
# SAS/OR® 15.1 User's Guide
## Mathematical Programming
# The Linear Programming Solver

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**SAS/OR® 15.1 User's Guide: Mathematical Programming**



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# Chapter 7
# The Linear Programming Solver

**Contents**

# Overview: LP Solver

The OPTMODEL procedure provides a framework for specifying and solving linear programs (LPs). A standard linear program has the following formulation:

$$\begin{aligned} \min \quad & \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \; \{\geq, =, \leq\} \; \mathbf{b} \\ & \mathbf{l} \leq \mathbf{x} \leq \mathbf{u} \end{aligned}$$

where

| | | | |
|---|---|---|---|
| $\mathbf{x}$ | $\in$ | $\mathbb{R}^n$ | is the vector of decision variables |
| $\mathbf{A}$ | $\in$ | $\mathbb{R}^{m \times n}$ | is the matrix of constraints |
| $\mathbf{c}$ | $\in$ | $\mathbb{R}^n$ | is the vector of objective function coefficients |
| $\mathbf{b}$ | $\in$ | $\mathbb{R}^m$ | is the vector of constraints' right-hand sides (RHS) |
| $\mathbf{l}$ | $\in$ | $\mathbb{R}^n$ | is the vector of lower bounds on variables |
| $\mathbf{u}$ | $\in$ | $\mathbb{R}^n$ | is the vector of upper bounds on variables |

The following LP algorithms are available in the OPTMODEL procedure:

- primal simplex algorithm
- dual simplex algorithm
- network simplex algorithm
- interior point algorithm

The primal and dual simplex algorithms implement the two-phase simplex method. In phase I, the algorithm tries to find a feasible solution. If no feasible solution is found, the LP is infeasible; otherwise, the algorithm enters phase II to solve the original LP. The network simplex algorithm extracts a network substructure, solves this using network simplex, and then constructs an advanced basis to feed to either primal or dual simplex. The interior point algorithm implements a primal-dual predictor-corrector interior point algorithm. If any of the decision variables are constrained to be integer-valued, then the relaxed version of the problem is solved.

# Getting Started: LP Solver

The following example illustrates how you can use the OPTMODEL procedure to solve linear programs. Suppose you want to solve the following problem:

$$\begin{array}{rlllllll} \max & x_1 & + & x_2 & + & x_3 & & \\ \text{subject to} & 3x_1 & + & 2x_2 & - & x_3 & \leq & 1 \\ & -2x_1 & - & 3x_2 & + & 2x_3 & \leq & 1 \\ & & & x_1, & & x_2, \; x_3 & \geq & 0 \end{array}$$

You can use the following statements to call the OPTMODEL procedure for solving linear programs:

```
proc optmodel;
   var x{i in 1..3} >= 0;
   max f =    x[1] +   x[2] +   x[3];
   con c1:  3*x[1] + 2*x[2] -   x[3] <= 1;
   con c2: -2*x[1] - 3*x[2] + 2*x[3] <= 1;
   solve with lp / algorithm = ps presolver = none logfreq = 1;
   print x;
quit;
```

The optimal solution and the optimal objective value are displayed in Figure 7.1.

**Figure 7.1** Solution Summary

**The OPTMODEL Procedure**

| Problem Summary | |
|---|---|
| **Objective Sense** | Maximization |
| **Objective Function** | f |
| **Objective Type** | Linear |
| **Number of Variables** | 3 |
| **Bounded Above** | 0 |
| **Bounded Below** | 3 |
| **Bounded Below and Above** | 0 |
| **Free** | 0 |
| **Fixed** | 0 |
| **Number of Constraints** | 2 |
| **Linear LE (<=)** | 2 |
| **Linear EQ (=)** | 0 |
| **Linear GE (>=)** | 0 |
| **Linear Range** | 0 |
| **Constraint Coefficients** | 6 |

| Solution Summary | |
|---|---|
| **Solver** | LP |
| **Algorithm** | Primal Simplex |
| **Objective Function** | f |
| **Solution Status** | Optimal |
| **Objective Value** | 8 |
| **Primal Infeasibility** | 0 |
| **Dual Infeasibility** | 1.776357E-15 |
| **Bound Infeasibility** | 0 |
| **Iterations** | 4 |
| **Presolve Time** | 0.00 |
| **Solution Time** | 0.00 |

**Figure 7.1** *continued*

| [1] | x |
|---|---|
| **1** | 0 |
| **2** | 3 |
| **3** | 5 |

The iteration log displaying problem statistics, progress of the solution, and the optimal objective value is shown in Figure 7.2.

**Figure 7.2** Log

```
NOTE: Problem generation will use 16 threads.
NOTE: The problem has 3 variables (0 free, 0 fixed).
NOTE: The problem has 2 linear constraints (2 LE, 0 EQ, 0 GE, 0 range).
NOTE: The problem has 6 linear constraint coefficients.
NOTE: The problem has 0 nonlinear constraints (0 LE, 0 EQ, 0 GE, 0 range).
NOTE: The LP presolver value NONE is applied.
NOTE: The LP solver is called.
NOTE: The Primal Simplex algorithm is used.
                          Objective                Entering      Leaving
      Phase Iteration       Value         Time     Variable      Variable
       P 2          1    0.000000E+00         0       x[1]            c1 (S)
       P 2          2    3.333333E-01         0       x[3]            c2 (S)
       P 2          3    2.000000E+00         0       x[2]          x[1]
       P 2          4    8.000000E+00         0
NOTE: Optimal.
NOTE: Objective = 8.
NOTE: The Primal Simplex solve time is 0.00 seconds.
```

# Syntax: LP Solver

The following statement is available in the OPTMODEL procedure:

**SOLVE WITH LP** < / *options* > **;**

## Functional Summary

Table 7.1 summarizes the list of options available for the SOLVE WITH LP statement, classified by function.

**Table 7.1** Options for the LP Solver

| Description | Option |
|---|---|
| **Solver Options** | |
| Specifies the type of algorithm | ALGORITHM= |
| Specifies the type of algorithm called after network simplex | ALGORITHM2= |
| Enables or disables IIS detection | IIS= |
| **Presolve Option** | |
| Specifies the type of presolve | PRESOLVER= |
| Controls the dualization of the problem | DUALIZE= |
| **Control Options** | |
| Specifies the feasibility tolerance | FEASTOL= |
| Specifies the frequency of printing solution progress | LOGFREQ= |
| Specifies the detail of solution progress printed in log | LOGLEVEL= |
| Specifies the maximum number of iterations | MAXITER= |
| Specifies the time limit for the optimization process | MAXTIME= |
| Specifies the optimality tolerance | OPTTOL= |
| Specifies units of CPU time or real time | TIMETYPE= |
| **Simplex Algorithm Options** | |
| Specifies the type of initial basis | BASIS= |
| Specifies the type of pricing strategy | PRICETYPE= |
| Specifies the queue size for determining entering variable | QUEUESIZE= |
| Enables or disables scaling of the problem | SCALE= |
| Specifies the initial seed for the random number generator | SEED= |
| **Interior Point Algorithm Options** | |
| Enables or disables interior crossover | CROSSOVER= |
| Specifies the stopping criterion based on duality gap | DUALITYGAP= |
| **Decomposition Algorithm Options** | |
| Enables decomposition algorithm and specifies general control options | DECOMP=() |
| Specifies options for the master problem | DECOMPMASTER=() |
| Specifies options for the subproblem | DECOMPSUBPROB=() |
| **Parallel Options** | |
| Enables the LP solver to run deterministically | DETERMINISTIC= |
| Specifies the number of threads for the parallel LP solver to use | NTHREADS= |

## LP Solver Options

This section describes the options recognized by the LP solver. These options can be specified after a forward slash (/) in the SOLVE statement, provided that the LP solver is explicitly specified using a WITH clause.

If the LP solver terminates before reaching an optimal solution, an intermediate solution is available. You can access this solution by using the .sol variable suffix in the OPTMODEL procedure. See the section "Suffixes" on page 135 for details.

## Solver Options

**ALGORITHM=***option*
**SOLVER=***option*
**SOL=***option*
specifies an LP algorithm. You can specify the following *options*:

| | |
|---|---|
| **PRIMAL (PS)** | uses the primal simplex algorithm. |
| **DUAL (DS)** | uses the dual simplex algorithm. |
| **NETWORK (NS)** | uses the network simplex algorithm. |
| **INTERIORPOINT (IP)** | uses the interior point algorithm. |
| **CONCURRENT (CON)** | uses several different algorithms in parallel. |

The valid abbreviated value for each option is indicated in parentheses. By default, ALGORITHM=DUAL.

**ALGORITHM2=***option*
**SOLVER2=***option*
specifies an LP algorithm if ALGORITHM=NS. You can specify the following *options*:

| | |
|---|---|
| **PRIMAL (PS)** | uses the primal simplex algorithm (after network simplex). |
| **DUAL (DS)** | uses the dual simplex algorithm (after network simplex). |

The valid abbreviated value for each option is indicated in parentheses. By default, the LP solver decides which algorithm is best to use after calling the network simplex algorithm on the extracted network.

**IIS=FALSE | TRUE**
specifies whether the LP solver attempts to identify a set of constraints and variables that form an irreducible infeasible set (IIS). You can specify the following values:

| | |
|---|---|
| **FALSE** | disables IIS detection. |
| **TRUE** | enables IIS detection. |

If an IIS is found, information about the infeasibilities can be found in the .status values of the constraints and variables. By default, IIS=FALSE. For more information about the IIS= option, see the section "Irreducible Infeasible Set" on page 275. For more information about the .status suffix, see the section "Suffixes" on page 135.

## Presolve Options

**DUALIZE=AUTOMATIC | OFF | ON**
controls the dualization of the problem. You can specify the following values:

**AUTOMATIC** the presolver uses a heuristic to decide whether to dualize the problem or not.

**OFF** disables dualization. The optimization problem is solved in the form that you specify.

**ON** the presolver formulates the dual of the linear optimization problem.

Dualization is usually helpful for problems that have many more constraints than variables. You can use this option with all simplex algorithms in the SOLVE WITH LP statement, but it is most effective with the primal and dual simplex algorithms.

By default, DUALIZE=AUTOMATIC.

**PRESOLVER=AUTOMATIC | NONE | BASIC | MODERATE | AGGRESSIVE**
specifies the presolve option. You can specify the following values:

**AUTOMATIC** applies the presolver by using default settings.

**NONE** disables the presolver.

**BASIC** performs a minimal presolve, such as removing empty rows, columns, and fixed variables.

**MODERATE** performs a basic presolve and applies other inexpensive presolve techniques.

**AGGRESSIVE** performs a moderate presolve and applies other aggressive (but computationally expensive) presolve techniques.

By default, PRESOLVER=AUTOMATIC. For more information, see the section "Presolve" on page 267.

## Control Options

**FEASTOL=$\epsilon$**
specifies the feasibility tolerance, $\epsilon \in$[1E–9, 1E–4], for determining the feasibility of a variable. The default value is 1E–6. To compute the feasibility tolerance, simplex algorithms use the absolute error, and interior point algorithms use the relative error.

**LOGFREQ=*k***

**PRINTFREQ=*k***
specifies that the printing of the solution progress to the iteration log is to occur after every $k$ iterations. The print frequency, $k$, is an integer between zero and the largest four-byte signed integer, which is $2^{31} - 1$.

The value $k = 0$ disables the printing of the progress of the solution. If the primal or dual simplex algorithms are used, the default value of this option is determined dynamically according to the problem size. If the network simplex algorithm is used, the default value of this option is 10,000. If the interior point algorithm is used, the default value of this option is 1.

**LOGLEVEL=*string***
controls the amount of information displayed in the SAS log by the LP solver, from a short description of presolve information and summary to details at each iteration. The valid values for this option are described below.

**NONE** turns off all solver-related messages to SAS log.

**BASIC** displays a solver summary after stopping.

**MODERATE** prints a solver summary and an iteration log by using the interval dictated by the LOGFREQ= option.

**AGGRESSIVE** prints a detailed solver summary and an iteration log by using the interval dictated by the LOGFREQ= option.

The default value is MODERATE.

**MAXITER=***k*

specifies the maximum number of iterations. The value *k* can be any integer between one and the largest four-byte signed integer, which is $2^{31} - 1$. If you do not specify this option, the procedure does not stop based on the number of iterations performed. For network simplex, this iteration limit corresponds to the algorithm called after network simplex (either primal or dual simplex).

**MAXTIME=***t*

specifies an upper limit of *t* units of time for the optimization process, including problem generation time and solution time. The value of the TIMETYPE= option determines the type of units used. If you do not specify the MAXTIME= option, the solver does not stop based on the amount of time elapsed. The value of *t* can be any positive number; the default value is the positive number that has the largest absolute value that can be represented in your operating environment.

**OPTTOL=**$\epsilon$

specifies the optimality tolerance, $\epsilon \in$ [1E–9, 1E–4], for declaring optimality. The default value is 1E–6. Simplex algorithms use the absolute error and interior point algorithms use the relative error for the computation of optimality tolerance.

**TIMETYPE=CPU | REAL**

specifies the units of time used by the MAXTIME= option and reported by the PRESOLVE_TIME and SOLUTION_TIME terms in the _OROPTMODEL_ macro variable. The valid values of the TIMETYPE= option are described below.

**CPU** specifies units of CPU time.

**REAL** specifies units of real time.

The "Optimization Statistics" table, an output of the OPTMODEL procedure if you specify PRINTLEVEL=2 in the PROC OPTMODEL statement, also includes the same time units for Presolver Time and Solver Time. The other times (such as Problem Generation Time) in the "Optimization Statistics" table are also in the same units.

By default, TIMETYPE=REAL.

## Simplex Algorithm Options

**BASIS=CRASH | SLACK | WARMSTART**

specifies the options for generating an initial basis. You can specify the following values:

**CRASH** generates an initial basis by using crash techniques (Maros 2003). The procedure creates a triangular basic matrix consisting of both decision variables and slack variables.

**SLACK** generates an initial basis by using all slack variables.

**WARMSTART** starts the primal and dual simplex algorithms with the available basis.

The default option is determined automatically based on the problem structure. For the network simplex algorithm, this option has no effect.

**PRICETYPE=HYBRID | PARTIAL | FULL | DEVEX | STEEPESTEDGE**
specifies the pricing strategy for the primal and dual simplex algorithms. You can specify the following values:

**HYBRID** uses hybrid Devex and steepest-edge pricing strategies. This strategy is available for the primal simplex algorithm only.

**PARTIAL** uses partial pricing strategy. Optionally, you can specify QUEUESIZE=. This strategy is available for the primal simplex algorithm only.

**FULL** uses the most negative reduced-cost pricing strategy.

**DEVEX** uses the Devex pricing strategy.

**STEEPESTEDGE** uses the steepest-edge pricing strategy.

The default option is determined automatically based on the problem structure. For the network simplex algorithm, this option applies only to the algorithm specified by the ALGORITHM2= option. For more information, see the section "Pricing Strategies for the Primal and Dual Simplex Algorithms" on page 267.

**QUEUESIZE=***k*
specifies the queue size, $k \in [1, n]$, where $n$ is the number of decision variables. This queue is used for finding an entering variable in the simplex iteration. The default value is chosen adaptively based on the number of decision variables. This option is used only when PRICETYPE=PARTIAL.

**SCALE=NONE | AUTOMATIC**
specifies the scaling option. You can specify the following values:

**NONE** disables scaling.

**AUTOMATIC** automatically applies scaling procedure if necessary.

By default, SCALE=AUTOMATIC.

**SEED=***number*
specifies the initial seed for the random number generator. Because the seed affects the perturbation in the simplex algorithms, the result might be a different optimal solution and a different solver path, but the effect is usually negligible. The value of *number* can be any positive integer up to the largest four-byte signed integer, which is $2^{31} - 1$. By default, SEED=100.

## Interior Point Algorithm Options

**CROSSOVER=FALSE | TRUE**
specifies whether to convert the interior point solution to a basic simplex solution. You can specify the following values:

| | |
|---|---|
| **FALSE** | disables crossover. |
| **TRUE** | applies the crossover algorithm to the interior point solution. |

If the interior point algorithm terminates with a solution, the crossover algorithm uses the interior point solution to create an initial basic solution. After performing primal fixing and dual fixing, the crossover algorithm calls a simplex algorithm to locate an optimal basic solution. By default, CROSSOVER=TRUE.

**DUALITYGAP=$\delta$**
specifies the desired relative duality gap, $\delta \in$[1E–9, 1E–4]. This is the relative difference between the primal and dual objective function values and is the primary solution quality parameter. The default value is 1E–6. For more information, see the section "The Interior Point Algorithm" on page 268.

## Parallel Options

**DETERMINISTIC=TRUE | FALSE**
specifies whether to run the LP solver in deterministic parallel or nondeterministic parallel mode. You can specify the following values:

| | |
|---|---|
| **TRUE** | runs in deterministic parallel mode. |
| **FALSE** | runs in nondeterministic parallel mode. |

By default, DETERMINISTIC=TRUE.

**NTHREADS=*number***
specifies the maximum number of threads for the LP solver to use for multithreaded processing. The value of *number* can be any integer between 1 and 256, inclusive. The default is the value of the NTHREADS= option in PROC OPTMODEL.

