 | 0 | 0 | 0.1628 | 0.3692 | 0 |
| Married | 0 | 1.0000 | 1.0000 | 0.7126 | 0.4525 | 0 |
| Boy | 0 | 1.0000 | 1.0000 | 0.5158 | 0.4998 | 0 |
| MomSmoke | 0 | 0 | 0 | 0.1307 | 0.3370 | 0 |
| CigsPerDay | 0 | 0 | 0 | 1.4766 | 4.6541 | 0 |
| MomAge | -4.0000 | 0 | 5.0000 | 0.4161 | 5.7285 | 5.9304 |
| MomAge*MomAge | 4.0000 | 16.0000 | 49.0000 | 32.9877 | 39.2861 | 22.2390 |
| MomWtGain | -8.0000 | 0 | 9.0000 | 0.7092 | 12.8761 | 11.8608 |
| MomWtGain*MomWtGain | 16.0000 | 64.0000 | 196.0 | 166.3 | 298.8 | 88.9561 |
| Weight | 3062.0 | 3402.0 | 3720.0 | 3370.8 | 566.4 | 504.1 |

Among the 11 independent variables, Black, Married, Boy, and MomSmoke are binary variables. For these variables, the mean represents the proportion in the category. The two continuous variables, MomAge and MomWtGain, are centered at their medians, which are 27 and 30, respectively.

The quantile plots for the intercept and the other 15 factors with nonzero degrees of freedom are shown in the following four panels. In each plot, the regression coefficient at a given quantile indicates the effect on birth weight of a unit change in that factor, assuming that the other factors are fixed. The bands represent 95% confidence intervals.

Although the data set used here is a subset of the Natality data set, the results are quite similar to those of Koenker and Hallock (2001) for the full data set.

In Output 81.3.3, the first plot is for the intercept. As explained by Koenker and Hallock (2001), the intercept "may be interpreted as the estimated conditional quantile function of the birth-weight distribution of a girl born to an unmarried, white mother with less than a high school education, who is 27 years old and had a weight gain of 30 pounds, didn't smoke, and had her first prenatal visit in the first trimester of the pregnancy."

The second plot shows that infants born to black mothers weigh less than infants born to white mothers, especially in the lower tail of the birth-weight distribution. The third plot shows that marital status has a large positive effect on birth weight, especially in the lower tail. The fourth plot shows that boys weigh more than girls for any chosen quantile; this difference is smaller in the lower quantiles of the distribution.

In Output 81.3.4, the first three plots deal with prenatal care. Compared with babies born to mothers who had a prenatal visit in the first trimester, babies born to mothers who received no prenatal care weigh less, especially in the lower quantiles of the birth-weight distributions. As noted by Koenker and Hallock (2001), "babies born to mothers who delayed prenatal visits until the second or third trimester have substantially *higher* birthweights in the lower tail than mothers who had a prenatal visit in the first trimester. This might be interpreted as the self-selection effect of mothers confident about favorable outcomes."

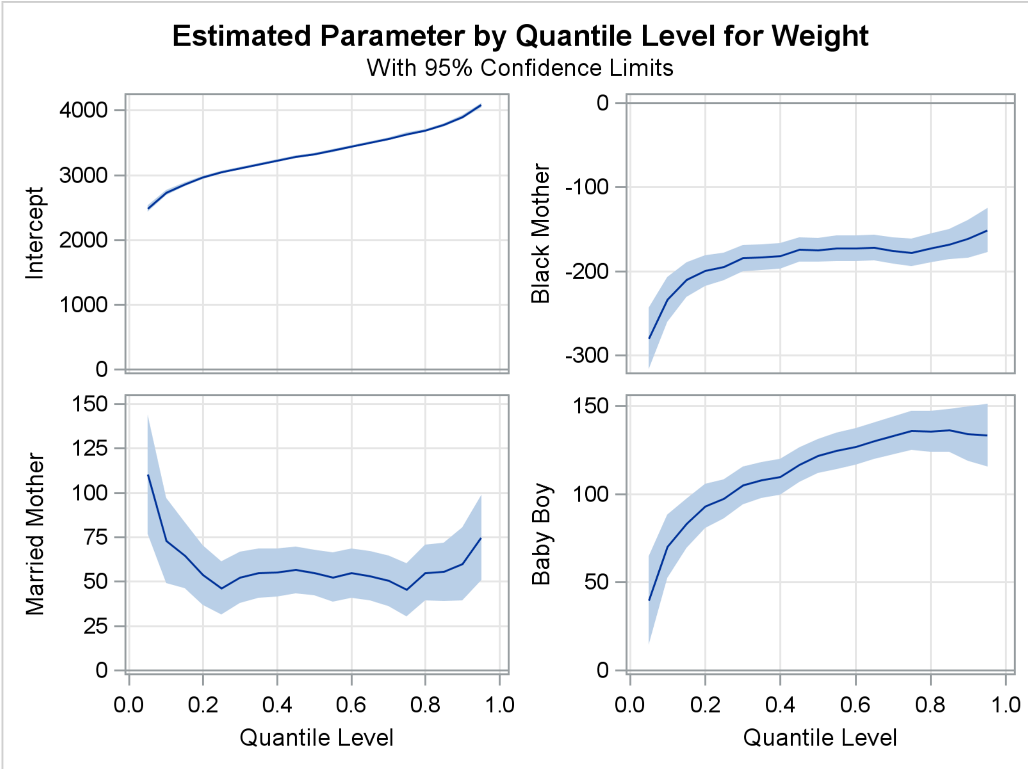
The fourth plot in Output 81.3.4 and the first two plots in Output 81.3.5 are for variables that are related to education. Education beyond high school is associated with a positive effect on birth weight. The effect of high school education is uniformly around 15 grams across the entire birth-weight distribution (this is a pure location shift effect), whereas the effect of some college and college education is more positive in the lower quantiles than the upper quantiles.

