
# SAS/OR® 14.2 User's Guide: Mathematical Programming

# The Network Solver

This document is an individual chapter from *SAS/OR® 14.2 User's Guide: Mathematical Programming*.

The correct bibliographic citation for this manual is as follows: SAS Institute Inc. 2016. *SAS/OR® 14.2 User's Guide: Mathematical Programming*. Cary, NC: SAS Institute Inc.

**SAS/OR® 14.2 User's Guide: Mathematical Programming**



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November 2016

# Chapter 9
# The Network Solver

## Contents

# Overview: Network Solver

The network solver includes a number of graph theory, combinatorial optimization, and network analysis algorithms. The algorithm classes are listed in Table 9.1.

**Table 9.1** Algorithm Classes in the Network solver

| Algorithm Class | SOLVE WITH NETWORK Option |
|---|---|
| Biconnected components | BICONCOMP |
| Maximal cliques | CLIQUE= |
| Connected components | CONCOMP |
| Cycle detection | CYCLE= |
| Linear assignment (matching) | LINEAR_ASSIGNMENT |
| Minimum-cost network flow | MINCOSTFLOW |
| Minimum cut | MINCUT= |
| Minimum spanning tree | MINSPANTREE |
| Shortest path | SHORTPATH= |
| Transitive closure | TRANSITIVE_CLOSURE |
| Traveling salesman | TSP= |

You can use the network solver to analyze relationships between entities. These relationships are typically defined by using a *graph*. A graph, $G = (N, A)$, is defined over a set $N$ of nodes, and a set $A$ of links. A *node* is an abstract representation of some entity (or object), and an *arc* defines some relationship (or connection) between two nodes. The terms *node* and *vertex* are often interchanged in describing an entity. The term *arc* is often interchanged with the term *edge* or *link* in describing a relationship.

Unlike other solvers that PROC OPTMODEL uses, the network solver operates directly on arrays and sets. You do not need to explicitly define variables, constraints, and objectives to use the network solver. PROC OPTMODEL declares the appropriate objects internally as needed. You specify the names of arrays and sets that define your inputs and outputs as options in the SOLVE WITH NETWORK statement.

# Getting Started: Network Solver

This section shows an introductory example for getting started with the network solver. For more information about the expected input formats and the various algorithms available, see the sections "Details: Network Solver" on page 398 and "Examples: Network Solver" on page 463.

Consider the following road network between a SAS employee's home in Raleigh, NC, and the SAS headquarters in Cary, NC.

In this road network (graph), the links are the roads and the nodes are intersections between roads. For each road, you assign a *link attribute* in the parameter time_to_travel to describe the number of minutes that it takes to drive from one node to another. The following data were collected using Google Maps (Google 2011):

```
data LinkSetInRoadNC10am;
   input start_inter $1-20 end_inter $20-40 miles miles_per_hour;
   datalines;
614CapitalBlvd      Capital/WadeAve      0.6  25
614CapitalBlvd      Capital/US70W        0.6  25
614CapitalBlvd      Capital/US440W       3.0  45
Capital/WadeAve     WadeAve/RaleighExpy  3.0  40
Capital/US70W       US70W/US440W         3.2  60
US70W/US440W        US440W/RaleighExpy   2.7  60
Capital/US440W      US440W/RaleighExpy   6.7  60
US440W/RaleighExpy  RaleighExpy/US40W    3.0  60
WadeAve/RaleighExpy RaleighExpy/US40W    3.0  60
RaleighExpy/US40W   US40W/HarrisonAve    1.3  55
US40W/HarrisonAve   SASCampusDrive       0.5  25
;
```

Using the network solver, you want to find the route that yields the shortest path between home (614 Capital Blvd) and the SAS headquarters (SAS Campus Drive). This can be done by using the SHORTPATH= option as follows:

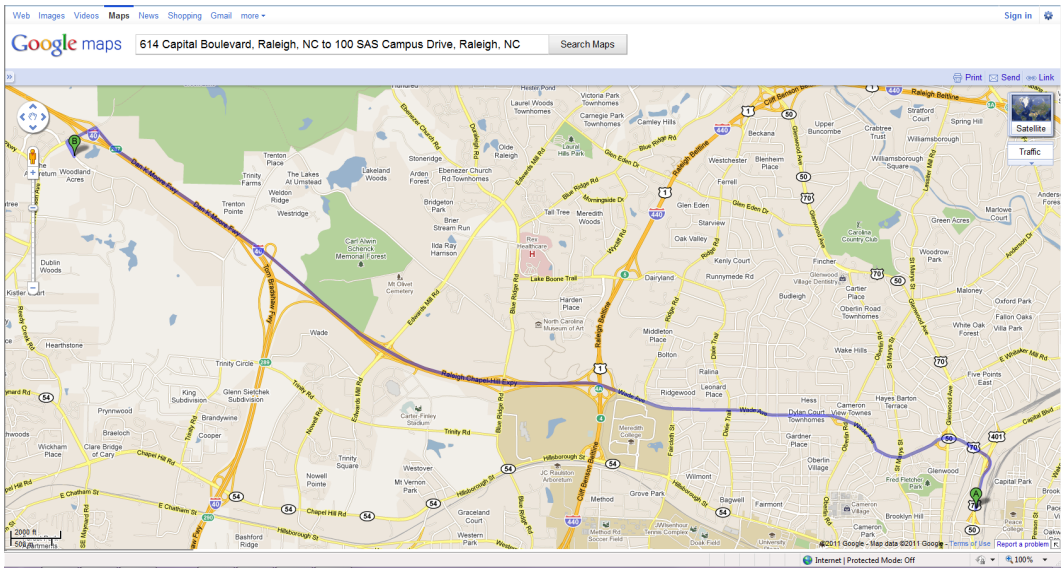
```
proc optmodel;
   set<str,str> LINKS;
   num miles{LINKS};
   num miles_per_hour{LINKS};
   num time_to_travel{<i,j> in LINKS} = miles[i,j]/ miles_per_hour[i,j] * 60;
   read data LinkSetInRoadNC10am into
      LINKS=[start_inter end_inter]
      miles miles_per_hour
   ;
   /* You can compute paths between many pairs of source and destination,
      so these parameters are declared as sets */
   set HOME = /"614CapitalBlvd"/;
   set WORK = /"SASCampusDrive"/;

   /* The path is stored as a set of: Start, End, Sequence, Tail, Head */
   set<str,str,num,str,str> PATH;

   solve with network /
      links     = ( weight = time_to_travel )
      shortpath = ( source = HOME
                    sink   = WORK )
      out       = ( sppaths = PATH )
   ;
   create data ShortPath from [s t order start_inter end_inter]=PATH
      time_to_travel[start_inter,end_inter];
quit;
```

For more information about shortest path algorithms in the network solver, see the section "Shortest Path" on page 436. Figure 9.1 displays the output data set ShortPath, which shows the best route to take to minimize travel time at 10:00 a.m. This route is also shown in Google Maps in Figure 9.2.

**Figure 9.1** Shortest Path for Road Network at 10:00 A.M.

| order | start_inter | end_inter | time_to_travel |
|---|---|---|---|
| 1 | 614CapitalBlvd | Capital/WadeAve | 1.4400 |
| 2 | Capital/WadeAve | WadeAve/RaleighExpy | 4.5000 |
| 3 | WadeAve/RaleighExpy | RaleighExpy/US40W | 3.0000 |
| 4 | RaleighExpy/US40W | US40W/HarrisonAve | 1.4182 |
| 5 | US40W/HarrisonAve | SASCampusDrive | 1.2000 |
| | | | **11.5582** |

**Figure 9.2** Shortest Path for Road Network at 10:00 A.M. in Google Maps




Now suppose that it is rush hour (5:00 p.m.) and the time to traverse the roads has changed because of traffic patterns. You want to find the route that is the shortest path for going home from SAS headquarters under different speed assumptions due to traffic.