## Decomposition Algorithm Options

The following options are available for the decomposition algorithm in the LP solver. For information about the decomposition algorithm, see Chapter 16, "The Decomposition Algorithm."

**DECOMP=(*options*)**
enables the decomposition algorithm and specifies overall control options for the algorithm. For more information about this option, see Chapter 16, "The Decomposition Algorithm."

**DECOMPMASTER=(*options*)**
specifies options for the master problem. For more information about this option, see Chapter 16, "The Decomposition Algorithm."

**DECOMPSUBPROB=(*options*)**
specifies option for the subproblem. For more information about this option, see Chapter 16, "The Decomposition Algorithm."

# Details: LP Solver

## Presolve

Presolve in the simplex LP algorithms of PROC OPTMODEL uses a variety of techniques to reduce the problem size, improve numerical stability, and detect infeasibility or unboundedness (Andersen and Andersen 1995; Gondzio 1997). During presolve, redundant constraints and variables are identified and removed. Presolve can further reduce the problem size by substituting variables. Variable substitution is a very effective technique, but it might occasionally increase the number of nonzero entries in the constraint matrix.

In most cases, using presolve is very helpful in reducing solution times. You can enable presolve at different levels or disable it by specifying the PRESOLVER= option.

## Pricing Strategies for the Primal and Dual Simplex Algorithms

Several pricing strategies for the primal and dual simplex algorithms are available. Pricing strategies determine which variable enters the basis at each simplex pivot. These can be controlled by specifying the PRICETYPE= option.

The primal simplex algorithm has the following five pricing strategies:

| | |
|---|---|
| PARTIAL | scans a queue of decision variables to find an entering variable. You can optionally specify the QUEUESIZE= option to control the length of this queue. |
| FULL | uses Dantzig's most violated reduced cost rule (Dantzig 1963). It compares the reduced cost of all decision variables, and selects the variable with the most violated reduced cost as the entering variable. |
| DEVEX | implements the Devex pricing strategy developed by Harris (1973). |
| STEEPESTEDGE | uses the steepest-edge pricing strategy developed by Forrest and Goldfarb (1992). |
| HYBRID | uses a hybrid of the Devex and steepest-edge pricing strategies. |

The dual simplex algorithm has only three pricing strategies available: FULL, DEVEX, and STEEPEST-EDGE.

## The Network Simplex Algorithm

The network simplex algorithm in PROC OPTMODEL attempts to leverage the speed of the network simplex algorithm to more efficiently solve linear programs by using the following process:

1. It heuristically extracts the largest possible network substructure from the original problem.
2. It uses the network simplex algorithm to solve for an optimal solution to this substructure.
3. It uses this solution to construct an advanced basis to warm-start either the primal or dual simplex algorithm on the original linear programming problem.

The network simplex algorithm is a specialized version of the simplex algorithm that uses spanning-tree bases to more efficiently solve linear programming problems that have a pure network form. Such LPs can be modeled using a formulation over a directed graph, as a minimum-cost flow problem. Let $G = (N, A)$ be a directed graph, where $N$ denotes the nodes and $A$ denotes the arcs of the graph. The decision variable $x_{ij}$ denotes the amount of flow sent from node $i$ to node $j$. The cost per unit of flow on the arcs is designated by $c_{ij}$, and the amount of flow sent across each arc is bounded to be within $[l_{ij}, u_{ij}]$. The demand (or supply) at each node is designated as $b_i$, where $b_i > 0$ denotes a supply node and $b_i < 0$ denotes a demand node. The corresponding linear programming problem is as follows:

$$\begin{array}{rlcll} \min & \sum_{(i,j)\in A} c_{ij} x_{ij} & & & \\ \text{subject to} & \sum_{(i,j)\in A} x_{ij} - \sum_{(j,i)\in A} x_{ji} & = & b_i & \forall i \in N \\ & x_{ij} & \leq & u_{ij} & \forall (i,j) \in A \\ & x_{ij} & \geq & l_{ij} & \forall (i,j) \in A. \end{array}$$

The network simplex algorithm used in PROC OPTMODEL is the primal network simplex algorithm. This algorithm finds the optimal primal feasible solution and a dual solution that satisfies complementary slackness. Sometimes the directed graph $G$ is disconnected. In this case, the problem can be decomposed into its weakly connected components, and each minimum-cost flow problem can be solved separately. After solving each component, the optimal basis for the network substructure is augmented with the non-network variables and constraints from the original problem. This advanced basis is then used as a starting point for the primal or dual simplex method. The solver automatically selects the algorithm to use after network simplex. However, you can override this selection with the ALGORITHM2= option.

The network simplex algorithm can be more efficient than the other algorithms on problems that have a large network substructure. The size of this network structure can be seen in the log.

## The Interior Point Algorithm

The interior point LP algorithm in PROC OPTMODEL implements an infeasible primal-dual predictor-corrector interior point algorithm. To illustrate the algorithm and the concepts of duality and dual infeasibility, consider the following LP formulation (the primal):

$$\begin{array}{rl} \min & \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} \geq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

The corresponding dual is as follows:

$$\begin{array}{rlcll} \max & \mathbf{b}^{\mathrm{T}}\mathbf{y} & & & \\ \text{subject to} & \mathbf{A}^{\mathrm{T}}\mathbf{y} + \mathbf{w} & = & \mathbf{c} & \\ & \mathbf{y} & \geq & \mathbf{0} & \\ & \mathbf{w} & \geq & \mathbf{0} & \end{array}$$

where $\mathbf{y} \in \mathbb{R}^m$ refers to the vector of dual variables and $\mathbf{w} \in \mathbb{R}^n$ refers to the vector of dual slack variables.

The dual makes an important contribution to the certificate of optimality for the primal. The primal and dual constraints combined with complementarity conditions define the first-order optimality conditions, also known as KKT (Karush-Kuhn-Tucker) conditions, which can be stated as follows:

$$\begin{array}{rcll} \mathbf{Ax} - \mathbf{s} & = & \mathbf{b} & (\text{Primal Feasibility}) \\ \mathbf{A}^{\mathrm{T}}\mathbf{y} + \mathbf{w} & = & \mathbf{c} & (\text{Dual Feasibility}) \\ \mathbf{WXe} & = & \mathbf{0} & (\text{Complementarity}) \\ \mathbf{SYe} & = & \mathbf{0} & (\text{Complementarity}) \\ \mathbf{x},\ \mathbf{y},\ \mathbf{w},\ \mathbf{s} & \geq & \mathbf{0} & \end{array}$$

where $\mathbf{e} \equiv (1, \ldots, 1)^{\mathrm{T}}$ of appropriate dimension and $\mathbf{s} \in \mathbb{R}^m$ is the vector of primal *slack* variables.

**NOTE:** Slack variables (the *s* vector) are automatically introduced by the algorithm when necessary; it is therefore recommended that you not introduce any slack variables explicitly. This enables the algorithm to handle slack variables much more efficiently.

The letters **X**, **Y**, **W**, and **S** denote matrices with corresponding *x*, *y*, *w*, and *s* on the main diagonal and zero elsewhere, as in the following example:

$$\mathbf{X} \equiv \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_n \end{bmatrix}$$

If $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{w}^*, \mathbf{s}^*)$ is a solution of the previously defined system of equations representing the KKT conditions, then $\mathbf{x}^*$ is also an optimal solution to the original LP model.

At each iteration the interior point algorithm solves a large, sparse system of linear equations as follows:

$$\begin{bmatrix} \mathbf{Y}^{-1}\mathbf{S} & \mathbf{A} \\ \mathbf{A}^{\mathrm{T}} & -\mathbf{X}^{-1}\mathbf{W} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{y} \\ \Delta\mathbf{x} \end{bmatrix} = \begin{bmatrix} \Xi \\ \Theta \end{bmatrix}$$

where $\Delta\mathbf{x}$ and $\Delta\mathbf{y}$ denote the vector of *search directions* in the primal and dual spaces, respectively; $\Theta$ and $\Xi$ constitute the vector of the right-hand sides.

The preceding system is known as the reduced KKT system. The interior point algorithm uses a preconditioned quasi-minimum residual algorithm to solve this system of equations efficiently.

An important feature of the interior point algorithm is that it takes full advantage of the sparsity in the constraint matrix, thereby enabling it to efficiently solve large-scale linear programs.

The interior point algorithm works simultaneously in the primal and dual spaces. It attains optimality when both primal and dual feasibility are achieved and when complementarity conditions hold. Therefore it is of interest to observe the following four measures:

- Relative primal infeasibility measure $\alpha$:

$$\alpha = \frac{\|\mathbf{Ax} - \mathbf{b} - \mathbf{s}\|_2}{\|\mathbf{b}\|_2 + 1}$$

- Relative dual infeasibility measure $\beta$:

$$\beta = \frac{\|\mathbf{c} - \mathbf{A}^{\mathrm{T}}\mathbf{y} - \mathbf{w}\|_2}{\|\mathbf{c}\|_2 + 1}$$

- Relative duality gap $\delta$:

$$\delta = \frac{|\mathbf{c}^{\mathrm{T}}\mathbf{x} - \mathbf{b}^{\mathrm{T}}\mathbf{y}|}{|\mathbf{c}^{\mathrm{T}}\mathbf{x}| + 1}$$

- Absolute complementarity $\gamma$:

$$\gamma = \sum_{i=1}^{n} x_i w_i + \sum_{i=1}^{m} y_i s_i$$

where $\|v\|_2$ is the Euclidean norm of the vector $v$. These measures are displayed in the iteration log.

## Iteration Log for the Primal and Dual Simplex Algorithms

The primal and dual simplex algorithms implement a two-phase simplex algorithm. Phase I finds a feasible solution, which phase II improves to an optimal solution.

When LOGFREQ=1, the following information is printed in the iteration log:

| | |
|---|---|
| Algorithm | indicates which simplex method is running by printing the letter P (primal) or D (dual). |
| Phase | indicates whether the algorithm is in phase I or phase II of the simplex method. |
| Iteration | indicates the iteration number. |
| Objective Value | indicates the current amount of infeasibility in phase I and the primal objective value of the current solution in phase II. |
| Time | indicates the time elapsed (in seconds). |
| Entering Variable | indicates the entering pivot variable. A slack variable that enters the basis is indicated by the corresponding row name followed by "(S)". If the entering nonbasic variable has distinct and finite lower and upper bounds, then a "bound swap" can take place in the primal simplex method. |
| Leaving Variable | indicates the leaving pivot variable. A slack variable that leaves the basis is indicated by the corresponding row name followed by "(S)". The leaving variable is the same as the entering variable if a bound swap has taken place. |

When you omit the LOGFREQ= option or specify a value larger than 1, only the algorithm, phase, iteration, objective value, and time information is printed in the iteration log.

The behavior of objective values in the iteration log depends on both the current phase and the chosen algorithm. In phase I, both simplex methods have artificial objective values that decrease to 0 when a feasible solution is found. For the dual simplex method, phase II maintains a dual feasible solution, so a minimization problem has increasing objective values in the iteration log. For the primal simplex method,

phase II maintains a primal feasible solution, so a minimization problem has decreasing objective values in the iteration log.

During the solution process, some elements of the LP model might be perturbed to improve performance. In this case the objective values that are printed correspond to the perturbed problem. After reaching optimality for the perturbed problem, the LP solver solves the original problem by switching from the primal simplex method to the dual simplex method (or from the dual simplex method to the primal simplex method). Because the problem might be perturbed again, this process can result in several changes between the two algorithms.

## Iteration Log for the Network Simplex Algorithm

After finding the embedded network and formulating the appropriate relaxation, the network simplex algorithm uses a primal network simplex algorithm. In the case of a connected network, with one (weakly connected) component, the log will show the progress of the simplex algorithm. The following information is displayed in the iteration log:

| | |
|---|---|
| Iteration | indicates the iteration number. |
| PrimalObj | indicates the primal objective value of the current solution. |
| Primal Infeas | indicates the maximum primal infeasibility of the current solution. |
| Time | indicates the time spent on the current component by network simplex. |

The frequency of the simplex iteration log is controlled by the LOGFREQ= option. The default value of the LOGFREQ= option is 10,000.

If the network relaxation is disconnected, the information in the iteration log shows progress at the component level. The following information is displayed in the iteration log:

| | |
|---|---|
| Component | indicates the component number being processed. |
| Nodes | indicates the number of nodes in this component. |
| Arcs | indicates the number of arcs in this component. |
| Iterations | indicates the number of simplex iterations needed to solve this component. |
| Time | indicates the time spent so far in network simplex. |

The frequency of the component iteration log is controlled by the LOGFREQ= option. In this case, the default value of the LOGFREQ= option is determined by the size of the network.

The LOGLEVEL= option adjusts the amount of detail shown. By default, LOGLEVEL=MODERATE and reports as in the preceding description. If LOGLEVEL=NONE, no information is shown. If LOGLEVEL=BASIC, the only information shown is a summary of the network relaxation and the time spent solving the relaxation. If LOGLEVEL=AGGRESSIVE, in the case of one component, the log displays as in the preceding description; in the case of multiple components, for each component, a separate simplex iteration log is displayed.

## Iteration Log for the Interior Point Algorithm

The interior point algorithm implements an infeasible primal-dual predictor-corrector interior point algorithm. The following information is displayed in the iteration log:

| | |
|---|---|
| Iter | indicates the iteration number |
| Complement | indicates the (absolute) complementarity |
| Duality Gap | indicates the (relative) duality gap |
| Primal Infeas | indicates the (relative) primal infeasibility measure |
| Bound Infeas | indicates the (relative) bound infeasibility measure |
| Dual Infeas | indicates the (relative) dual infeasibility measure |
| Time | indicates the time elapsed (in seconds). |

If the sequence of solutions converges to an optimal solution of the problem, you should see all columns in the iteration log converge to zero or very close to zero. If they do not, it can be the result of insufficient iterations being performed to reach optimality. In this case, you might need to increase the value specified in the option MAXITER= or MAXTIME=. If the complementarity and/or the duality gap do not converge, the problem might be infeasible or unbounded. If the infeasibility columns do not converge, the problem might be infeasible.

## Iteration Log for the Crossover Algorithm

The crossover algorithm takes an optimal solution from the interior point algorithm and transforms it into an optimal basic solution. The iterations of the crossover algorithm are similar to simplex iterations; this similarity is reflected in the format of the iteration logs.

When LOGFREQ=1, the following information is printed in the iteration log:

| | |
|---|---|
| Phase | indicates whether the primal crossover (PC) or dual crossover (DC) technique is used. |
| Iteration | indicates the iteration number. |
| Objective Value | indicates the total amount by which the superbasic variables are off their bound. This value decreases to 0 as the crossover algorithm progresses. |
| Time | indicates the time elapsed (in seconds). |
| Entering Variable | indicates the entering pivot variable. A slack variable that enters the basis is indicated by the corresponding row name followed by “(S)”. |
| Leaving Variable | indicates the leaving pivot variable. A slack variable that leaves the basis is indicated by the corresponding row name followed by “(S)”. |

When you omit the LOGFREQ= option or specify a value greater than 1, only the phase, iteration, objective value, and time information are printed in the iteration log.

After all the superbasic variables have been eliminated, the crossover algorithm continues with regular primal or dual simplex iterations.

## Concurrent LP

The ALGORITHM=CON option starts several different linear optimization algorithms in parallel in a single-machine mode. The LP solver automatically determines which algorithms to run and how many threads to assign to each algorithm. If sufficient resources are available, the solver runs all four standard algorithms. When the first algorithm finishes, the LP solver returns the results from that algorithm and terminates any other algorithms that are still running. If you specify a value of TRUE for the DETERMINISTIC= option, the algorithm for which the results are returned is not necessarily the one that finished first. The LP solver deterministically selects the algorithm for which the results are returned. Regardless of which mode (deterministic or nondeterministic) is in effect, terminating algorithms that are still running might take a significant amount of time.

During concurrent optimization, the procedure displays the iteration log for the dual simplex algorithm. See the section "Iteration Log for the Primal and Dual Simplex Algorithms" on page 270 for more information about this iteration log. Upon termination, the solver displays the iteration log for the algorithm that finishes first, unless the dual simplex algorithm finishes first. If you specify LOGLEVEL=AGGRESSIVE, the LP solver displays the iteration logs for all algorithms that were run concurrently.

If you specify PRINTLEVEL=2 in the PROC OPTMODEL statement and ALGORITHM=CON in the SOLVE WITH LP statement, the LP solver produces an ODS table called ConcurrentSummary. This table contains a summary of the solution statuses of all algorithms that are run concurrently.

## Parallel Processing

The interior point, concurrent LP, and decomposition algorithms can be run in single-machine mode; in this mode, the computation is executed by multiple threads on a single computer.

## Problem Statistics

Optimizers can encounter difficulty when solving poorly formulated models. Information about data magnitude provides a simple gauge to determine how well a model is formulated. For example, a model whose constraint matrix contains one very large entry (on the order of $10^9$) can cause difficulty when the remaining entries are single-digit numbers. The PRINTLEVEL=2 option in the OPTMODEL procedure causes the ODS table "ProblemStatistics" to be generated when the LP solver is called. This table provides basic data magnitude information that enables you to improve the formulation of your models.

The example output in Figure 7.3 demonstrates the contents of the ODS table "ProblemStatistics."