The remaining two plots in Output 81.3.5 show that smoking is associated with a large negative effect on birth weight.

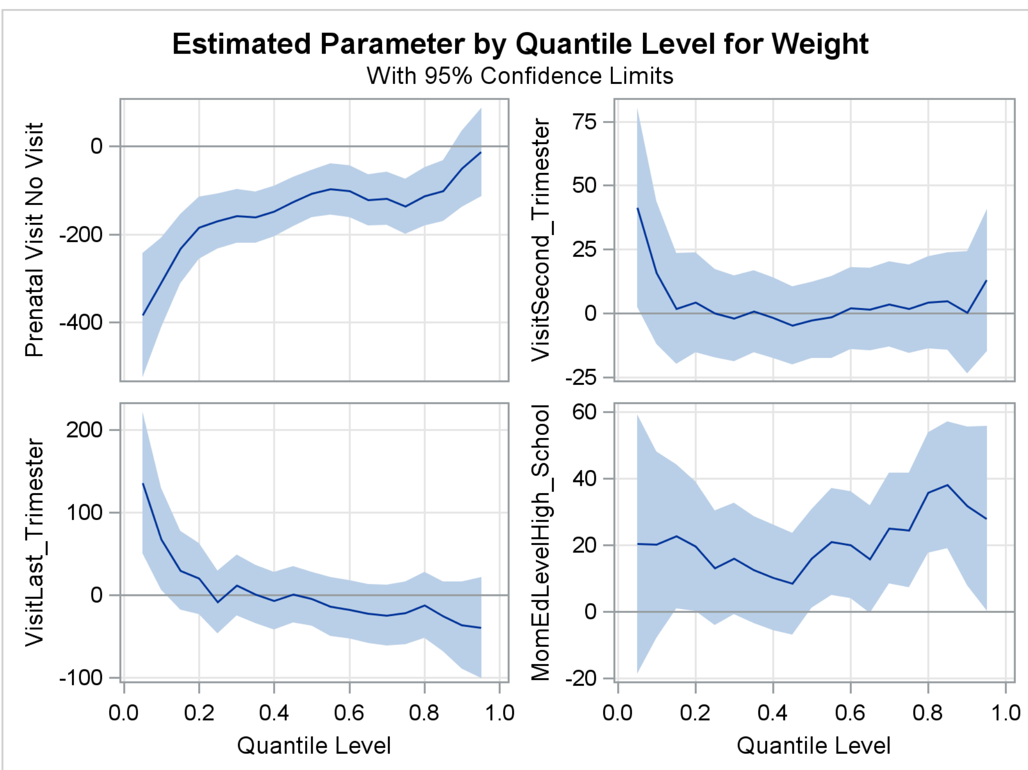
The linear and quadratic effects for the two continuous variables are shown in Output 81.3.6. Both of these variables are centered at their median. At the lower quantiles, the quadratic effect of the mother's age is more concave. The optimal age at the first quantile is about 33, and the optimal age at the third quantile is about 38. The effect of the mother's weight gain is clearly positive, as indicated by the narrow confidence bands for both linear and quadratic coefficients.

For more information about the covariate effects that are discovered by using quantile regression, see Koenker and Hallock (2001).

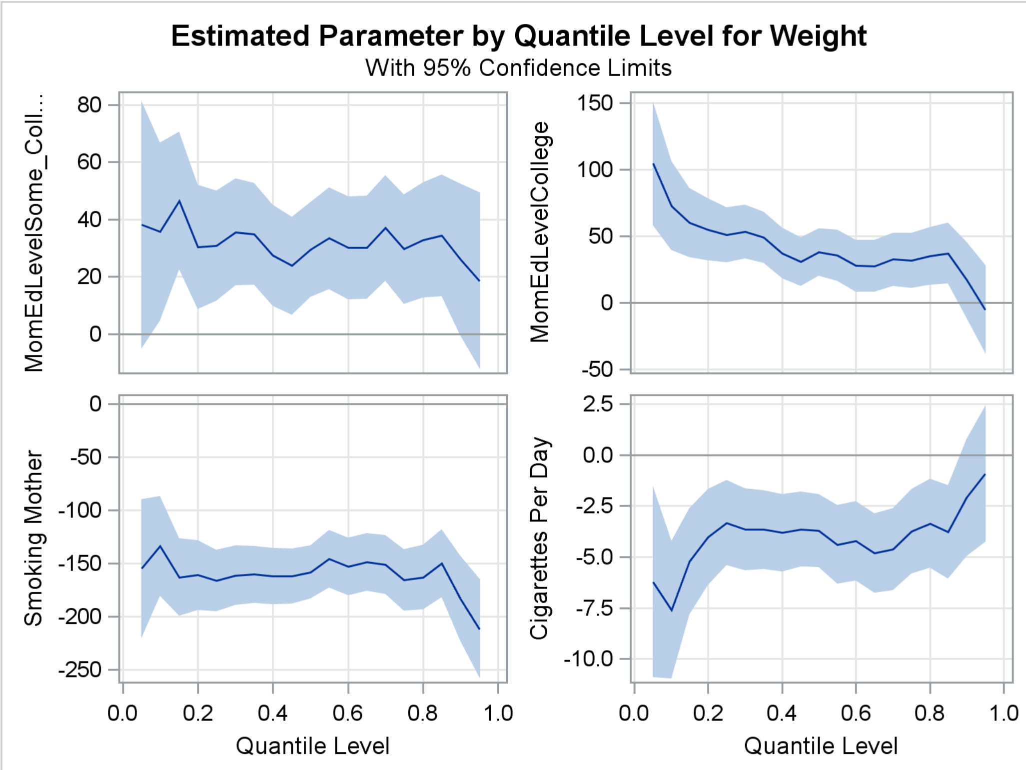
**Output 81.3.3** Quantile Processes with 95% Confidence Bands