**Figure 7.3** ODS Table ProblemStatistics

**The OPTMODEL Procedure**

| Problem Statistics | |
|---|---|
| Number of Constraint Matrix Nonzeros | 6 |
| Maximum Constraint Matrix Coefficient | 3 |
| Minimum Constraint Matrix Coefficient | 1 |
| Average Constraint Matrix Coefficient | 2.1666666667 |
| Number of Objective Nonzeros | 3 |
| Maximum Objective Coefficient | 1 |
| Minimum Objective Coefficient | 1 |
| Average Objective Coefficient | 1 |
| Number of RHS Nonzeros | 2 |
| Maximum RHS | 1 |
| Minimum RHS | 1 |
| Average RHS | 1 |
| Maximum Number of Nonzeros per Column | 2 |
| Minimum Number of Nonzeros per Column | 2 |
| Average Number of Nonzeros per Column | 2 |
| Maximum Number of Nonzeros per Row | 3 |
| Minimum Number of Nonzeros per Row | 3 |
| Average Number of Nonzeros per Row | 3 |

## Variable and Constraint Status

Upon termination of the LP solver, the .status suffix of each decision variable and constraint stores information about the status of that variable or constraint. For more information about suffixes in the OPTMODEL procedure, see the section "Suffixes" on page 135.

### Variable Status

The .status suffix of a decision variable specifies the status of that decision variable. The suffix can take one of the following values:

B basic variable

L nonbasic variable at its lower bound

U nonbasic variable at its upper bound

F free variable

A superbasic variable (a nonbasic variable that has a value strictly between its bounds)

I LP model infeasible (all decision variables have .status equal to I)

For the interior point algorithm with IIS=FALSE, .status is blank.

The following values can appear only if IIS=TRUE. See the section "Irreducible Infeasible Set" on page 275 for details.

I_L the lower bound of the variable is needed for the IIS

I_U the upper bound of the variable is needed for the IIS

I_F both bounds of the variable are needed for the IIS (the variable is fixed or has conflicting bounds)

### Constraint Status

The .status suffix of a constraint specifies the status of the slack variable for that constraint. The suffix can take one of the following values:

B basic variable

L nonbasic variable at its lower bound

U nonbasic variable at its upper bound

F free variable

A superbasic variable (a nonbasic variable that has a value strictly between its bounds)

I LP model infeasible (all decision variables have .status equal to I)

The following values can appear only if option IIS=TRUE. See the section "Irreducible Infeasible Set" on page 275 for details.

I_L the "GE" ($\geq$) condition of the constraint is needed for the IIS

I_U the "LE" ($\leq$) condition of the constraint is needed for the IIS

I_F both conditions of the constraint are needed for the IIS (the constraint is an equality or a range constraint with conflicting bounds)

## Irreducible Infeasible Set

For a linear programming problem, an irreducible infeasible set (IIS) is an infeasible subset of constraints and variable bounds that will become feasible if any single constraint or variable bound is removed. It is possible to have more than one IIS in an infeasible LP. Identifying an IIS can help isolate the structural infeasibility in an LP.

The presolver in the LP algorithms can detect infeasibility, but it identifies only the variable bound or constraint that triggers the infeasibility.

The IIS=TRUE option directs the LP solver to search for an IIS in a specified LP. When you specify IIS=TRUE, you should also specify the PRESOLVER=NONE option in the OPTMODEL statement; otherwise the IIS results can be incomplete. The LP solver does not apply the LP presolver to the problem during the IIS search. If the LP solver detects an IIS, it updates the .status suffix of the decision variables and constraints, and then it stops. The number of iterations that are reported in the macro variable and the ODS table is the

total number of simplex iterations. This total includes the initial LP solve and all subsequent iterations during the constraint deletion phase.

The IIS= option can add special values to the .status suffixes of variables and constraints. (For more information, see the section "Variable and Constraint Status" on page 274.) For constraints, a status of "I_L", "I_U", or "I_F" indicates that the "GE" ($\geq$), "LE" ($\leq$), or "EQ" (=) constraint, respectively, is part of the IIS. For range constraints, a status of "I_L" or "I_U" indicates that the lower or upper bound, respectively, of the constraint is needed for the IIS, and "I_F" indicates that the bounds in the constraint are conflicting. For variables, a status of "I_L", "I_U", or "I_F" indicates that the lower, upper, or both bounds, respectively, of the variable are needed for the IIS. From this information, you can identify both the names of the constraints (variables) in the IIS and the corresponding bound where infeasibility occurs.

Making any one of the constraints or variable bounds in the IIS nonbinding removes the infeasibility from the IIS. In some cases, changing a right-hand side or bound by a finite amount removes the infeasibility. However, the only way to guarantee removal of the infeasibility is to set the appropriate right-hand side or bound to $\infty$ or $-\infty$. Because it is possible for an LP to have multiple irreducible infeasible sets, simply removing the infeasibility from one set might not make the entire problem feasible. To make the entire problem feasible, you can specify IIS=TRUE and rerun the LP solver after removing the infeasibility from an IIS. Repeating this process until the LP solver no longer detects an IIS results in a feasible problem. This approach to infeasibility repair can produce different end problems depending on which right-hand sides and bounds you choose to relax.

The IIS= option in the LP solver uses two different methods to identify an IIS:

1. Based on the result of the initial solve, the *sensitivity filter* removes several constraints and variable bounds immediately while still maintaining infeasibility. This phase is quick and dramatically reduces the size of the IIS.

2. Next, the *deletion filter* removes each remaining constraint and variable bound one by one to check which of them are needed to obtain an infeasible system. This second phase is more time consuming, but it ensures that the IIS set that the LP solver returns is indeed irreducible. The progress of the deletion filter is reported at regular intervals. The sensitivity filter might be called again during the deletion filter to improve performance.

See Example 7.4 for an example that demonstrates the use of the IIS= option in locating and removing infeasibilities.

## Macro Variable _OROPTMODEL_

The OPTMODEL procedure always creates and initializes a SAS macro called _OROPTMODEL_. This variable contains a character string. After each PROC OROPTMODEL run, you can examine this macro by specifying `%put &_OROPTMODEL_;` and check the execution of the most recently invoked solver from the value of the macro variable. The various terms of the variable after the LP solver is called are interpreted as follows.

**STATUS**
indicates the solver status at termination. It can take one of the following values:

| | |
|---|---|
| OK | The solver terminated normally. |
| SYNTAX_ERROR | Incorrect syntax was used. |
| DATA_ERROR | The input data were inconsistent. |
| OUT_OF_MEMORY | Insufficient memory was allocated to the procedure. |
| IO_ERROR | A problem occurred in reading or writing data. |
| SEMANTIC_ERROR | An evaluation error, such as an invalid operand type, occurred. |
| ERROR | The status cannot be classified into any of the preceding categories. |

**ALGORITHM**

indicates the algorithm that produces the solution data in the macro variable. This term appears only when STATUS=OK. It can take one of the following values:

| | |
|---|---|
| PS | The primal simplex algorithm produced the solution data. |
| DS | The dual simplex algorithm produced the solution data. |
| NS | The network simplex algorithm produced the solution data. |
| IP | The interior point algorithm produced the solution data. |
| DECOMP | The decomposition algorithm produced the solution data. |

When you run algorithms concurrently (ALGORITHM=CON), this term indicates which algorithm is the first to terminate.

**SOLUTION_STATUS**

indicates the solution status at termination. It can take one of the following values:

| | |
|---|---|
| OPTIMAL | The solution is optimal. |
| CONDITIONAL_OPTIMAL | The solution is optimal, but some infeasibilities (primal, dual or bound) exceed tolerances due to scaling or pre-processing. |
| FEASIBLE | The problem is feasible. |
| INFEASIBLE | The problem is infeasible. |
| UNBOUNDED | The problem is unbounded. |
| INFEASIBLE_OR_UNBOUNDED | The problem is infeasible or unbounded. |
| BAD_PROBLEM_TYPE | The problem type is unsupported by the solver. |
| ITERATION_LIMIT_REACHED | The maximum allowable number of iterations was reached. |
| TIME_LIMIT_REACHED | The solver reached its execution time limit. |
| FUNCTION_CALL_LIMIT_REACHED | The solver reached its limit on function evaluations. |
| INTERRUPTED | The solver was interrupted externally. |
| FAILED | The solver failed to converge, possibly due to numerical issues. |

When SOLUTION_STATUS has a value of OPTIMAL, CONDITIONAL_OPTIMAL, ITERATION_LIMIT_REACHED, or TIME_LIMIT_REACHED, all terms of the _OROPTMODEL_ macro variable are present; for other values of SOLUTION_STATUS, some terms do not appear.

**OBJECTIVE**
indicates the objective value obtained by the solver at termination.

**PRIMAL_INFEASIBILITY**
indicates, for the primal simplex and dual simplex algorithms, the maximum (absolute) violation of the primal constraints by the primal solution. For the interior point algorithm, this term indicates the relative violation of the primal constraints by the primal solution.

**DUAL_INFEASIBILITY**
indicates, for the primal simplex and dual simplex algorithms, the maximum (absolute) violation of the dual constraints by the dual solution. For the interior point algorithm, this term indicates the relative violation of the dual constraints by the dual solution.

**BOUND_INFEASIBILITY**
indicates, for the primal simplex and dual simplex algorithms, the maximum (absolute) violation of the lower or upper bounds by the primal solution. For the interior point algorithm, this term indicates the relative violation of the lower or upper bounds by the primal solution.

**DUALITY_GAP**
indicates the (relative) duality gap. This term appears only if the option ALGORITHM=INTERIORPOINT is specified in the SOLVE statement.

**COMPLEMENTARITY**
indicates the (absolute) complementarity. This term appears only if the option ALGORITHM=INTERIORPOINT is specified in the SOLVE statement.

**ITERATIONS**
indicates the number of iterations taken to solve the problem. When the network simplex algorithm is used, this term indicates the number of network simplex iterations taken to solve the network relaxation. When crossover is enabled, this term indicates the number of interior point iterations taken to solve the problem.

**ITERATIONS2**
indicates the number of simplex iterations performed by the secondary algorithm. The network simplex algorithm selects the secondary algorithm automatically unless a value has been specified for the ALGORITHM2= option. When crossover is enabled, the secondary algorithm is selected automatically. This term appears only if the network simplex algorithm is used or if crossover is enabled.

**PRESOLVE_TIME**
indicates the time (in seconds) used in preprocessing.

**SOLUTION_TIME**
indicates the time (in seconds) taken to solve the problem, including preprocessing time.

**NOTE:** The time reported in PRESOLVE_TIME and SOLUTION_TIME is either CPU time or real time. The type is determined by the TIMETYPE= option.

When SOLUTION_STATUS has a value of OPTIMAL, CONDITIONAL_OPTIMAL, ITERATION_LIMIT_REACHED, or TIME_LIMIT_REACHED, all terms of the _OROPTMODEL_ macro variable are present; for other values of SOLUTION_STATUS, some terms do not appear.

# Examples: LP Solver

## Example 7.1: Diet Problem

Consider the problem of diet optimization. There are six different foods: bread, milk, cheese, potato, fish, and yogurt. The cost and nutrition values per unit are displayed in Table 7.2.

**Table 7.2** Cost and Nutrition Values

| | Bread | Milk | Cheese | Potato | Fish | Yogurt |
|---|---|---|---|---|---|---|
| **Cost** | 2.0 | 3.5 | 8.0 | 1.5 | 11.0 | 1.0 |
| **Protein, g** | 4.0 | 8.0 | 7.0 | 1.3 | 8.0 | 9.2 |
| **Fat, g** | 1.0 | 5.0 | 9.0 | 0.1 | 7.0 | 1.0 |
| **Carbohydrates, g** | 15.0 | 11.7 | 0.4 | 22.6 | 0.0 | 17.0 |
| **Calories** | 90 | 120 | 106 | 97 | 130 | 180 |

The following SAS code creates the data set fooddata of Table 7.2:

```
data fooddata;
   infile datalines;
   input name $  cost  prot  fat  carb  cal;
   datalines;
Bread   2     4     1    15    90
Milk    3.5   8     5    11.7  120
Cheese  8     7     9    0.4   106
Potato  1.5   1.3   0.1  22.6  97
Fish    11    8     7    0     130
Yogurt  1     9.2   1    17    180
;
```

The objective is to find a minimum-cost diet that contains at least 300 calories, not more than 10 grams of protein, not less than 10 grams of carbohydrates, and not less than 8 grams of fat. In addition, the diet should contain at least 0.5 unit of fish and no more than 1 unit of milk.

You can model the problem and solve it by using PROC OPTMODEL as follows:

```
proc optmodel;
   /* declare index set */
   set<str> FOOD;

   /* declare variables */
   var diet{FOOD} >= 0;

   /* objective function */
```

```
num cost{FOOD};
min f=sum{i in FOOD}cost[i]*diet[i];

/* constraints */
num prot{FOOD};
num fat{FOOD};
num carb{FOOD};
num cal{FOOD};
num min_cal, max_prot, min_carb, min_fat;
con cal_con: sum{i in FOOD}cal[i]*diet[i] >= 300;
con prot_con: sum{i in FOOD}prot[i]*diet[i] <= 10;
con carb_con: sum{i in FOOD}carb[i]*diet[i] >= 10;
con fat_con: sum{i in FOOD}fat[i]*diet[i] >= 8;

/* read parameters */
read data fooddata into FOOD=[name] cost prot fat carb cal;

/* bounds on variables */
diet['Fish'].lb = 0.5;
diet['Milk'].ub = 1.0;

/* solve and print the optimal solution */
solve with lp/logfreq=1; /* print each iteration to log */
print diet;
```

The optimal solution and the optimal objective value are displayed in Output 7.1.1.

**Output 7.1.1** Optimal Solution to the Diet Problem

**The OPTMODEL Procedure**

| Problem Summary | |
|---|---|
| **Objective Sense** | Minimization |
| **Objective Function** | f |
| **Objective Type** | Linear |
| | |
| **Number of Variables** | 6 |
| **Bounded Above** | 0 |
| **Bounded Below** | 5 |
| **Bounded Below and Above** | 1 |
| **Free** | 0 |
| **Fixed** | 0 |
| | |
| **Number of Constraints** | 4 |
| **Linear LE (<=)** | 1 |
| **Linear EQ (=)** | 0 |
| **Linear GE (>=)** | 3 |
| **Linear Range** | 0 |
| | |
| **Constraint Coefficients** | 23 |

**Output 7.1.1** *continued*

| Solution Summary | |
|---|---|
| **Solver** | LP |
| **Algorithm** | Dual Simplex |
| **Objective Function** | f |
| **Solution Status** | Optimal |
| **Objective Value** | 12.081337881 |
| | |
| **Primal Infeasibility** | 1.776357E-15 |
| **Dual Infeasibility** | 1.110223E-16 |
| **Bound Infeasibility** | 0 |
| | |
| **Iterations** | 5 |
| **Presolve Time** | 0.00 |
| **Solution Time** | 0.00 |

| [1] | diet |
|---|---|
| **Bread** | 0.000000 |
| **Cheese** | 0.449499 |
| **Fish** | 0.500000 |
| **Milk** | 0.053599 |
| **Potato** | 1.865168 |
| **Yogurt** | 0.000000 |

## Example 7.2: Reoptimizing the Diet Problem Using BASIS=WARMSTART

After an LP is solved, you might want to change a set of the parameters of the LP and solve the problem again. This can be done efficiently in PROC OPTMODEL. The warm start technique uses the optimal solution of the solved LP as a starting point and solves the modified LP problem faster than it can be solved again from scratch. This example illustrates reoptimizing the diet problem described in Example 7.1.

Assume the optimal solution is found by the SOLVE statement. Instead of quitting the OPTMODEL procedure, you can continue to solve several variations of the original problem.

Suppose the cost of cheese increases from 8 to 10 per unit and the cost of fish decreases from 11 to 7 per serving unit. You can change the parameters and solve the modified problem by submitting the following code:

```
cost['Cheese']=10; cost['Fish']=7;
solve with lp/presolver=none
             basis=warmstart
             algorithm=ps
             logfreq=1;
print diet;
```

Note that the primal simplex algorithm is preferred because the primal solution to the last-solved LP is still feasible for the modified problem in this case. The solutions to the original diet problem and the modified problem are shown in Output 7.2.1.