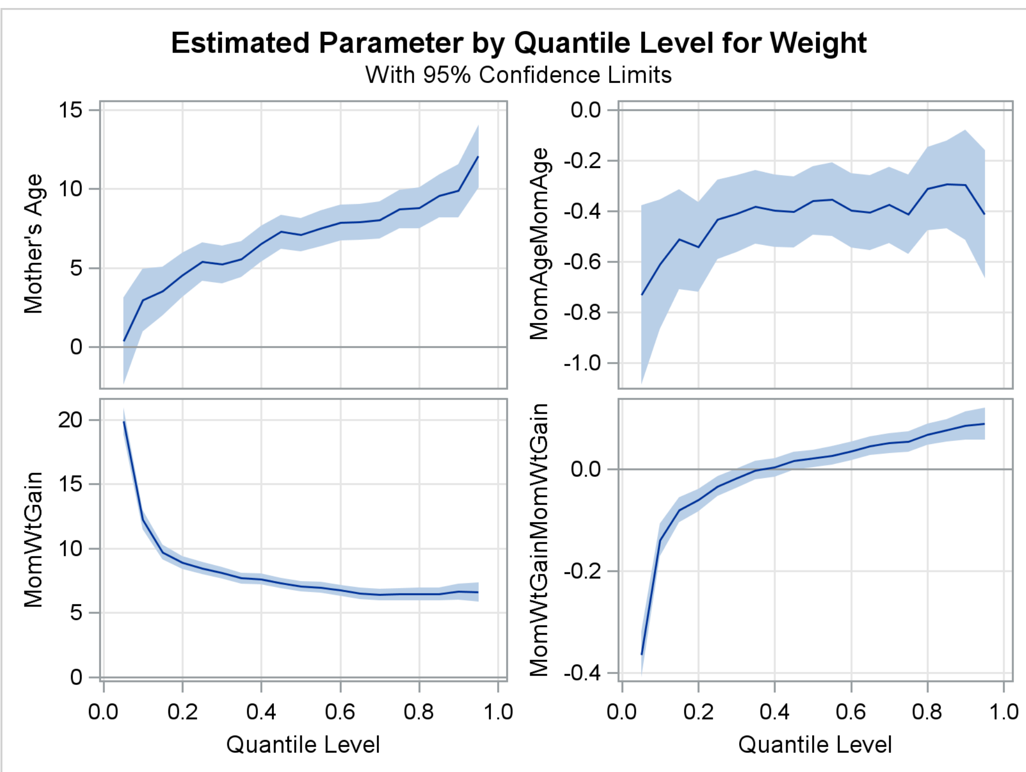

**Output 81.3.4** Quantile Processes with 95% Confidence Bands



**Output 81.3.5** Quantile Processes with 95% Confidence Bands




**Output 81.3.6** Quantile Processes with 95% Confidence Bands



## Example 81.4: Nonparametric Quantile Regression for Ozone Levels

Tracing seasonal trends in the level of tropospheric ozone is essential for predicting high-level periods, observing long-term trends, and discovering potential changes in pollution. Traditional methods for modeling seasonal effects are based on the conditional mean of ozone concentration. However, the upper conditional quantiles are more critical from a public-health perspective. In this example, the QUANTREG procedure fits conditional quantile curves for seasonal effects by using nonparametric quantile regression with cubic B-splines.

The data used here are from Chock, Winkler, and Chen (2000), who studied the association between daily mortality and ambient air pollutant concentrations in Pittsburgh, Pennsylvania. The data set ozone contains the following two variables: Ozone, which represents the daily maximum one-hour ozone concentration (ppm) and Days, which is an index of 1,095 days (3 years).

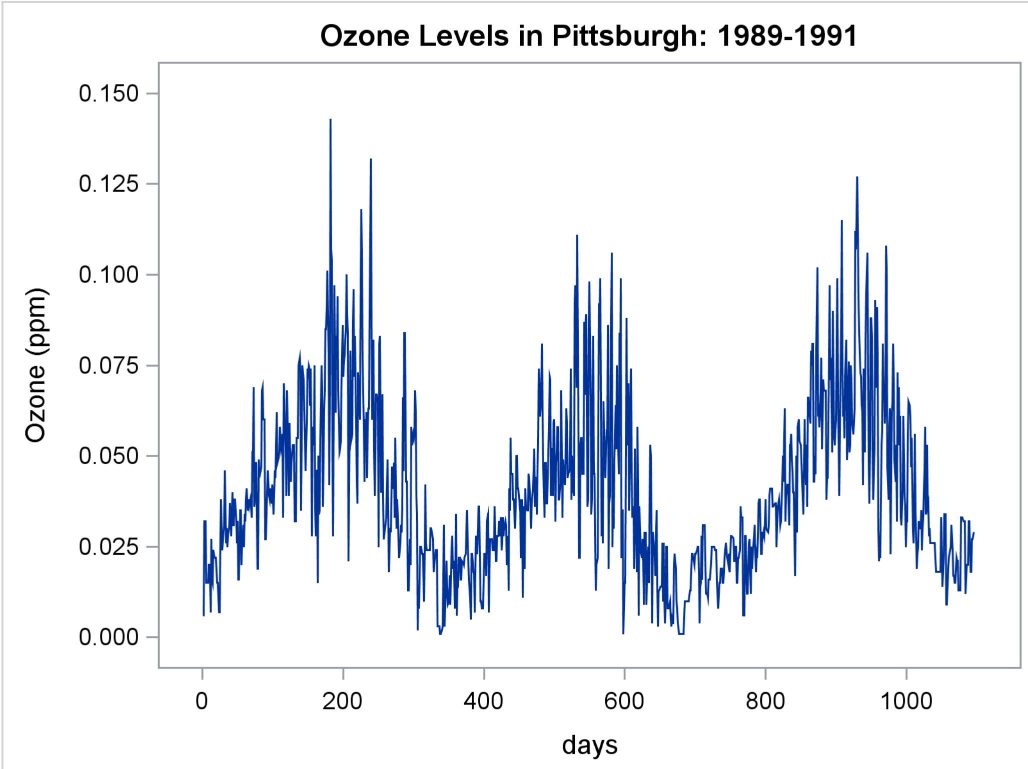
```
data ozone;
   days = _n_;
   input ozone @@;
   datalines;
0.0060 0.0060 0.0320 0.0320 0.0320 0.0150 0.0150 0.0150 0.0200 0.0200
0.0160 0.0070 0.0270 0.0160 0.0150 0.0240 0.0220 0.0220 0.0220 0.0185
0.0150 0.0150 0.0110 0.0070 0.0070 0.0240 0.0380 0.0240 0.0265 0.0290
0.0310 0.0460 0.0360 0.0260 0.0300 0.0250 0.0280 0.0310 0.0370 0.0325

   ... more lines ...

0.0220 0.0210 0.0210 0.0130 0.0130 0.0130 0.0330 0.0330 0.0330 0.0325
0.0320 0.0320 0.0320 0.0120 0.0200 0.0200 0.0200 0.0320 0.0320 0.0250
0.0180 0.0180 0.0270 0.0270 0.0290
;
```