**Output 7.2.1** Optimal Solutions to the Original Diet Problem and the Diet Problem with Modified Objective Function

**The OPTMODEL Procedure**

| Problem Summary | |
|---|---|
| **Objective Sense** | Minimization |
| **Objective Function** | f |
| **Objective Type** | Linear |
| **Number of Variables** | 6 |
| **Bounded Above** | 0 |
| **Bounded Below** | 5 |
| **Bounded Below and Above** | 1 |
| **Free** | 0 |
| **Fixed** | 0 |
| **Number of Constraints** | 4 |
| **Linear LE (<=)** | 1 |
| **Linear EQ (=)** | 0 |
| **Linear GE (>=)** | 3 |
| **Linear Range** | 0 |
| **Constraint Coefficients** | 23 |

| Solution Summary | |
|---|---|
| **Solver** | LP |
| **Algorithm** | Dual Simplex |
| **Objective Function** | f |
| **Solution Status** | Optimal |
| **Objective Value** | 12.081337881 |
| **Primal Infeasibility** | 1.776357E-15 |
| **Dual Infeasibility** | 1.110223E-16 |
| **Bound Infeasibility** | 0 |
| **Iterations** | 5 |
| **Presolve Time** | 0.00 |
| **Solution Time** | 0.00 |

| [1] | diet |
|---|---|
| **Bread** | 0.000000 |
| **Cheese** | 0.449499 |
| **Fish** | 0.500000 |
| **Milk** | 0.053599 |
| **Potato** | 1.865168 |
| **Yogurt** | 0.000000 |

**Output 7.2.1** *continued*

| Problem Summary | |
|---|---|
| **Objective Sense** | Minimization |
| **Objective Function** | f |
| **Objective Type** | Linear |
| **Number of Variables** | 6 |
| **Bounded Above** | 0 |
| **Bounded Below** | 5 |
| **Bounded Below and Above** | 1 |
| **Free** | 0 |
| **Fixed** | 0 |
| **Number of Constraints** | 4 |
| **Linear LE (<=)** | 1 |
| **Linear EQ (=)** | 0 |
| **Linear GE (>=)** | 3 |
| **Linear Range** | 0 |
| **Constraint Coefficients** | 23 |

| Solution Summary | |
|---|---|
| **Solver** | LP |
| **Algorithm** | Primal Simplex |
| **Objective Function** | f |
| **Solution Status** | Optimal |
| **Objective Value** | 10.980335514 |
| **Primal Infeasibility** | 1.776357E-15 |
| **Dual Infeasibility** | 8.881784E-16 |
| **Bound Infeasibility** | 0 |
| **Iterations** | 1 |
| **Presolve Time** | 0.00 |
| **Solution Time** | 0.00 |

| [1] | diet |
|---|---|
| **Bread** | 0.000000 |
| **Cheese** | 0.449499 |
| **Fish** | 0.500000 |
| **Milk** | 0.053599 |
| **Potato** | 1.865168 |
| **Yogurt** | 0.000000 |

The following iteration log indicates that it takes the LP solver no more iterations to solve the modified problem by using BASIS=WARMSTART, since the optimal solution to the original problem remains optimal after the objective function is changed.

**Output 7.2.2** Log

```
NOTE: There were 6 observations read from the data set WORK.FOODDATA.
NOTE: Problem generation will use 16 threads.
NOTE: The problem has 6 variables (0 free, 0 fixed).
NOTE: The problem has 4 linear constraints (1 LE, 0 EQ, 3 GE, 0 range).
NOTE: The problem has 23 linear constraint coefficients.
NOTE: The problem has 0 nonlinear constraints (0 LE, 0 EQ, 0 GE, 0 range).
NOTE: The LP presolver value AUTOMATIC is applied.
NOTE: The LP presolver time is 0.00 seconds.
NOTE: The LP presolver removed 0 variables and 0 constraints.
NOTE: The LP presolver removed 0 constraint coefficients.
NOTE: The presolved problem has 6 variables, 4 constraints, and 23 constraint
      coefficients.
NOTE: The LP solver is called.
NOTE: The Dual Simplex algorithm is used.
                          Objective
      Phase Iteration       Value         Time
       D 2          1    5.500000E+00         0
       D 2          5    1.208134E+01         0
NOTE: Optimal.
NOTE: Objective = 12.081337881.
NOTE: The Dual Simplex solve time is 0.00 seconds.
NOTE: Problem generation will use 16 threads.
NOTE: The problem has 6 variables (0 free, 0 fixed).
NOTE: The problem has 4 linear constraints (1 LE, 0 EQ, 3 GE, 0 range).
NOTE: The problem has 23 linear constraint coefficients.
NOTE: The problem has 0 nonlinear constraints (0 LE, 0 EQ, 0 GE, 0 range).
NOTE: The LP presolver value NONE is applied.
NOTE: The LP solver is called.
NOTE: The Primal Simplex algorithm is used.
                          Objective               Entering          Leaving
      Phase Iteration       Value         Time     Variable          Variable
       P 2          1    1.098034E+01         0
NOTE: Optimal.
NOTE: Objective = 10.980335514.
NOTE: The Primal Simplex solve time is 0.00 seconds.
```

Next, restore the original coefficients of the objective function and consider the case that you need a diet that supplies at least 150 calories. You can change the parameters and solve the modified problem by submitting the following code:

```
cost['Cheese']=8; cost['Fish']=11;cal_con.lb=150;
solve with lp/presolver=none
             basis=warmstart
             algorithm=ds
             logfreq=1;
print diet;
```

Note that the dual simplex algorithm is preferred because the dual solution to the last-solved LP is still feasible for the modified problem in this case. The solution is shown in Output 7.2.3.

**Output 7.2.3** Optimal Solution to the Diet Problem with Modified RHS

**The OPTMODEL Procedure**

| Problem Summary | |
|---|---|
| **Objective Sense** | Minimization |
| **Objective Function** | f |
| **Objective Type** | Linear |
| | |
| **Number of Variables** | 6 |
| **Bounded Above** | 0 |
| **Bounded Below** | 5 |
| **Bounded Below and Above** | 1 |
| **Free** | 0 |
| **Fixed** | 0 |
| | |
| **Number of Constraints** | 4 |
| **Linear LE (<=)** | 1 |
| **Linear EQ (=)** | 0 |
| **Linear GE (>=)** | 3 |
| **Linear Range** | 0 |
| | |
| **Constraint Coefficients** | 23 |

| Solution Summary | |
|---|---|
| **Solver** | LP |
| **Algorithm** | Dual Simplex |
| **Objective Function** | f |
| **Solution Status** | Optimal |
| **Objective Value** | 9.1744131985 |
| | |
| **Primal Infeasibility** | 0 |
| **Dual Infeasibility** | 1.387779E-16 |
| **Bound Infeasibility** | 0 |
| | |
| **Iterations** | 2 |
| **Presolve Time** | 0.00 |
| **Solution Time** | 0.00 |

| [1] | diet |
|---|---|
| **Bread** | 0.00000 |
| **Cheese** | 0.18481 |
| **Fish** | 0.50000 |
| **Milk** | 0.56440 |
| **Potato** | 0.14702 |
| **Yogurt** | 0.00000 |

The following iteration log indicates that it takes the LP solver just one more phase II iteration to solve the modified problem by using BASIS=WARMSTART.

**Output 7.2.4** Log

```
NOTE: There were 6 observations read from the data set WORK.FOODDATA.
NOTE: Problem generation will use 16 threads.
NOTE: The problem has 6 variables (0 free, 0 fixed).
NOTE: The problem has 4 linear constraints (1 LE, 0 EQ, 3 GE, 0 range).
NOTE: The problem has 23 linear constraint coefficients.
NOTE: The problem has 0 nonlinear constraints (0 LE, 0 EQ, 0 GE, 0 range).
NOTE: Problem generation will use 16 threads.
NOTE: The problem has 6 variables (0 free, 0 fixed).
NOTE: The problem has 4 linear constraints (1 LE, 0 EQ, 3 GE, 0 range).
NOTE: The problem has 23 linear constraint coefficients.
NOTE: The problem has 0 nonlinear constraints (0 LE, 0 EQ, 0 GE, 0 range).
NOTE: The LP presolver value NONE is applied.
NOTE: The LP solver is called.
NOTE: The Dual Simplex algorithm is used.
                          Objective                Entering          Leaving
      Phase Iteration        Value         Time     Variable          Variable
       D 2          1    8.813205E+00         0        cal_con (S)
      carb_con (S)
       D 2          2    9.174413E+00         0
NOTE: Optimal.
NOTE: Objective = 9.1744131985.
NOTE: The Dual Simplex solve time is 0.00 seconds.
```

Next, restore the original constraint on calories and consider the case that you need a diet that supplies no more than 550 mg of sodium per day. The following row is appended to Table 7.2.

| | Bread | Milk | Cheese | Potato | Fish | Yogurt |
|---|---|---|---|---|---|---|
| **sodium, mg** | 148 | 122 | 337 | 186 | 56 | 132 |

You can change the parameters, add the new constraint, and solve the modified problem by submitting the following code:

```
cal_con.lb=300;
num sod{FOOD}=[148 122 337 186 56 132];
con sodium: sum{i in FOOD}sod[i]*diet[i] <= 550;
solve with lp/presolver=none
             basis=warmstart
             logfreq=1;
print diet;
```

The solution is shown in Output 7.2.5.

**Output 7.2.5** Optimal Solution to the Diet Problem with Additional Constraint

**The OPTMODEL Procedure**

| Problem Summary | |
|---|---|
| **Objective Sense** | Minimization |
| **Objective Function** | f |
| **Objective Type** | Linear |
| **Number of Variables** | 6 |
| **Bounded Above** | 0 |
| **Bounded Below** | 5 |
| **Bounded Below and Above** | 1 |
| **Free** | 0 |
| **Fixed** | 0 |
| **Number of Constraints** | 5 |
| **Linear LE (<=)** | 2 |
| **Linear EQ (=)** | 0 |
| **Linear GE (>=)** | 3 |
| **Linear Range** | 0 |
| **Constraint Coefficients** | 29 |

| Solution Summary | |
|---|---|
| **Solver** | LP |
| **Algorithm** | Dual Simplex |
| **Objective Function** | f |
| **Solution Status** | Optimal |
| **Objective Value** | 12.081337881 |
| **Primal Infeasibility** | 1.776357E-15 |
| **Dual Infeasibility** | 1.776357E-15 |
| **Bound Infeasibility** | 0 |
| **Iterations** | 1 |
| **Presolve Time** | 0.00 |
| **Solution Time** | 0.00 |

| [1] | diet |
|---|---|
| **Bread** | 0.000000 |
| **Cheese** | 0.449499 |
| **Fish** | 0.500000 |
| **Milk** | 0.053599 |
| **Potato** | 1.865168 |
| **Yogurt** | 0.000000 |

The following iteration log indicates that it takes the LP solver no more iterations to solve the modified problem by using the BASIS=WARMSTART option, since the optimal solution to the original problem remains optimal after one more constraint is added.

**Output 7.2.6** Log

```
NOTE: There were 6 observations read from the data set WORK.FOODDATA.
NOTE: Problem generation will use 16 threads.
NOTE: The problem has 6 variables (0 free, 0 fixed).
NOTE: The problem has 4 linear constraints (1 LE, 0 EQ, 3 GE, 0 range).
NOTE: The problem has 23 linear constraint coefficients.
NOTE: The problem has 0 nonlinear constraints (0 LE, 0 EQ, 0 GE, 0 range).
NOTE: Problem generation will use 16 threads.
NOTE: The problem has 6 variables (0 free, 0 fixed).
NOTE: The problem has 5 linear constraints (2 LE, 0 EQ, 3 GE, 0 range).
NOTE: The problem has 29 linear constraint coefficients.
NOTE: The problem has 0 nonlinear constraints (0 LE, 0 EQ, 0 GE, 0 range).
NOTE: The LP presolver value NONE is applied.
NOTE: The LP solver is called.
NOTE: The Dual Simplex algorithm is used.
                            Objective                Entering          Leaving
      Phase Iteration         Value         Time     Variable          Variable
       D 2          1    1.208134E+01         0
NOTE: Optimal.
NOTE: Objective = 12.081337881.
NOTE: The Dual Simplex solve time is 0.00 seconds.
```

## Example 7.3: Two-Person Zero-Sum Game

Consider a two-person zero-sum game (where one person wins what the other person loses). The players make moves simultaneously, and each has a choice of actions. There is a *payoff* matrix that indicates the amount one player gives to the other under each combination of actions:

$$\begin{array}{c} \\ \text{Player I plays } i \end{array} \begin{array}{c} \text{Player II plays } j \\ \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{pmatrix} -5 & 3 & 1 & 8 \\ 5 & 5 & 4 & 6 \\ -4 & 6 & 0 & 5 \end{pmatrix} \end{array}$$

If player I makes move $i$ and player II makes move $j$, then player I wins (and player II loses) $a_{ij}$. What is the best strategy for the two players to adopt? This example is simple enough to be analyzed from observation. Suppose player I plays 1 or 3; the best response of player II is to play 1. In both cases, player I loses and player II wins. So the best action for player I is to play 2. In this case, the best response for player II is to play 3, which minimizes the loss. In this case, (2, 3) is a *pure-strategy Nash equilibrium* in this game.

For illustration, consider the following mixed strategy case. Assume that player I selects $i$ with probability $p_i$, $i = 1, 2, 3$, and player II selects $j$ with probability $q_j$, $j = 1, 2, 3, 4$. Consider player II's problem of minimizing the maximum expected payout:

$$\min_{\mathbf{q}} \left\{ \max_{i} \sum_{j=1}^{4} a_{ij} q_j \right\} \quad \text{subject to} \quad \sum_{j=1}^{4} q_{ij} = 1, \quad \mathbf{q} \geq 0$$

This is equivalent to

$$\begin{aligned} \min_{\mathbf{q},v} \; v \quad \text{subject to} \quad \sum_{j=1}^{4} a_{ij} q_j &\leq v \quad \forall i \\ \sum_{j=1}^{4} q_j &= 1 \\ \mathbf{q} &\geq 0 \end{aligned}$$

The problem can be transformed into a more standard format by making a simple change of variables: $x_j = q_j / v$. The preceding LP formulation now becomes

$$\begin{aligned} \min_{\mathbf{x},v} \; v \quad \text{subject to} \quad \sum_{j=1}^{4} a_{ij} x_j &\leq 1 \quad \forall i \\ \sum_{j=1}^{4} x_j &= 1/v \\ \mathbf{q} &\geq 0 \end{aligned}$$

which is equivalent to

$$\max_{\mathbf{x}} \sum_{j=1}^{4} x_j \quad \text{subject to} \quad A\mathbf{x} \leq \mathbf{1}, \quad \mathbf{x} \geq 0$$

where $A$ is the payoff matrix and $\mathbf{1}$ is a vector of 1's. It turns out that the corresponding optimization problem from player I's perspective can be obtained by solving the dual problem, which can be written as

$$\min_{\mathbf{y}} \sum_{i=1}^{3} y_i \quad \text{subject to} \quad A^{\mathrm{T}}\mathbf{y} \geq \mathbf{1}, \quad \mathbf{y} \geq 0$$

You can model the problem and solve it by using PROC OPTMODEL as follows:

```
proc optmodel;
   num a{1..3, 1..4}=[-5 3 1 8
                       5 5 4 6
                      -4 6 0 5];
   var x{1..4} >= 0;
   max f = sum{i in 1..4}x[i];
   con c{i in 1..3}: sum{j in 1..4}a[i,j]*x[j] <= 1;
   solve with lp / algorithm = ps presolver = none logfreq = 1;
   print x;
   print c.dual;
quit;
```

The optimal solution is displayed in Output 7.3.1.

**Output 7.3.1** Optimal Solutions to the Two-Person Zero-Sum Game

**The OPTMODEL Procedure**

| Problem Summary | |
|---|---|
| **Objective Sense** | Maximization |
| **Objective Function** | f |
| **Objective Type** | Linear |
| **Number of Variables** | 4 |
| **Bounded Above** | 0 |
| **Bounded Below** | 4 |
| **Bounded Below and Above** | 0 |
| **Free** | 0 |
| **Fixed** | 0 |
| **Number of Constraints** | 3 |
| **Linear LE (<=)** | 3 |
| **Linear EQ (=)** | 0 |
| **Linear GE (>=)** | 0 |
| **Linear Range** | 0 |
| **Constraint Coefficients** | 11 |

| Solution Summary | |
|---|---|
| **Solver** | LP |
| **Algorithm** | Primal Simplex |
| **Objective Function** | f |
| **Solution Status** | Optimal |
| **Objective Value** | 0.25 |
| **Primal Infeasibility** | 0 |
| **Dual Infeasibility** | 0 |
| **Bound Infeasibility** | 0 |
| **Iterations** | 2 |
| **Presolve Time** | 0.00 |
| **Solution Time** | 0.00 |

| [1] | x |
|---|---|
| **1** | 0.00 |
| **2** | 0.00 |
| **3** | 0.25 |
| **4** | 0.00 |

| [1] | c.DUAL |
|---|---|
| **1** | 0.00 |
| **2** | 0.25 |
| **3** | 0.00 |

The optimal solution $\mathbf{x}^* = (0, 0, 0.25, 0)$ with an optimal value of 0.25. Therefore the optimal strategy for player II is $\mathbf{q}^* = \mathbf{x}^*/0.25 = (0, 0, 1, 0)$. You can check the optimal solution of the dual problem by using the constraint suffix ".dual". So $\mathbf{y}^* = (0, 0.25, 0)$ and player I's optimal strategy is (0, 1, 0). The solution is consistent with our intuition from observation.

## Example 7.4: Finding an Irreducible Infeasible Set

This example demonstrates the use of the IIS= option to locate an irreducible infeasible set. Suppose you want to solve a linear program that has the following simple formulation:

$$
\begin{array}{rrcrcrcrl}
\min & & x_1 & + & x_2 & + & x_3 & & (\text{cost}) \\
\text{subject to} & & x_1 & + & x_2 & & & \geq \quad 10 & (\text{con1}) \\
 & & x_1 & & & + & x_3 & \leq \quad 4 & (\text{con2}) \\
 & 4 \leq & & & x_2 & + & x_3 & \leq \quad 5 & (\text{con3}) \\
 & & & & & x_1, & x_2 & \geq \quad 0 & \\
 & & & & 0 & \leq & x_3 & \leq \quad 3 &
\end{array}
$$

It is easy to verify that the following three constraints (or rows) and one variable (or column) bound form an IIS for this problem:

$$
\begin{array}{rcrcrcrl}
x_1 & + & x_2 & & & \geq & 10 & (\text{con1}) \\
x_1 & & & + & x_3 & \leq & 4 & (\text{con2}) \\
 & & x_2 & + & x_3 & \leq & 5 & (\text{con3}) \\
 & & & & x_3 & \geq & 0 &
\end{array}
$$

You can formulate the problem and call the LP solver by using the following statements:

```
proc optmodel presolver=none;
   /* declare variables */
   var x{1..3} >=0;

   /* upper bound on variable x[3] */
   x[3].ub = 3;

   /* objective function */
   min obj = x[1] + x[2] + x[3];

   /* constraints */
   con c1: x[1] + x[2] >= 10;
   con c2: x[1] + x[3] <= 4;
   con c3: 4 <= x[2] + x[3] <= 5;

   solve with lp / iis=true;

   print x.status;
   print c1.status c2.status c3.status;

   expand / IIS;
```

The notes printed in the log appear in Output 7.4.1.

**Output 7.4.1** Finding an IIS: Log

```
NOTE: Problem generation will use 16 threads.
NOTE: The problem has 3 variables (0 free, 0 fixed).
NOTE: The problem has 3 linear constraints (1 LE, 0 EQ, 1 GE, 1 range).
NOTE: The problem has 6 linear constraint coefficients.
NOTE: The problem has 0 nonlinear constraints (0 LE, 0 EQ, 0 GE, 0 range).
NOTE: The LP solver is called.
NOTE: The IIS= option is enabled.
                          Objective
      Phase Iteration        Value         Time
       P 1          1    6.000000E+00         0
       P 1          3    1.000000E+00         0
NOTE: Applying the IIS sensitivity filter.
NOTE: The sensitivity filter removed 1 constraints and 3 variable bounds.
NOTE: Applying the IIS deletion filter.
NOTE: Processing constraints.
      Processed     Removed      Time
              0           0         0
              1           0         0
              2           0         0
              3           0         0
NOTE: Processing variable bounds.
      Processed     Removed      Time
              0           0         0
              1           0         0
              2           0         0
              3           0         0
NOTE: The deletion filter removed 0 constraints and 0 variable bounds.
NOTE: The IIS= option found this problem to be infeasible.
NOTE: The IIS= option found an irreducible infeasible set with 1 variables and
      3 constraints.
NOTE: The IIS solve time is 0.00 seconds.
```

There are two ways to display the rows and columns that are included in the IIS. One way is to use the PRINT statement to print the value of the .status suffix for each variable and constraint. The more straightforward approach is to use the EXPAND statement with the IIS option.

The output of both approaches appears in Output 7.4.2. The value of the .status suffix for the variables x[1] and x[2] is missing, so the variable bounds for x[1] and x[2] are not in the IIS.

**Output 7.4.2** Solution Summary, Variable Status, and Constraint Status

**The OPTMODEL Procedure**

| Solution Summary | |
|---|---|
| **Solver** | LP |
| **Algorithm** | IIS |
| **Objective Function** | obj |
| **Solution Status** | Infeasible |
| **Iterations** | 10 |
| **Presolve Time** | 0.00 |
| **Solution Time** | 0.00 |

| [1] | x.STATUS |
|---|---|
| **1** | |
| **2** | |
| **3** | I_L |

| c1.STATUS | c2.STATUS | c3.STATUS |
|---|---|---|
| I_L | I_U | I_U |

The OPTMODEL Procedure

```
Var x[3] >= 0 <= 3
Constraint c1: x[1] + x[2] >= 10
Constraint c2: x[1] + x[3] <= 4
Constraint c3: 4 <= x[2] + x[3] <= 5
```

The value of c3.status is I_U, which indicates that $x_2 + x_3 \leq 5$ is an element of the IIS. The original constraint is c3, a range constraint with a lower bound of 4. If you choose to remove the constraint $x_2 + x_3 \leq 5$, you can change the value of c3.ub to the largest positive number representable in your operating environment. You can specify this number by using the CONSTANT function.

The modified LP problem is specified and solved by adding the following lines to the original PROC OPTMODEL call.

```
/* relax upper bound on constraint c3 */
c3.ub = constant('BIG');

solve with lp / iis=true;
```

Because one element of the IIS has been removed, the modified LP problem should no longer contain the infeasible set. Due to the size of this problem, there should be no additional irreducible infeasible sets.

The notes shown in Output 7.4.3 are printed to the log.

**Output 7.4.3** Infeasibility Removed: Log

```
NOTE: Problem generation will use 16 threads.
NOTE: The problem has 3 variables (0 free, 0 fixed).
NOTE: The problem has 3 linear constraints (1 LE, 0 EQ, 2 GE, 0 range).
NOTE: The problem has 6 linear constraint coefficients.
NOTE: The problem has 0 nonlinear constraints (0 LE, 0 EQ, 0 GE, 0 range).
NOTE: The LP solver is called.
NOTE: The IIS= option is enabled.
                          Objective
      Phase Iteration         Value         Time
       P 1          1    1.400000E+01          0
       P 1          2    0.000000E+00          0
NOTE: The IIS= option found this problem to be feasible.
NOTE: The IIS solve time is 0.00 seconds.
```

The solution summary and primal solution are displayed in Output 7.4.4.

**Output 7.4.4** Infeasibility Removed: Solution

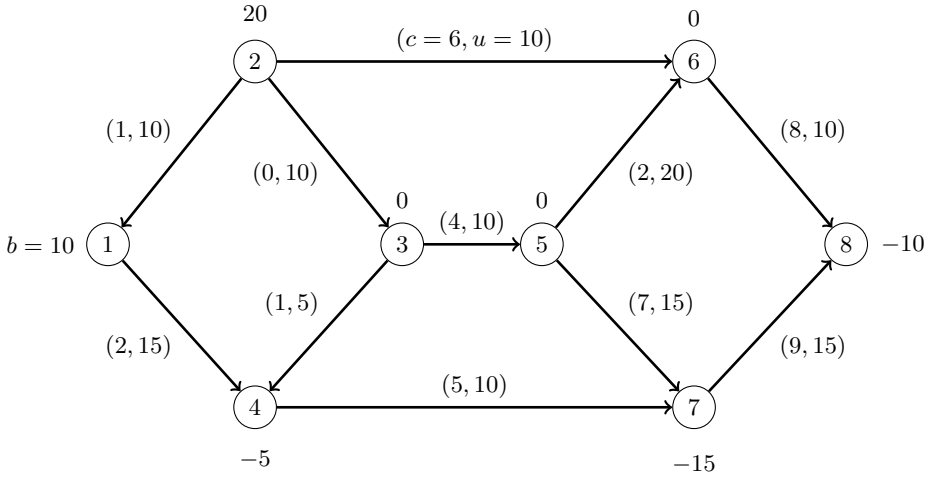
**The OPTMODEL Procedure**

| Solution Summary | |
|---|---|
| **Solver** | LP |
| **Algorithm** | IIS |
| **Objective Function** | obj |
| **Solution Status** | Feasible |
| | |
| **Iterations** | 2 |
| **Presolve Time** | 0.00 |
| **Solution Time** | 0.00 |

## Example 7.5: Using the Network Simplex Algorithm

This example demonstrates how you can use the network simplex algorithm to find the minimum-cost flow in a directed graph. Consider the directed graph in Figure 7.4, which appears in Ahuja, Magnanti, and Orlin (1993).

**Figure 7.4** Minimum-Cost Network Flow Problem: Data




You can use the following SAS statements to create the input data sets nodedata and arcdata:

```
data nodedata;
   input _node_ $ _sd_;
   datalines;
1    10
2    20
3     0
4    -5
5     0
6     0
7   -15
8   -10
;

data arcdata;
   input _tail_ $ _head_ $ _lo_ _capac_ _cost_;
   datalines;
1    4    0    15    2
2    1    0    10    1
2    3    0    10    0
2    6    0    10    6
3    4    0     5    1
3    5    0    10    4
4    7    0    10    5
5    6    0    20    2
5    7    0    15    7
6    8    0    10    8
7    8    0    15    9
;
```

You can use the following call to PROC OPTMODEL to find the minimum-cost flow:

```
proc optmodel;
   set <str> NODES;
   num supply_demand {NODES};

   set <str,str> ARCS;
   num arcLower  {ARCS};
   num arcUpper  {ARCS};
   num arcCost   {ARCS};

   read data arcdata into ARCS=[_tail_ _head_]
      arcLower=_lo_ arcUpper=_capac_ arcCost=_cost_;
   read data nodedata into NODES=[_node_] supply_demand=_sd_;

   var flow {<i,j> in ARCS} >= arcLower[i,j] <= arcUpper[i,j];
   min obj = sum {<i,j> in ARCS} arcCost[i,j] * flow[i,j];
   con balance {i in NODES}:
      sum {<(i),j> in ARCS} flow[i,j] - sum {<j,(i)> in ARCS} flow[j,i]
         = supply_demand[i];
   solve with lp / algorithm=ns scale=none logfreq=1;
   print flow;
quit;
%put &_OROPTMODEL_;
```

The optimal solution is displayed in Output 7.5.1.

**Output 7.5.1** Network Simplex Algorithm: Primal Solution Output

**The OPTMODEL Procedure**

| Problem Summary | |
|---|---|
| **Objective Sense** | Minimization |
| **Objective Function** | obj |
| **Objective Type** | Linear |
| | |
| **Number of Variables** | 11 |
| **Bounded Above** | 0 |
| **Bounded Below** | 0 |
| **Bounded Below and Above** | 11 |
| **Free** | 0 |
| **Fixed** | 0 |
| | |
| **Number of Constraints** | 8 |
| **Linear LE (<=)** | 0 |
| **Linear EQ (=)** | 8 |
| **Linear GE (>=)** | 0 |
| **Linear Range** | 0 |
| | |
| **Constraint Coefficients** | 22 |

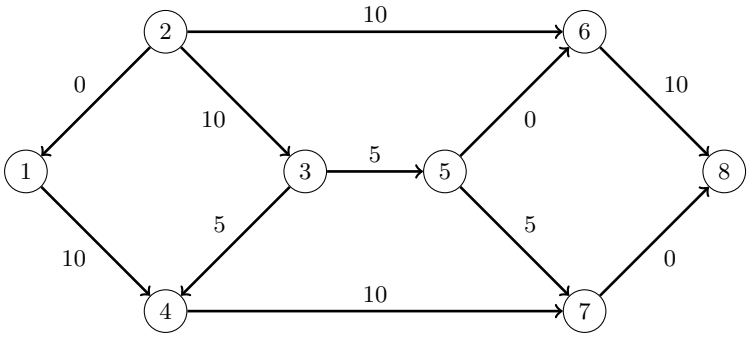
**Output 7.5.1** *continued*

| Solution Summary | |
|---|---|
| **Solver** | LP |
| **Algorithm** | Network Simplex |
| **Objective Function** | obj |
| **Solution Status** | Optimal |
| **Objective Value** | 270 |
| | |
| **Primal Infeasibility** | 0 |
| **Dual Infeasibility** | 0 |
| **Bound Infeasibility** | 0 |
| | |
| **Iterations** | 8 |
| **Iterations2** | 0 |
| **Presolve Time** | 0.00 |
| **Solution Time** | 0.01 |

| [1] | [2] | flow |
|---|---|---|
| **1** | **4** | 10 |
| **2** | **1** | 0 |
| **2** | **3** | 10 |
| **2** | **6** | 10 |
| **3** | **4** | 5 |
| **3** | **5** | 5 |
| **4** | **7** | 10 |
| **5** | **6** | 0 |
| **5** | **7** | 5 |
| **6** | **8** | 10 |
| **7** | **8** | 0 |

The optimal solution is represented graphically in Figure 7.5.

**Figure 7.5** Minimum-Cost Network Flow Problem: Optimal Solution



The iteration log is displayed in Output 7.5.2.

**Output 7.5.2** Log: Solution Progress

```
NOTE: There were 11 observations read from the data set WORK.ARCDATA.
NOTE: There were 8 observations read from the data set WORK.NODEDATA.
NOTE: Problem generation will use 16 threads.
NOTE: The problem has 11 variables (0 free, 0 fixed).
NOTE: The problem has 8 linear constraints (0 LE, 8 EQ, 0 GE, 0 range).
NOTE: The problem has 22 linear constraint coefficients.
NOTE: The problem has 0 nonlinear constraints (0 LE, 0 EQ, 0 GE, 0 range).
NOTE: The problem is a pure network instance; PRESOLVER=NONE is used.
NOTE: The LP presolver value NONE is applied.
NOTE: The LP solver is called.
NOTE: The Network Simplex algorithm is used.
NOTE: The network has 8 rows (100.00%), 11 columns (100.00%), and 1 component.
NOTE: The network extraction and setup time is 0.00 seconds.
                        Primal         Primal           Dual
      Iteration      Objective  Infeasibility  Infeasibility     Time
              1   0.000000E+00   2.000000E+01   8.900000E+01     0.00
              2   0.000000E+00   2.000000E+01   8.900000E+01     0.00
              3   5.000000E+00   1.500000E+01   8.400000E+01     0.00
              4   5.000000E+00   1.500000E+01   8.300000E+01     0.00
              5   7.500000E+01   1.500000E+01   8.300000E+01     0.00
              6   7.500000E+01   1.500000E+01   7.900000E+01     0.00
              7   1.300000E+02   1.000000E+01   7.600000E+01     0.00
              8   2.700000E+02   0.000000E+00   0.000000E+00     0.00
NOTE: The Network Simplex solve time is 0.00 seconds.
NOTE: The total Network Simplex solve time is 0.00 seconds.
NOTE: Optimal.
NOTE: Objective = 270.
STATUS=OK ALGORITHM=NS SOLUTION_STATUS=OPTIMAL OBJECTIVE=270
PRIMAL_INFEASIBILITY=0 DUAL_INFEASIBILITY=0 BOUND_INFEASIBILITY=0 ITERATIONS=8
ITERATIONS2=0 PRESOLVE_TIME=0.00 SOLUTION_TIME=0.00
```

Now, suppose there is a budget on the flow that comes out of arc 2: the total arc cost of flow that comes out of arc 2 cannot exceed 50. You can use the following call to PROC OPTMODEL to find the minimum-cost flow:

```
proc optmodel;
   set <str> NODES;
   num supply_demand {NODES};

   set <str,str> ARCS;
   num arcLower  {ARCS};
   num arcUpper  {ARCS};
   num arcCost   {ARCS};

   read data arcdata into ARCS=[_tail_ _head_]
      arcLower=_lo_ arcUpper=_capac_ arcCost=_cost_;
```

```
   read data nodedata into NODES=[_node_] supply_demand=_sd_;

   var flow {<i,j> in ARCS} >= arcLower[i,j] <= arcUpper[i,j];
   min obj = sum {<i,j> in ARCS} arcCost[i,j] * flow[i,j];
   con balance {i in NODES}:
      sum {<(i),j> in ARCS} flow[i,j] - sum {<j,(i)> in ARCS} flow[j,i]
         = supply_demand[i];
   con budgetOn2:
      sum {<i,j> in ARCS: i='2'} arcCost[i,j] * flow[i,j] <= 50;
   solve with lp / algorithm=ns scale=none logfreq=1;
   print flow;
quit;
%put &_OROPTMODEL_;
```

The optimal solution is displayed in Output 7.5.3.

**Output 7.5.3** Network Simplex Algorithm: Primal Solution Output

**The OPTMODEL Procedure**

| Problem Summary | |
|---|---|
| **Objective Sense** | Minimization |
| **Objective Function** | obj |
| **Objective Type** | Linear |
| | |
| **Number of Variables** | 11 |
| **Bounded Above** | 0 |
| **Bounded Below** | 0 |
| **Bounded Below and Above** | 11 |
| **Free** | 0 |
| **Fixed** | 0 |
| | |
| **Number of Constraints** | 9 |
| **Linear LE (<=)** | 1 |
| **Linear EQ (=)** | 8 |
| **Linear GE (>=)** | 0 |
| **Linear Range** | 0 |
| | |
| **Constraint Coefficients** | 24 |

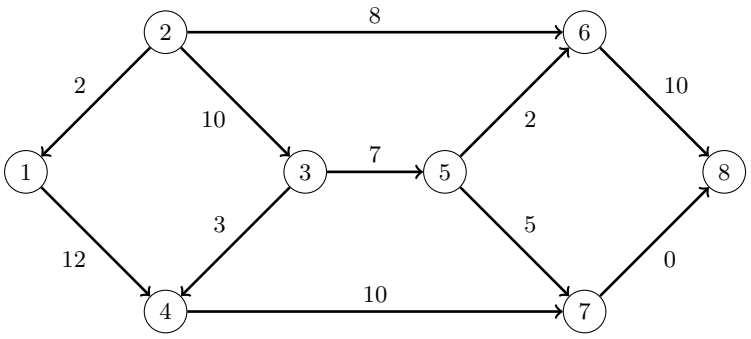
**Output 7.5.3** *continued*

| Solution Summary | |
|---|---|
| **Solver** | LP |
| **Algorithm** | Network Simplex |
| **Objective Function** | obj |
| **Solution Status** | Optimal |
| **Objective Value** | 274 |
| | |
| **Primal Infeasibility** | 0 |
| **Dual Infeasibility** | 4.440892E-16 |
| **Bound Infeasibility** | 0 |
| | |
| **Iterations** | 3 |
| **Iterations2** | 4 |
| **Presolve Time** | 0.00 |
| **Solution Time** | 0.00 |

| [1] | [2] | flow |
|---|---|---|
| **1** | **4** | 12 |
| **2** | **1** | 2 |
| **2** | **3** | 10 |
| **2** | **6** | 8 |
| **3** | **4** | 3 |
| **3** | **5** | 7 |
| **4** | **7** | 10 |
| **5** | **6** | 2 |
| **5** | **7** | 5 |
| **6** | **8** | 10 |
| **7** | **8** | 0 |

The optimal solution is represented graphically in Figure 7.6.