Output 81.4.1, which displays the time series plot of ozone concentration for the three years, shows a clear seasonal pattern.

In this example, cubic B-splines are used to fit the seasonal effect. These splines are generated with 11 knots, which split the 3 years into 12 seasons. The following statements create the spline basis and fit multiple quantile regression spline curves:

```
ods graphics on;

proc quantreg data=ozone algorithm=smooth ci=none plot=fitplot(nodata);
   effect sp = spline( days / knotmethod = list
          (90 182 272 365 455 547 637 730 820 912 1002) );
   model ozone = sp / quantile = 0.5 0.75 0.90 0.95 seed=1268;
run;
```

**Output 81.4.1** Time Series of Ozone Levels in Pittsburgh, Pennsylvania

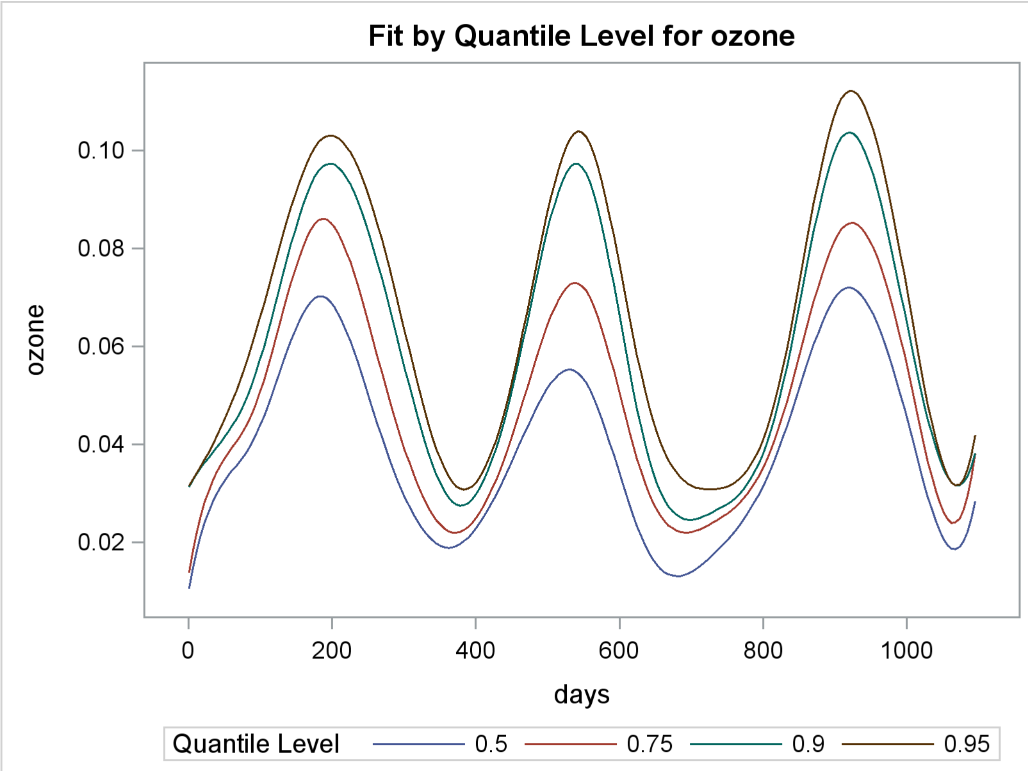



The EFFECT statement creates spline bases for the variable Days. The KNOTMETHOD=LIST option provides all internal knots for these bases. Cubic spline bases are generated by default. These bases are treated as components of the spline effect *sp*, which is specified in the MODEL statement. Spline fits for four quantiles are requested in the QUANTILE= option.

When ODS Graphics is enabled, the QUANTREG procedure automatically generates a fit plot, which includes all fitted curves.