**Figure 7.6** Minimum-Cost Network Flow Problem: Optimal Solution (with Budget Constraint)



The iteration log is displayed in Output 7.5.4. Note that the network simplex algorithm extracts a subnetwork in this case.

**Output 7.5.4** Log: Solution Progress

```
NOTE: There were 11 observations read from the data set WORK.ARCDATA.
NOTE: There were 8 observations read from the data set WORK.NODEDATA.
NOTE: Problem generation will use 16 threads.
NOTE: The problem has 11 variables (0 free, 0 fixed).
NOTE: The problem has 9 linear constraints (1 LE, 8 EQ, 0 GE, 0 range).
NOTE: The problem has 24 linear constraint coefficients.
NOTE: The problem has 0 nonlinear constraints (0 LE, 0 EQ, 0 GE, 0 range).
NOTE: The LP presolver value AUTOMATIC is applied.
NOTE: The LP presolver time is 0.00 seconds.
NOTE: The LP presolver removed 9 variables and 7 constraints.
NOTE: The LP presolver removed 20 constraint coefficients.
NOTE: The presolved problem has 2 variables, 2 constraints, and 4 constraint
      coefficients.
NOTE: The LP solver is called.
NOTE: The Network Simplex algorithm is used.
NOTE: The network has 1 rows (50.00%), 2 columns (100.00%), and 1 component.
NOTE: The network extraction and setup time is 0.00 seconds.
                        Primal         Primal           Dual
      Iteration      Objective  Infeasibility  Infeasibility     Time
              1   2.784000E+02   0.000000E+00   2.000000E+00     0.00
              2   2.724000E+02   0.000000E+00   0.000000E+00     0.00
              3   2.724000E+02   0.000000E+00   0.000000E+00     0.00
NOTE: The Network Simplex solve time is 0.00 seconds.
NOTE: The total Network Simplex solve time is 0.00 seconds.
NOTE: The Dual Simplex algorithm is used.
                           Objective                Entering           Leaving
      Phase Iteration        Value         Time     Variable           Variable
       D 2          1    2.724000E+02         0    balance['6'] (S)
      budgetOn2 (S)
       D 2          2    2.724000E+02         0   flow['3','4']
      flow['2','3']
       D 2          3    2.740000E+02         0
       P 2          4    2.740000E+02         0
NOTE: Optimal.
NOTE: Objective = 274.
NOTE: The Simplex solve time is 0.00 seconds.
STATUS=OK ALGORITHM=NS SOLUTION_STATUS=OPTIMAL OBJECTIVE=274
PRIMAL_INFEASIBILITY=0 DUAL_INFEASIBILITY=4.440892E-16 BOUND_INFEASIBILITY=0
ITERATIONS=3 ITERATIONS2=4 PRESOLVE_TIME=0.00 SOLUTION_TIME=0.00
```

## Example 7.6: Migration to OPTMODEL: Generalized Networks

The following example shows how to use PROC OPTMODEL to solve the example "Generalized Networks: Using the EXCESS= Option" in Chapter 5, "The NETFLOW Procedure" (*SAS/OR User's Guide: Mathematical Programming Legacy Procedures*). The input data sets are the same as in the PROC NETFLOW example.

```
title 'Generalized Networks';

data garcs;
   input _from_ $ _to_ $ _cost_ _mult_;
   datalines;
s1 d1 1 .
s1 d2 8 .
s2 d1 4 2
s2 d2 2 2
s2 d3 1 2
s3 d2 5 0.5
s3 d3 4 0.5
;

data gnodes;
   input _node_ $ _sd_ ;
   datalines;
s1 5
s2 20
s3 10
d1 -5
d2 -10
d3 -20
;
```

The following PROC OPTMODEL statements read the data sets, build the linear programming model, solve the model, and output the optimal solution to a SAS data set called GNETOUT:

```
proc optmodel;
   set <str> NODES;
   num _sd_ {NODES} init 0;
   read data gnodes into NODES=[_node_] _sd_;

   set <str,str> ARCS;
   num _lo_ {ARCS} init 0;
   num _capac_ {ARCS} init .;
   num _cost_ {ARCS};
   num _mult_ {ARCS} init 1;
   read data garcs nomiss into ARCS=[_from_ _to_] _cost_ _mult_;
   NODES = NODES union (union {<i,j> in ARCS} {i,j});
```

```
   var Flow {<i,j> in ARCS} >= _lo_[i,j];
   min obj = sum {<i,j> in ARCS} _cost_[i,j] * Flow[i,j];
   con balance {i in NODES}: sum {<(i),j> in ARCS} Flow[i,j]
      - sum {<j,(i)> in ARCS} _mult_[j,i] * Flow[j,i] = _sd_[i];

   num infinity = constant('BIG');
   /* change equality constraint to le constraint for supply nodes */
   for {i in NODES: _sd_[i] > 0} balance[i].lb = -infinity;

   solve;

   num _supply_ {<i,j> in ARCS} = (if _sd_[i] ne 0 then _sd_[i] else .);
   num _demand_ {<i,j> in ARCS} = (if _sd_[j] ne 0 then -_sd_[j] else .);
   num _fcost_ {<i,j> in ARCS} = _cost_[i,j] * Flow[i,j].sol;

   create data gnetout from [_from_ _to_]
      _cost_ _capac_ _lo_ _mult_ _supply_ _demand_ _flow_=Flow _fcost_;
quit;
```

To solve a generalized network flow problem, the usual balance constraint is altered to include the arc multiplier "_mult_[i,j]" in the second sum. The balance constraint is initially declared as an equality, but to mimic the EXCESS=SUPPLY option in PROC NETFLOW, the sense of this constraint is changed to "≤" by relaxing the constraint's lower bound for supply nodes. The output data set is displayed in Output 7.6.1.

**Output 7.6.1** Optimal Solution with Excess Supply

| Obs | _from_ | _to_ | _cost_ | _capac_ | _lo_ | _mult_ | _supply_ | _demand_ | _flow_ | _fcost_ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | s1 | d1 | 1 | . | 0 | 1.0 | 5 | 5 | 5 | 5 |
| 2 | s1 | d2 | 8 | . | 0 | 1.0 | 5 | 10 | 0 | 0 |
| 3 | s2 | d1 | 4 | . | 0 | 2.0 | 20 | 5 | 0 | 0 |
| 4 | s2 | d2 | 2 | . | 0 | 2.0 | 20 | 10 | 5 | 10 |
| 5 | s2 | d3 | 1 | . | 0 | 2.0 | 20 | 20 | 10 | 10 |
| 6 | s3 | d2 | 5 | . | 0 | 0.5 | 10 | 10 | 0 | 0 |
| 7 | s3 | d3 | 4 | . | 0 | 0.5 | 10 | 20 | 0 | 0 |

The log is displayed in Output 7.6.2.

**Output 7.6.2** OPTMODEL Log

```
NOTE: There were 6 observations read from the data set WORK.GNODES.
NOTE: There were 7 observations read from the data set WORK.GARCS.
NOTE: Problem generation will use 16 threads.
NOTE: The problem has 7 variables (0 free, 0 fixed).
NOTE: The problem has 6 linear constraints (3 LE, 3 EQ, 0 GE, 0 range).
NOTE: The problem has 14 linear constraint coefficients.
NOTE: The problem has 0 nonlinear constraints (0 LE, 0 EQ, 0 GE, 0 range).
NOTE: The OPTMODEL presolver is disabled for linear problems.
NOTE: The LP presolver value AUTOMATIC is applied.
NOTE: The LP presolver time is 0.00 seconds.
NOTE: The LP presolver removed all variables and constraints.
NOTE: Optimal.
NOTE: Objective = 25.
NOTE: The data set WORK.GNETOUT has 7 observations and 10 variables.
```

Now consider the previous example but with a slight modification to the arc multipliers, as in the PROC NETFLOW example:

```
data garcs1;
   input _from_ $ _to_ $ _cost_ _mult_;
   datalines;
s1 d1 1 0.5
s1 d2 8 0.5
s2 d1 4 .
s2 d2 2 .
s2 d3 1 .
s3 d2 5 0.5
s3 d3 4 0.5
;
```

The following PROC OPTMODEL statements are identical to the preceding example, except for the balance constraint. The balance constraint is still initially declared as an equality, but to mimic the PROC NETFLOW EXCESS=DEMAND option, the sense of this constraint is changed to "≥" by relaxing the constraint's upper bound for demand nodes.

```
proc optmodel;
   set <str> NODES;
   num _sd_ {NODES} init 0;
   read data gnodes into NODES=[_node_] _sd_;

   set <str,str> ARCS;
   num _lo_ {ARCS} init 0;
   num _capac_ {ARCS} init .;
   num _cost_ {ARCS};
   num _mult_ {ARCS} init 1;
   read data garcs1 nomiss into ARCS=[_from_ _to_] _cost_ _mult_;
   NODES = NODES union (union {<i,j> in ARCS} {i,j});

   var Flow {<i,j> in ARCS} >= _lo_[i,j];
   for {<i,j> in ARCS: _capac_[i,j] ne .} Flow[i,j].ub = _capac_[i,j];
   min obj = sum {<i,j> in ARCS} _cost_[i,j] * Flow[i,j];
   con balance {i in NODES}: sum {<(i),j> in ARCS} Flow[i,j]
      - sum {<j,(i)> in ARCS} _mult_[j,i] * Flow[j,i] = _sd_[i];

   num infinity = constant('BIG');
   /* change equality constraint to ge constraint */
   for {i in NODES: _sd_[i] < 0} balance[i].ub = infinity;

   solve;

   num _supply_ {<i,j> in ARCS} = (if _sd_[i] ne 0 then _sd_[i] else .);
   num _demand_ {<i,j> in ARCS} = (if _sd_[j] ne 0 then -_sd_[j] else .);
   num _fcost_ {<i,j> in ARCS} = _cost_[i,j] * Flow[i,j].sol;

   create data gnetout1 from [_from_ _to_]
      _cost_ _capac_ _lo_ _mult_ _supply_ _demand_ _flow_=Flow _fcost_;
quit;
```

The output data set is displayed in Output 7.6.3.

**Output 7.6.3** Optimal Solution with Excess Demand

| Obs | _from_ | _to_ | _cost_ | _capac_ | _lo_ | _mult_ | _supply_ | _demand_ | _flow_ | _fcost_ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | s1 | d1 | 1 | . | 0 | 0.5 | 5 | 5 | 5 | 5 |
| 2 | s1 | d2 | 8 | . | 0 | 0.5 | 5 | 10 | 0 | 0 |
| 3 | s2 | d1 | 4 | . | 0 | 1.0 | 20 | 5 | 0 | 0 |
| 4 | s2 | d2 | 2 | . | 0 | 1.0 | 20 | 10 | 5 | 10 |
| 5 | s2 | d3 | 1 | . | 0 | 1.0 | 20 | 20 | 15 | 15 |
| 6 | s3 | d2 | 5 | . | 0 | 0.5 | 10 | 10 | 0 | 0 |
| 7 | s3 | d3 | 4 | . | 0 | 0.5 | 10 | 20 | 10 | 40 |

The log is displayed in Output 7.6.4.

**Output 7.6.4** OPTMODEL Log

```
NOTE: There were 6 observations read from the data set WORK.GNODES.
NOTE: There were 7 observations read from the data set WORK.GARCS1.
NOTE: Problem generation will use 16 threads.
NOTE: The problem has 7 variables (0 free, 0 fixed).
NOTE: The problem has 6 linear constraints (0 LE, 3 EQ, 3 GE, 0 range).
NOTE: The problem has 14 linear constraint coefficients.
NOTE: The problem has 0 nonlinear constraints (0 LE, 0 EQ, 0 GE, 0 range).
NOTE: The OPTMODEL presolver is disabled for linear problems.
NOTE: The LP presolver value AUTOMATIC is applied.
NOTE: The LP presolver time is 0.00 seconds.
NOTE: The LP presolver removed 3 variables and 3 constraints.
NOTE: The LP presolver removed 6 constraint coefficients.
NOTE: The presolved problem has 4 variables, 3 constraints, and 8 constraint
      coefficients.
NOTE: The LP solver is called.
NOTE: The Dual Simplex algorithm is used.
                          Objective
      Phase Iteration       Value         Time
       D 2          1    6.500000E+01         0
       D 2          2    7.000000E+01         0
NOTE: Optimal.
NOTE: Objective = 70.
NOTE: The Dual Simplex solve time is 0.00 seconds.
NOTE: The data set WORK.GNETOUT1 has 7 observations and 10 variables.
```

## Example 7.7: Migration to OPTMODEL: Maximum Flow

The following example shows how to use PROC OPTMODEL to solve the example "Maximum Flow Problem" in Chapter 5, "The NETFLOW Procedure" (*SAS/OR User's Guide: Mathematical Programming Legacy Procedures*). The input data set is the same as in that example.

```
title 'Maximum Flow Problem';

data arcs;
   input _from_ $ _to_ $ _cost_ _capac_;
   datalines;
S a . .
S b . .
a c 1 7
b c 2 9
a d 3 5
b d 4 8
c e 5 15
d f 6 20
e g 7 11
f g 8 6
e h 9 12
f h 10 4
g T . .
h T . .
;
```

The following PROC OPTMODEL statements read the data sets, build the linear programming model, solve the model, and output the optimal solution to a SAS data set called GOUT3:

```
proc optmodel;
   str source = 'S';
   str sink = 'T';

   set <str> NODES;
   num _supdem_ {i in NODES} = (if i in {source, sink} then . else 0);

   set <str,str> ARCS;
   num _lo_ {ARCS} init 0;
   num _capac_ {ARCS} init .;
   num _cost_ {ARCS} init 0;
   read data arcs nomiss into ARCS=[_from_ _to_] _cost_ _capac_;
   NODES = (union {<i,j> in ARCS} {i,j});

   var Flow {<i,j> in ARCS} >= _lo_[i,j];
   for {<i,j> in ARCS: _capac_[i,j] ne .} Flow[i,j].ub = _capac_[i,j];
   max obj = sum {<i,j> in ARCS: j = sink} Flow[i,j];
   con balance {i in NODES diff {source, sink}}:
      sum {<(i),j> in ARCS} Flow[i,j]
      - sum {<j,(i)> in ARCS} Flow[j,i] = _supdem_[i];

   solve;

   num _supply_ {<i,j> in ARCS} =
      (if _supdem_[i] ne 0 then _supdem_[i] else .);
   num _demand_ {<i,j> in ARCS} =
      (if _supdem_[j] ne 0 then -_supdem_[j] else .);
   num _fcost_ {<i,j> in ARCS} = _cost_[i,j] * Flow[i,j].sol;

   create data gout3 from [_from_ _to_]
      _cost_ _capac_ _lo_ _supply_ _demand_ _flow_=Flow _fcost_;
quit;
```

To solve a maximum flow problem, you solve a network flow problem that has a zero supply or demand at all nodes other than the source and sink nodes, as specified in the declaration of the _SUPDEM_ numeric parameter and the balance constraint. The objective declaration uses the logical condition J = SINK to maximize the flow into the sink node. The output data set is displayed in Output 7.7.1.

**Output 7.7.1** Optimal Solution

| Obs | _from_ | _to_ | _cost_ | _capac_ | _lo_ | _supply_ | _demand_ | _flow_ | _fcost_ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | S | a | 0 | . | 0 | . | . | 12 | 0 |
| 2 | S | b | 0 | . | 0 | . | . | 13 | 0 |
| 3 | a | c | 1 | 7 | 0 | . | . | 7 | 7 |
| 4 | b | c | 2 | 9 | 0 | . | . | 8 | 16 |
| 5 | a | d | 3 | 5 | 0 | . | . | 5 | 15 |
| 6 | b | d | 4 | 8 | 0 | . | . | 5 | 20 |
| 7 | c | e | 5 | 15 | 0 | . | . | 15 | 75 |
| 8 | d | f | 6 | 20 | 0 | . | . | 10 | 60 |
| 9 | e | g | 7 | 11 | 0 | . | . | 11 | 77 |
| 10 | f | g | 8 | 6 | 0 | . | . | 6 | 48 |
| 11 | e | h | 9 | 12 | 0 | . | . | 4 | 36 |
| 12 | f | h | 10 | 4 | 0 | . | . | 4 | 40 |
| 13 | g | T | 0 | . | 0 | . | . | 17 | 0 |
| 14 | h | T | 0 | . | 0 | . | . | 8 | 0 |

The log is displayed in Output 7.7.2.