Output 81.4.2 displays these curves. The curves show that peak ozone levels occur in the summer. For the three years 1989–1991, the median curve (labeled 50%) does not cross the 0.08 ppm line, which is the 1997 EPA eight-hour standard. The median curve and the 75% curve show a drop for the ozone concentration levels in 1990. However, for the 90% and 95% curves, peak ozone levels tend to increase. This indicates that there might have been more days with low ozone concentration in 1990, but the top 10% and 5% tend to have higher ozone concentration levels.

**Output 81.4.2** Quantiles of Ozone Levels in Pittsburgh, Pennsylvania




The quantile curves also show that high ozone concentration in 1989 had a longer duration than in 1990 and 1991. This is indicated by the wider spread of the quantile curves in 1989.

## Example 81.5: Quantile Polynomial Regression for Salary Data

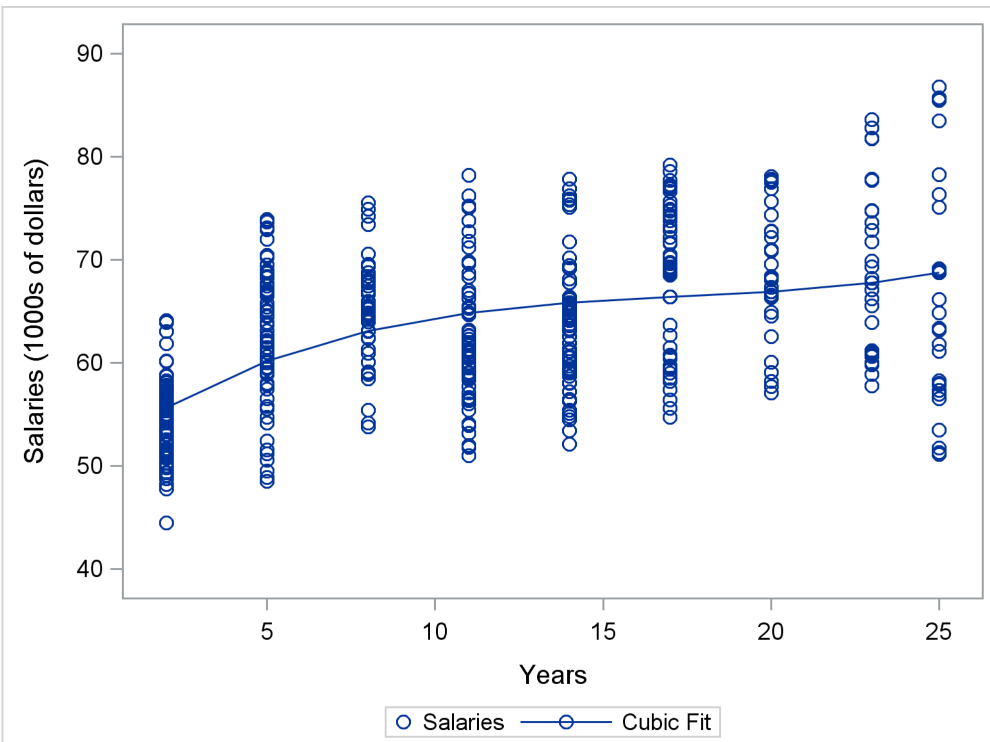
This example uses the data set from a university union survey of salaries of professors in 1991. The survey covered departments in US colleges and universities that list programs in statistics. The goal of this example is to examine the relationship between faculty salaries and years of service.

The data include salaries and years of service for 459 professors. The scatter plot in Output 81.5.1 shows that the relationship is not linear and that a quadratic or cubic regression curve is appropriate. Output 81.5.1 shows a cubic curve.

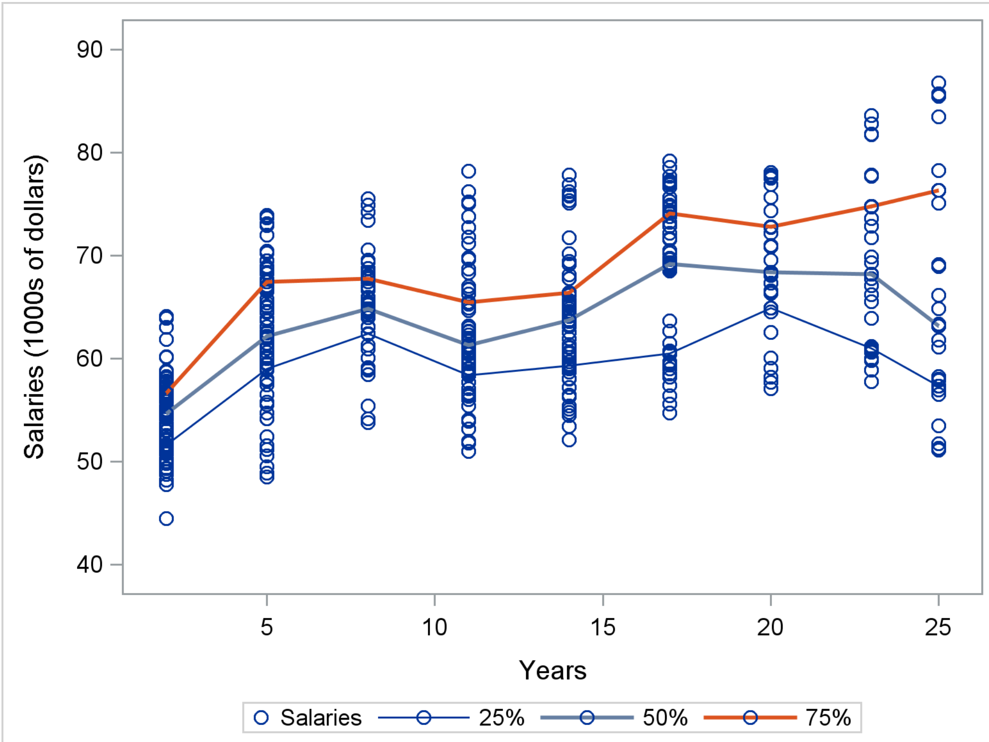
The curve in Output 81.5.1 does not adequately describe the conditional salary distributions and how they change with length of service. Output 81.5.2 shows the 25th, 50th, and 75th percentiles for each number of years, which gives a better picture of the conditional distributions.

```
data salary;
   input Salaries Years @@;
   label Salaries='Salaries (1000s of dollars)';
   datalines;
54.94   2  58.24   2  58.11   2  52.23   2  52.98    2  57.62    2
44.48   2  57.22   2  54.24   2  54.79   2  56.42    2  61.90    2
63.90   2  64.10   2  47.77   2  54.86   2  49.31    2  53.37    2
51.69   2  53.66   2  58.77   2  56.77   2  53.06    2  54.86    2
50.96   2  56.46   2  51.67   2  49.37   2  56.86    2  49.85    2

   ... more lines ...

85.72  25  64.87  25  51.76  25  51.11  25  51.31   25  78.28   25
57.91  25  86.78  25  58.27  25  56.56  25  76.33   25  61.83   25
69.13  25  63.15  25  66.13  25
;
```

**Output 81.5.1** Salary and Years as Professor: Cubic Fit



**Output 81.5.2** Salary and Years as Professor: Sample Quantiles




These descriptive percentiles do not clearly show trends with length of service. The following statements use polynomial quantile regression to obtain a smooth version.

```
ods graphics on;

proc quantreg data=salary ci=sparsity;
   model salaries = years years*years years*years*years
                    /quantile=0.25 0.5 0.75
                    plot=fitplot(showlimits);

   test  years/QINTERACT;

run;
```

The results are shown in Output 81.5.3 and Output 81.5.5. Output 81.5.3 displays the regression coefficients for the three quantiles, from which you can see a difference among the estimated parameters of the variable *years* across the three quantiles. To test whether the difference is significant, you can specify the option QINTERACT in the TEST statement. Output 81.5.4 indicates that the difference is not significant (the *p*-value is greater than 0.05).

**Output 81.5.3** Regression Coefficients

BMI Percentiles for Men: 2-80 Years Old

The QUANTREG Procedure

Parameter Estimates

| Parameter | DF | Estimate | Standard Error | 95% Confidence Limits | | t Value | Pr > \|t\| |
|---|---|---|---|---|---|---|---|
| Intercept | 1 | 48.2509 | 1.3484 | 45.6011 | 50.9007 | 35.78 | <.0001 |
| Years | 1 | 2.2234 | 0.5455 | 1.1514 | 3.2953 | 4.08 | <.0001 |
| Years*Years | 1 | -0.1292 | 0.0500 | -0.2275 | -0.0308 | -2.58 | 0.0101 |
| Years*Years*Years | 1 | 0.0024 | 0.0013 | -0.0001 | 0.0049 | 1.86 | 0.0634 |

Parameter Estimates

| Parameter | DF | Estimate | Standard Error | 95% Confidence Limits | | t Value | Pr > \|t\| |
|---|---|---|---|---|---|---|---|
| Intercept | 1 | 50.2512 | 1.2812 | 47.7334 | 52.7690 | 39.22 | <.0001 |
| Years | 1 | 2.7173 | 0.5947 | 1.5485 | 3.8860 | 4.57 | <.0001 |
| Years*Years | 1 | -0.1632 | 0.0632 | -0.2873 | -0.0390 | -2.58 | 0.0101 |
| Years*Years*Years | 1 | 0.0034 | 0.0018 | -0.0002 | 0.0070 | 1.85 | 0.0647 |