**Output 7.7.2** OPTMODEL Log

```
NOTE: There were 14 observations read from the data set WORK.ARCS.
NOTE: Problem generation will use 16 threads.
NOTE: The problem has 14 variables (0 free, 0 fixed).
NOTE: The problem has 8 linear constraints (0 LE, 8 EQ, 0 GE, 0 range).
NOTE: The problem has 24 linear constraint coefficients.
NOTE: The problem has 0 nonlinear constraints (0 LE, 0 EQ, 0 GE, 0 range).
NOTE: The OPTMODEL presolver is disabled for linear problems.
NOTE: The problem is a pure network instance. The ALGORITHM=NETWORK option is
      recommended for solving problems with this structure.
NOTE: The LP presolver value AUTOMATIC is applied.
NOTE: The LP presolver time is 0.00 seconds.
NOTE: The LP presolver removed 9 variables and 5 constraints.
NOTE: The LP presolver removed 18 constraint coefficients.
NOTE: The presolved problem has 5 variables, 3 constraints, and 6 constraint
      coefficients.
NOTE: The LP solver is called.
NOTE: The Dual Simplex algorithm is used.
                            Objective
      Phase Iteration         Value         Time
       D 2          1    3.300000E+01         0
       P 2          5    2.500000E+01         0
NOTE: Optimal.
NOTE: Objective = 25.
NOTE: The Dual Simplex solve time is 0.00 seconds.
NOTE: The data set WORK.GOUT3 has 14 observations and 9 variables.
```

## Example 7.8: Migration to OPTMODEL: Production, Inventory, Distribution

The following example shows how to use PROC OPTMODEL to solve the example "Production, Inventory, Distribution Problem" in Chapter 5, "The NETFLOW Procedure" (*SAS/OR User's Guide: Mathematical Programming Legacy Procedures*). The input data sets are the same as in that example.

```
title 'Minimum-Cost Flow Problem';
title2 'Production Planning/Inventory/Distribution';

data node0;
   input _node_ $ _supdem_ ;
   datalines;
fact1_1 1000
fact2_1 850
fact1_2 1000
fact2_2 1500
shop1_1 -900
shop2_1 -900
shop1_2 -900
shop2_2 -1450
;

data arc0;
   input _tail_ $ _head_ $ _cost_ _capac_ _lo_
      diagonal factory key_id $10. mth_made $ _name_&$17.;
   datalines;
fact1_1  f1_mar_1 127.9   500 50 19 1 production March prod f1 19 mar
fact1_1  f1_apr_1 78.6    600 50 19 1 production April prod f1 19 apl
fact1_1  f1_may_1 95.1    400 50 19 1 production May   .
f1_mar_1 f1_apr_1 15      50  .  19 1 storage    March .
f1_apr_1 f1_may_1 12      50  .  19 1 storage    April .
f1_apr_1 f1_mar_1 28      20  .  19 1 backorder  April back f1 19 apl
f1_may_1 f1_apr_1 28      20  .  19 1 backorder  May   back f1 19 may
f1_mar_1 f2_mar_1 11      .   .  19 . f1_to_2    March .
f1_apr_1 f2_apr_1 11      .   .  19 . f1_to_2    April .
f1_may_1 f2_may_1 16      .   .  19 . f1_to_2    May   .
f1_mar_1 shop1_1  -327.65 250 .  19 1 sales      March .
f1_apr_1 shop1_1  -300    250 .  19 1 sales      April .
f1_may_1 shop1_1  -285    250 .  19 1 sales      May   .
f1_mar_1 shop2_1  -362.74 250 .  19 1 sales      March .
f1_apr_1 shop2_1  -300    250 .  19 1 sales      April .
f1_may_1 shop2_1  -245    250 .  19 1 sales      May   .
fact2_1  f2_mar_1 88.0    450 35 19 2 production March prod f2 19 mar
fact2_1  f2_apr_1 62.4    480 35 19 2 production April prod f2 19 apl
fact2_1  f2_may_1 133.8   250 35 19 2 production May   .
f2_mar_1 f2_apr_1 18      30  .  19 2 storage    March .
f2_apr_1 f2_may_1 20      30  .  19 2 storage    April .
f2_apr_1 f2_mar_1 17      15  .  19 2 backorder  April back f2 19 apl
f2_may_1 f2_apr_1 25      15  .  19 2 backorder  May   back f2 19 may
f2_mar_1 f1_mar_1 10      40  .  19 . f2_to_1    March .
f2_apr_1 f1_apr_1 11      40  .  19 . f2_to_1    April .
f2_may_1 f1_may_1 13      40  .  19 . f2_to_1    May   .
```

```
f2_mar_1 shop1_1  -297.4  250 .  19 2 sales      March .
f2_apr_1 shop1_1  -290    250 .  19 2 sales      April .
f2_may_1 shop1_1  -292    250 .  19 2 sales      May   .
f2_mar_1 shop2_1  -272.7  250 .  19 2 sales      March .
f2_apr_1 shop2_1  -312    250 .  19 2 sales      April .
f2_may_1 shop2_1  -299    250 .  19 2 sales      May   .
fact1_2  f1_mar_2 217.9   400 40 25 1 production March prod f1 25 mar
fact1_2  f1_apr_2 174.5   550 50 25 1 production April prod f1 25 apl
fact1_2  f1_may_2 133.3   350 40 25 1 production May   .
f1_mar_2 f1_apr_2 20      40  .  25 1 storage    March .
f1_apr_2 f1_may_2 18      40  .  25 1 storage    April .
f1_apr_2 f1_mar_2 32      30  .  25 1 backorder  April back f1 25 apl
f1_may_2 f1_apr_2 41      15  .  25 1 backorder  May   back f1 25 may
f1_mar_2 f2_mar_2 23      .   .  25 . f1_to_2    March .
f1_apr_2 f2_apr_2 23      .   .  25 . f1_to_2    April .
f1_may_2 f2_may_2 26      .   .  25 . f1_to_2    May   .
f1_mar_2 shop1_2  -559.76 .   .  25 1 sales      March .
f1_apr_2 shop1_2  -524.28 .   .  25 1 sales      April .
f1_may_2 shop1_2  -475.02 .   .  25 1 sales      May   .
f1_mar_2 shop2_2  -623.89 .   .  25 1 sales      March .
f1_apr_2 shop2_2  -549.68 .   .  25 1 sales      April .
f1_may_2 shop2_2  -460.00 .   .  25 1 sales      May   .
fact2_2  f2_mar_2 182.0   650 35 25 2 production March prod f2 25 mar
fact2_2  f2_apr_2 196.7   680 35 25 2 production April prod f2 25 apl
fact2_2  f2_may_2 201.4   550 35 25 2 production May   .
f2_mar_2 f2_apr_2 28      50  .  25 2 storage    March .
f2_apr_2 f2_may_2 38      50  .  25 2 storage    April .
f2_apr_2 f2_mar_2 31      15  .  25 2 backorder  April back f2 25 apl
f2_may_2 f2_apr_2 54      15  .  25 2 backorder  May   back f2 25 may
f2_mar_2 f1_mar_2 20      25  .  25 . f2_to_1    March .
f2_apr_2 f1_apr_2 21      25  .  25 . f2_to_1    April .
f2_may_2 f1_may_2 43      25  .  25 . f2_to_1    May   .
f2_mar_2 shop1_2  -567.83 500 .  25 2 sales      March .
f2_apr_2 shop1_2  -542.19 500 .  25 2 sales      April .
f2_may_2 shop1_2  -461.56 500 .  25 2 sales      May   .
f2_mar_2 shop2_2  -542.83 500 .  25 2 sales      March .
f2_apr_2 shop2_2  -559.19 500 .  25 2 sales      April .
f2_may_2 shop2_2  -489.06 500 .  25 2 sales      May   .
;
```

The following PROC OPTMODEL statements read the data sets, build the linear programming model, solve the model, and output the optimal solution to SAS data sets called ARC1 and NODE2:

```
proc optmodel;
   set <str> NODES;
   num _supdem_ {NODES} init 0;
   read data node0 into NODES=[_node_] _supdem_;

   set <str,str> ARCS;
   num _lo_ {ARCS} init 0;
   num _capac_ {ARCS} init .;
   num _cost_ {ARCS};
   num diagonal {ARCS};
   num factory {ARCS};
   str key_id {ARCS};
   str mth_made {ARCS};
   str _name_ {ARCS};
   read data arc0 nomiss into ARCS=[_tail_ _head_] _lo_ _capac_ _cost_
      diagonal factory key_id mth_made _name_;
   NODES = NODES union (union {<i,j> in ARCS} {i,j});

   var Flow {<i,j> in ARCS} >= _lo_[i,j];
   for {<i,j> in ARCS: _capac_[i,j] ne .} Flow[i,j].ub = _capac_[i,j];
   min obj = sum {<i,j> in ARCS} _cost_[i,j] * Flow[i,j];
   con balance {i in NODES}: sum {<(i),j> in ARCS} Flow[i,j]
      - sum {<j,(i)> in ARCS} Flow[j,i] = _supdem_[i];

   num infinity = constant('BIG');
   num excess = sum {i in NODES} _supdem_[i];
   if (excess > 0) then do;
      /* change equality constraint to le constraint for supply nodes */
      for {i in NODES: _supdem_[i] > 0} balance[i].lb = -infinity;
   end;
   else if (excess < 0) then do;
      /* change equality constraint to ge constraint for demand nodes */
      for {i in NODES: _supdem_[i] < 0} balance[i].ub = infinity;
   end;

   solve;

   num _supply_ {<i,j> in ARCS} =
      (if _supdem_[i] ne 0 then _supdem_[i] else .);
   num _demand_ {<i,j> in ARCS} =
      (if _supdem_[j] ne 0 then -_supdem_[j] else .);
   num _fcost_ {<i,j> in ARCS} = _cost_[i,j] * Flow[i,j].sol;

   create data arc1 from [_tail_ _head_]
      _cost_ _capac_ _lo_ _name_ _supply_ _demand_ _flow_=Flow _fcost_
      _rcost_ =
         (if Flow[_tail_,_head_].rc ne 0 then Flow[_tail_,_head_].rc else .)
      _status_ = Flow.status diagonal factory key_id mth_made;
   create data node2 from [_node_]
      _supdem_ = (if _supdem_[_node_] ne 0 then _supdem_[_node_] else .)
      _dual_ = balance.dual;
quit;
```

The PROC OPTMODEL statements use both single-dimensional (NODES) and multiple-dimensional (ARCS) index sets, which are populated from the corresponding data set variables in the READ DATA statements. The _SUPDEM_, _LO_, and _CAPAC_ parameters are given initial values, and the NOMISS option in the READ DATA statement tells PROC OPTMODEL to read only the nonmissing values from the input data set. The balance constraint is initially declared as an equality, but depending on the total supply or demand, the sense of this constraint is changed to "≤" or "≥" by relaxing the constraint's lower or upper bound, respectively. The ARC1 output data set contains most of the same information as in the NETFLOW example, including reduced cost, basis status, and dual values. The _ANUMB_ and _TNUMB_ values do not apply here.

The PROC PRINT statements are similar to the PROC NETFLOW example:

```
options ls=80 ps=54;
proc print data=arc1 heading=h width=min;
   var _tail_ _head_ _cost_ _capac_ _lo_ _name_
   _supply_ _demand_ _flow_ _fcost_;
   sum _fcost_;
run;
proc print data=arc1 heading=h width=min;
   var _rcost_ _status_ diagonal factory key_id mth_made;
run;
proc print data=node2;
run;
```

The output data sets are displayed in Output 7.8.1.

**Output 7.8.1** Output Data Sets

| Obs | _tail_ | _head_ | _cost_ | _capac_ | _lo_ | _name_ | _supply_ | _demand_ | _flow_ | _fcost_ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | fact1_1 | f1_mar_1 | 127.90 | 500 | 50 | prod f1 19 mar | 1000 | . | 345 | 44125.50 |
| 2 | fact1_1 | f1_apr_1 | 78.60 | 600 | 50 | prod f1 19 apl | 1000 | . | 600 | 47160.00 |
| 3 | fact1_1 | f1_may_1 | 95.10 | 400 | 50 | | 1000 | . | 50 | 4755.00 |
| 4 | f1_mar_1 | f1_apr_1 | 15.00 | 50 | 0 | | . | . | 0 | 0.00 |
| 5 | f1_apr_1 | f1_may_1 | 12.00 | 50 | 0 | | . | . | 50 | 600.00 |
| 6 | f1_apr_1 | f1_mar_1 | 28.00 | 20 | 0 | back f1 19 apl | . | . | 20 | 560.00 |
| 7 | f1_may_1 | f1_apr_1 | 28.00 | 20 | 0 | back f1 19 may | . | . | 0 | 0.00 |
| 8 | f1_mar_1 | f2_mar_1 | 11.00 | . | 0 | | . | . | 0 | 0.00 |
| 9 | f1_apr_1 | f2_apr_1 | 11.00 | . | 0 | | . | . | 30 | 330.00 |
| 10 | f1_may_1 | f2_may_1 | 16.00 | . | 0 | | . | . | 100 | 1600.00 |
| 11 | f1_mar_1 | shop1_1 | -327.65 | 250 | 0 | | . | 900 | 155 | -50785.75 |
| 12 | f1_apr_1 | shop1_1 | -300.00 | 250 | 0 | | . | 900 | 250 | -75000.00 |
| 13 | f1_may_1 | shop1_1 | -285.00 | 250 | 0 | | . | 900 | 0 | 0.00 |
| 14 | f1_mar_1 | shop2_1 | -362.74 | 250 | 0 | | . | 900 | 250 | -90685.00 |
| 15 | f1_apr_1 | shop2_1 | -300.00 | 250 | 0 | | . | 900 | 250 | -75000.00 |
| 16 | f1_may_1 | shop2_1 | -245.00 | 250 | 0 | | . | 900 | 0 | 0.00 |
| 17 | fact2_1 | f2_mar_1 | 88.00 | 450 | 35 | prod f2 19 mar | 850 | . | 290 | 25520.00 |
| 18 | fact2_1 | f2_apr_1 | 62.40 | 480 | 35 | prod f2 19 apl | 850 | . | 480 | 29952.00 |
| 19 | fact2_1 | f2_may_1 | 133.80 | 250 | 35 | | 850 | . | 35 | 4683.00 |
| 20 | f2_mar_1 | f2_apr_1 | 18.00 | 30 | 0 | | . | . | 0 | 0.00 |
| 21 | f2_apr_1 | f2_may_1 | 20.00 | 30 | 0 | | . | . | 15 | 300.00 |
| 22 | f2_apr_1 | f2_mar_1 | 17.00 | 15 | 0 | back f2 19 apl | . | . | 0 | 0.00 |
| 23 | f2_may_1 | f2_apr_1 | 25.00 | 15 | 0 | back f2 19 may | . | . | 0 | 0.00 |
| 24 | f2_mar_1 | f1_mar_1 | 10.00 | 40 | 0 | | . | . | 40 | 400.00 |
| 25 | f2_apr_1 | f1_apr_1 | 11.00 | 40 | 0 | | . | . | 0 | 0.00 |
| 26 | f2_may_1 | f1_may_1 | 13.00 | 40 | 0 | | . | . | 0 | 0.00 |
| 27 | f2_mar_1 | shop1_1 | -297.40 | 250 | 0 | | . | 900 | 250 | -74350.00 |
| 28 | f2_apr_1 | shop1_1 | -290.00 | 250 | 0 | | . | 900 | 245 | -71050.00 |
| 29 | f2_may_1 | shop1_1 | -292.00 | 250 | 0 | | . | 900 | 0 | 0.00 |
| 30 | f2_mar_1 | shop2_1 | -272.70 | 250 | 0 | | . | 900 | 0 | 0.00 |
| 31 | f2_apr_1 | shop2_1 | -312.00 | 250 | 0 | | . | 900 | 250 | -78000.00 |
| 32 | f2_may_1 | shop2_1 | -299.00 | 250 | 0 | | . | 900 | 150 | -44850.00 |
| 33 | fact1_2 | f1_mar_2 | 217.90 | 400 | 40 | prod f1 25 mar | 1000 | . | 400 | 87160.00 |
| 34 | fact1_2 | f1_apr_2 | 174.50 | 550 | 50 | prod f1 25 apl | 1000 | . | 550 | 95975.00 |
| 35 | fact1_2 | f1_may_2 | 133.30 | 350 | 40 | | 1000 | . | 40 | 5332.00 |
| 36 | f1_mar_2 | f1_apr_2 | 20.00 | 40 | 0 | | . | . | 0 | 0.00 |
| 37 | f1_apr_2 | f1_may_2 | 18.00 | 40 | 0 | | . | . | 0 | 0.00 |
| 38 | f1_apr_2 | f1_mar_2 | 32.00 | 30 | 0 | back f1 25 apl | . | . | 30 | 960.00 |
| 39 | f1_may_2 | f1_apr_2 | 41.00 | 15 | 0 | back f1 25 may | . | . | 15 | 615.00 |
| 40 | f1_mar_2 | f2_mar_2 | 23.00 | . | 0 | | . | . | 0 | 0.00 |
| 41 | f1_apr_2 | f2_apr_2 | 23.00 | . | 0 | | . | . | 0 | 0.00 |
| 42 | f1_may_2 | f2_may_2 | 26.00 | . | 0 | | . | . | 0 | 0.00 |
| 43 | f1_mar_2 | shop1_2 | -559.76 | . | 0 | | . | 900 | 0 | 0.00 |
| 44 | f1_apr_2 | shop1_2 | -524.28 | . | 0 | | . | 900 | 0 | 0.00 |
| 45 | f1_may_2 | shop1_2 | -475.02 | . | 0 | | . | 900 | 25 | -11875.50 |
| 46 | f1_mar_2 | shop2_2 | -623.89 | . | 0 | | . | 1450 | 455 | -283869.95 |
| 47 | f1_apr_2 | shop2_2 | -549.68 | . | 0 | | . | 1450 | 535 | -294078.80 |
| 48 | f1_may_2 | shop2_2 | -460.00 | . | 0 | | . | 1450 | 0 | 0.00 |