Parameter Estimates

| Parameter | DF | Estimate | Standard Error | 95% Confidence Limits | | t Value | Pr > \|t\| |
|---|---|---|---|---|---|---|---|
| Intercept | 1 | 51.0298 | 1.5886 | 47.9078 | 54.1517 | 32.12 | <.0001 |
| Years | 1 | 3.6513 | 0.7594 | 2.1590 | 5.1436 | 4.81 | <.0001 |
| Years*Years | 1 | -0.2390 | 0.0764 | -0.3892 | -0.0888 | -3.13 | 0.0019 |
| Years*Years*Years | 1 | 0.0055 | 0.0021 | 0.0013 | 0.0096 | 2.60 | 0.0098 |

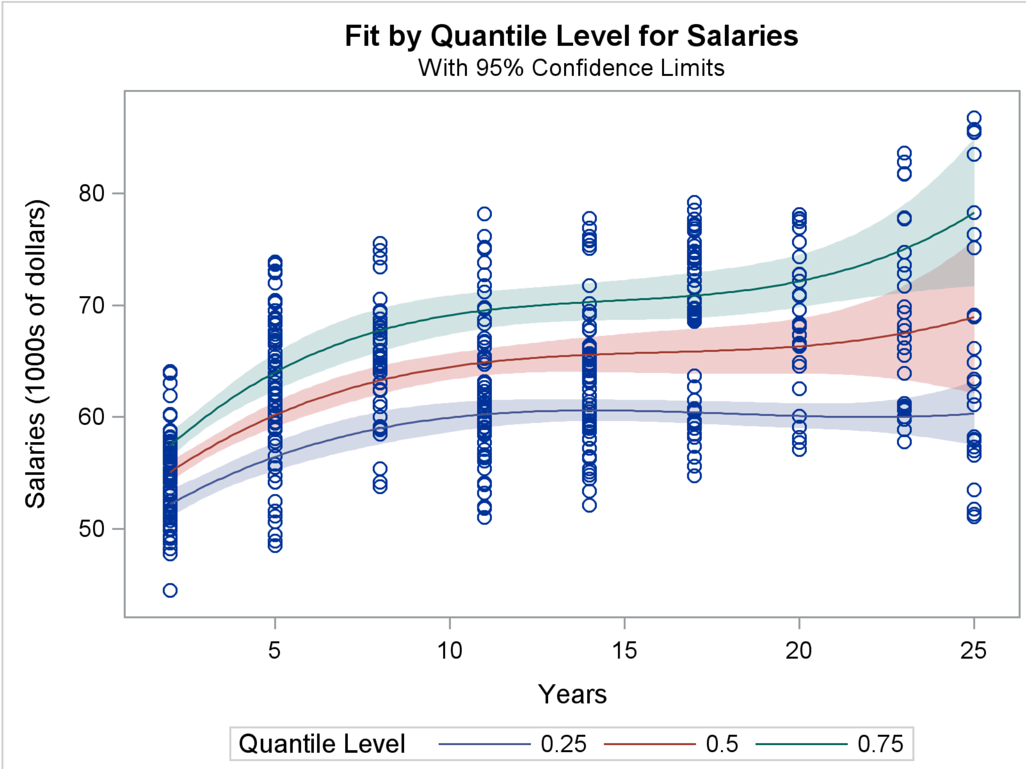
**Output 81.5.4** Tests for Heteroscedasticity

Test Results Equal Coefficients Across Quantiles

| Chi-Square | DF | Pr > ChiSq |
|---|---|---|
| 3.4026 | 2 | 0.1825 |

The three fitted quantile curves and their 95% confidence limits in the Output 81.5.5 clearly show that salary dispersion increases gradually with length of service. After 15 years, a salary more than $70,000 is relatively high, whereas a salary less than $60,000 is relatively low. Percentile curves of this type are useful in medical science as reference curves (Yu, Lu, and Stabder 2003).

**Output 81.5.5** Salary and Years as Professor: Regression Quantiles

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Barro, R. and Lee, J. W. (1994), "Data Set for a Panel of 138 Countries," `http://www.nber.org/pub/barro.lee`, discussion paper, National Bureau of Econometric Research.

Barrodale, I. and Roberts, F. D. K. (1973), "An Improved Algorithm for Discrete $l_1$ Linear Approximation," *SIAM Journal on Numerical Analysis*, 10, 839–848.

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