**Output 7.8.1** *continued*

| Obs | _tail_ | _head_ | _cost_ | _capac_ | _lo_ | _name_ | _supply_ | _demand_ | _flow_ | _fcost_ |
|---|---|---|---|---|---|---|---|---|---|---|
| **49** | fact2_2 | f2_mar_2 | 182.00 | 650 | 35 | prod f2 25 mar | 1500 | . | 645 | 117390.00 |
| **50** | fact2_2 | f2_apr_2 | 196.70 | 680 | 35 | prod f2 25 apl | 1500 | . | 680 | 133756.00 |
| **51** | fact2_2 | f2_may_2 | 201.40 | 550 | 35 | | 1500 | . | 35 | 7049.00 |
| **52** | f2_mar_2 | f2_apr_2 | 28.00 | 50 | 0 | | . | . | 0 | 0.00 |
| **53** | f2_apr_2 | f2_may_2 | 38.00 | 50 | 0 | | . | . | 0 | 0.00 |
| **54** | f2_apr_2 | f2_mar_2 | 31.00 | 15 | 0 | back f2 25 apl | . | . | 0 | 0.00 |
| **55** | f2_may_2 | f2_apr_2 | 54.00 | 15 | 0 | back f2 25 may | . | . | 15 | 810.00 |
| **56** | f2_mar_2 | f1_mar_2 | 20.00 | 25 | 0 | | . | . | 25 | 500.00 |
| **57** | f2_apr_2 | f1_apr_2 | 21.00 | 25 | 0 | | . | . | 0 | 0.00 |
| **58** | f2_may_2 | f1_may_2 | 43.00 | 25 | 0 | | . | . | 0 | 0.00 |
| **59** | f2_mar_2 | shop1_2 | -567.83 | 500 | 0 | | . | 900 | 500 | -283915.00 |
| **60** | f2_apr_2 | shop1_2 | -542.19 | 500 | 0 | | . | 900 | 375 | -203321.25 |
| **61** | f2_may_2 | shop1_2 | -461.56 | 500 | 0 | | . | 900 | 0 | 0.00 |
| **62** | f2_mar_2 | shop2_2 | -542.83 | 500 | 0 | | . | 1450 | 120 | -65139.60 |
| **63** | f2_apr_2 | shop2_2 | -559.19 | 500 | 0 | | . | 1450 | 320 | -178940.80 |
| **64** | f2_may_2 | shop2_2 | -489.06 | 500 | 0 | | . | 1450 | 20 | -9781.20 |
| | | | | | | | | | | **-1281110.35** |

**Output 7.8.1** *continued*

| Obs | _rcost_ | _status_ | diagonal | factory | key_id | mth_made |
|---|---|---|---|---|---|---|
| **1** | . | B | 19 | 1 | production | March |
| **2** | -0.65 | U | 19 | 1 | production | April |
| **3** | 0.85 | L | 19 | 1 | production | May |
| **4** | 63.65 | L | 19 | 1 | storage | March |
| **5** | -3.00 | U | 19 | 1 | storage | April |
| **6** | -20.65 | U | 19 | 1 | backorder | April |
| **7** | 43.00 | L | 19 | 1 | backorder | May |
| **8** | 50.90 | L | 19 | . | f1_to_2 | March |
| **9** | . | B | 19 | . | f1_to_2 | April |
| **10** | . | B | 19 | . | f1_to_2 | May |
| **11** | . | B | 19 | 1 | sales | March |
| **12** | -21.00 | U | 19 | 1 | sales | April |
| **13** | 9.00 | L | 19 | 1 | sales | May |
| **14** | -46.09 | U | 19 | 1 | sales | March |
| **15** | -32.00 | U | 19 | 1 | sales | April |
| **16** | 38.00 | L | 19 | 1 | sales | May |
| **17** | . | B | 19 | 2 | production | March |
| **18** | -27.85 | U | 19 | 2 | production | April |
| **19** | 23.55 | L | 19 | 2 | production | May |
| **20** | 15.75 | L | 19 | 2 | storage | March |
| **21** | . | B | 19 | 2 | storage | April |
| **22** | 19.25 | L | 19 | 2 | backorder | April |
| **23** | 45.00 | L | 19 | 2 | backorder | May |
| **24** | -29.90 | U | 19 | . | f2_to_1 | March |
| **25** | 22.00 | L | 19 | . | f2_to_1 | April |
| **26** | 29.00 | L | 19 | . | f2_to_1 | May |
| **27** | -9.65 | U | 19 | 2 | sales | March |
| **28** | . | B | 19 | 2 | sales | April |
| **29** | 18.00 | L | 19 | 2 | sales | May |
| **30** | 4.05 | L | 19 | 2 | sales | March |
| **31** | -33.00 | U | 19 | 2 | sales | April |
| **32** | . | B | 19 | 2 | sales | May |
| **33** | -45.16 | U | 25 | 1 | production | March |
| **34** | -14.35 | U | 25 | 1 | production | April |
| **35** | 2.11 | L | 25 | 1 | production | May |
| **36** | 94.21 | L | 25 | 1 | storage | March |
| **37** | 75.66 | L | 25 | 1 | storage | April |
| **38** | -42.21 | U | 25 | 1 | backorder | April |
| **39** | -16.66 | U | 25 | 1 | backorder | May |
| **40** | 104.06 | L | 25 | . | f1_to_2 | March |
| **41** | 13.49 | L | 25 | . | f1_to_2 | April |
| **42** | 28.96 | L | 25 | . | f1_to_2 | May |
| **43** | 47.13 | L | 25 | 1 | sales | March |
| **44** | 8.40 | L | 25 | 1 | sales | April |
| **45** | . | B | 25 | 1 | sales | May |
| **46** | . | B | 25 | 1 | sales | March |
| **47** | . | B | 25 | 1 | sales | April |
| **48** | 32.02 | L | 25 | 1 | sales | May |

**Output 7.8.1** *continued*

| Obs | _rcost_ | _status_ | diagonal | factory | key_id | mth_made |
|---|---|---|---|---|---|---|
| **49** | . | B | 25 | 2 | production | March |
| **50** | -1.66 | U | 25 | 2 | production | April |
| **51** | 73.17 | L | 25 | 2 | production | May |
| **52** | 11.64 | L | 25 | 2 | storage | March |
| **53** | 108.13 | L | 25 | 2 | storage | April |
| **54** | 47.36 | L | 25 | 2 | backorder | April |
| **55** | -16.13 | U | 25 | 2 | backorder | May |
| **56** | -61.06 | U | 25 | . | f2_to_1 | March |
| **57** | 30.51 | L | 25 | . | f2_to_1 | April |
| **58** | 40.04 | L | 25 | . | f2_to_1 | May |
| **59** | -42.00 | U | 25 | 2 | sales | March |
| **60** | . | B | 25 | 2 | sales | April |
| **61** | 10.50 | L | 25 | 2 | sales | May |
| **62** | . | B | 25 | 2 | sales | March |
| **63** | . | B | 25 | 2 | sales | April |
| **64** | . | B | 25 | 2 | sales | May |

| Obs | _node_ | _supdem_ | _dual_ |
|---|---|---|---|
| **1** | fact1_1 | 1000 | 0.00 |
| **2** | fact2_1 | 850 | 0.00 |
| **3** | fact1_2 | 1000 | 0.00 |
| **4** | fact2_2 | 1500 | 0.00 |
| **5** | shop1_1 | -900 | 199.75 |
| **6** | shop2_1 | -900 | 188.75 |
| **7** | shop1_2 | -900 | 343.83 |
| **8** | shop2_2 | -1450 | 360.83 |
| **9** | f1_mar_1 | . | -127.90 |
| **10** | f1_apr_1 | . | -79.25 |
| **11** | f1_may_1 | . | -94.25 |
| **12** | f2_mar_1 | . | -88.00 |
| **13** | f2_apr_1 | . | -90.25 |
| **14** | f2_may_1 | . | -110.25 |
| **15** | f1_mar_2 | . | -263.06 |
| **16** | f1_apr_2 | . | -188.85 |
| **17** | f1_may_2 | . | -131.19 |
| **18** | f2_mar_2 | . | -182.00 |
| **19** | f2_apr_2 | . | -198.36 |
| **20** | f2_may_2 | . | -128.23 |

The log is displayed in Output 7.8.2.

**Output 7.8.2** OPTMODEL Log

```
NOTE: There were 8 observations read from the data set WORK.NODE0.
NOTE: There were 64 observations read from the data set WORK.ARC0.
NOTE: Problem generation will use 16 threads.
NOTE: The problem has 64 variables (0 free, 0 fixed).
NOTE: The problem has 20 linear constraints (4 LE, 16 EQ, 0 GE, 0 range).
NOTE: The problem has 128 linear constraint coefficients.
NOTE: The problem has 0 nonlinear constraints (0 LE, 0 EQ, 0 GE, 0 range).
NOTE: The OPTMODEL presolver is disabled for linear problems.
NOTE: The LP presolver value AUTOMATIC is applied.
NOTE: The LP presolver time is 0.00 seconds.
NOTE: The LP presolver removed 0 variables and 0 constraints.
NOTE: The LP presolver removed 0 constraint coefficients.
NOTE: The presolved problem has 64 variables, 20 constraints, and 128
      constraint coefficients.
NOTE: The LP solver is called.
NOTE: The Dual Simplex algorithm is used.
                          Objective
      Phase Iteration        Value         Time
       D 2          1   -4.333564E+06         0
       D 2         32   -1.281110E+06         0
NOTE: Optimal.
NOTE: Objective = -1281110.35.
NOTE: The Dual Simplex solve time is 0.00 seconds.
NOTE: The data set WORK.ARC1 has 64 observations and 16 variables.
NOTE: The data set WORK.NODE2 has 20 observations and 3 variables.
```

## Example 7.9: Migration to OPTMODEL: Shortest Path

The following example shows how to use PROC OPTMODEL to solve the example "Shortest Path Problem" in Chapter 5, "The NETFLOW Procedure" (*SAS/OR User's Guide: Mathematical Programming Legacy Procedures*). The input data set is the same as in that example.

```
title 'Shortest Path Problem';
title2 'How to get Hawaiian Pineapples to a London Restaurant';

data aircost1;
   input ffrom&$13. tto&$15. _cost_;
   datalines;
Honolulu       Chicago         105
Honolulu       San Francisco    75
Honolulu       Los Angeles      68
Chicago        Boston           45
Chicago        New York         56
San Francisco  Boston           71
San Francisco  New York         48
San Francisco  Atlanta          63
Los Angeles    New York         44
Los Angeles    Atlanta          57
Boston         Heathrow London  88
New York       Heathrow London  65
Atlanta        Heathrow London  76
;
```

The following PROC OPTMODEL statements read the data sets, build the linear programming model, solve the model, and output the optimal solution to a SAS data set called SPATH:

```
proc optmodel;
   str sourcenode = 'Honolulu';
   str sinknode = 'Heathrow London';

   set <str> NODES;
   num _supdem_ {i in NODES} = (if i = sourcenode then 1
      else if i = sinknode then -1 else 0);

   set <str,str> ARCS;
   num _lo_ {ARCS} init 0;
   num _capac_ {ARCS} init .;
   num _cost_ {ARCS};
   read data aircost1 into ARCS=[ffrom tto] _cost_;
   NODES = (union {<i,j> in ARCS} {i,j});

   var Flow {<i,j> in ARCS} >= _lo_[i,j];
   min obj = sum {<i,j> in ARCS} _cost_[i,j] * Flow[i,j];
   con balance {i in NODES}: sum {<(i),j> in ARCS} Flow[i,j]
      - sum {<j,(i)> in ARCS} Flow[j,i] = _supdem_[i];
   solve;

   num _supply_ {<i,j> in ARCS} =
      (if _supdem_[i] ne 0 then _supdem_[i] else .);
   num _demand_ {<i,j> in ARCS} =
      (if _supdem_[j] ne 0 then -_supdem_[j] else .);
   num _fcost_ {<i,j> in ARCS} = _cost_[i,j] * Flow[i,j].sol;

   create data spath from [ffrom tto]
      _cost_ _capac_ _lo_ _supply_ _demand_ _flow_=Flow _fcost_
      _rcost_=(if Flow[ffrom,tto].rc ne 0 then Flow[ffrom,tto].rc else .)
      _status_=Flow.status;
quit;
```

The statements use both single-dimensional (NODES) and multiple-dimensional (ARCS) index sets. The ARCS index set is populated from the ffrom and tto data set variables in the READ DATA statement. To solve a shortest path problem, you solve a minimum-cost network flow problem that has a supply of one unit at the source node, a demand of one unit at the sink node, and zero supply or demand at all other nodes, as specified in the declaration of the _SUPDEM_ numeric parameter. The SPATH output data set contains most of the same information as in the PROC NETFLOW example, including reduced cost and basis status. The _ANUMB_ and _TNUMB_ values do not apply here.

The PROC PRINT statements are similar to the PROC NETFLOW example:

```
proc print data=spath;
   sum _fcost_;
run;
```

The output is displayed in Output 7.9.1.

**Output 7.9.1** Output Data Set

| Obs | ffrom | tto | _cost_ | _capac_ | _lo_ | _supply_ | _demand_ | _flow_ | _fcost_ | _rcost_ | _status_ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | Honolulu | Chicago | 105 | . | 0 | 1 | . | 0 | 0 | . | B |
| 2 | Honolulu | San Francisco | 75 | . | 0 | 1 | . | 0 | 0 | . | B |
| 3 | Honolulu | Los Angeles | 68 | . | 0 | 1 | . | 1 | 68 | . | B |
| 4 | Chicago | Boston | 45 | . | 0 | . | . | 0 | 0 | 61 | L |
| 5 | Chicago | New York | 56 | . | 0 | . | . | 0 | 0 | 49 | L |
| 6 | San Francisco | Boston | 71 | . | 0 | . | . | 0 | 0 | 57 | L |
| 7 | San Francisco | New York | 48 | . | 0 | . | . | 0 | 0 | 11 | L |
| 8 | San Francisco | Atlanta | 63 | . | 0 | . | . | 0 | 0 | 37 | L |
| 9 | Los Angeles | New York | 44 | . | 0 | . | . | 1 | 44 | . | B |
| 10 | Los Angeles | Atlanta | 57 | . | 0 | . | . | 0 | 0 | 24 | L |
| 11 | Boston | Heathrow London | 88 | . | 0 | . | 1 | 0 | 0 | . | B |
| 12 | New York | Heathrow London | 65 | . | 0 | . | 1 | 1 | 65 | . | B |
| 13 | Atlanta | Heathrow London | 76 | . | 0 | . | 1 | 0 | 0 | . | B |
| | | | | | | | | | 177 | | |

The log is displayed in Output 7.9.2.

**Output 7.9.2** OPTMODEL Log

```
NOTE: There were 13 observations read from the data set WORK.AIRCOST1.
NOTE: Problem generation will use 16 threads.
NOTE: The problem has 13 variables (0 free, 0 fixed).
NOTE: The problem has 8 linear constraints (0 LE, 8 EQ, 0 GE, 0 range).
NOTE: The problem has 26 linear constraint coefficients.
NOTE: The problem has 0 nonlinear constraints (0 LE, 0 EQ, 0 GE, 0 range).
NOTE: The OPTMODEL presolver is disabled for linear problems.
NOTE: The problem is a pure network instance. The ALGORITHM=NETWORK option is
      recommended for solving problems with this structure.
NOTE: The LP presolver value AUTOMATIC is applied.
NOTE: The LP presolver time is 0.00 seconds.
NOTE: The LP presolver removed 7 variables and 7 constraints.
NOTE: The LP presolver removed 20 constraint coefficients.
NOTE: The presolved problem has 6 variables, 1 constraints, and 6 constraint
      coefficients.
NOTE: The LP solver is called.
NOTE: The Dual Simplex algorithm is used.
                            Objective
      Phase Iteration         Value         Time
       D 2          1    1.240000E+01         0
       D 2          2    1.770000E+02         0
NOTE: Optimal.
NOTE: Objective = 177.
NOTE: The Dual Simplex solve time is 0.00 seconds.
NOTE: The data set WORK.SPATH has 13 observations and 11 variables.
```

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# Subject Index

# Syntax Index