The following statements are similar to the first network solver run, except that one *miles_per_hour* value is modified and the SOURCE= and SINK= option values are reversed:

```
proc optmodel;
   set<str,str> LINKS;
   num miles{LINKS};
   num miles_per_hour{LINKS};
   num time_to_travel{<i,j> in LINKS} = miles[i,j]/ miles_per_hour[i,j] * 60;
   read data LinkSetInRoadNC10am into
      LINKS=[start_inter end_inter]
      miles miles_per_hour
   ;
   /* high traffic */
   miles_per_hour['Capital/WadeAve','WadeAve/RaleighExpy'] = 25;

   /* You can compute paths between many pairs of source and destination,
      so these parameters are declared as sets */
   set HOME = /"614CapitalBlvd"/;
   set WORK = /"SASCampusDrive"/;

   /* The path is stored as a set of: Start, End, Sequence, Tail, Head */
   set<str,str,num,str,str> PATH;

   solve with network /
      links     = ( weight = time_to_travel )
      shortpath = ( source = WORK
                    sink   = HOME )
      out       = ( sppaths = PATH )
   ;
   create data ShortPath from [s t order start_inter end_inter]=PATH
      time_to_travel[start_inter,end_inter];
quit;
```

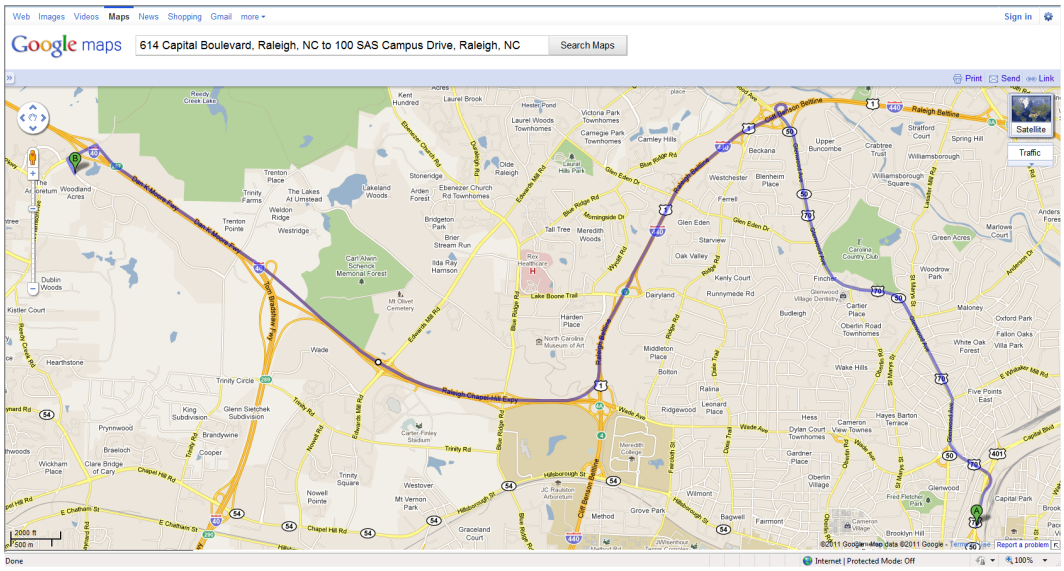
Now, the output data set ShortPath, shown in Figure 9.3, shows the best route for going home at 5:00 p.m. Because the traffic on Wade Avenue is usually heavy at this time of day, the route home is different from the route to work.

**Figure 9.3** Shortest Path for Road Network at 5:00 P.M.

| order | start_inter | end_inter | time_to_travel |
|---|---|---|---|
| 1 | US40W/HarrisonAve | SASCampusDrive | 1.2000 |
| 2 | RaleighExpy/US40W | US40W/HarrisonAve | 1.4182 |
| 3 | US440W/RaleighExpy | RaleighExpy/US40W | 3.0000 |
| 4 | US70W/US440W | US440W/RaleighExpy | 2.7000 |
| 5 | Capital/US70W | US70W/US440W | 3.2000 |
| 6 | 614CapitalBlvd | Capital/US70W | 1.4400 |
| | | | **12.9582** |

This new route is shown in Google Maps in Figure 9.4.

**Figure 9.4** Shortest Path for Road Network at 5:00 P.M. in Google Maps



# Syntax: Network Solver

**SOLVE WITH NETWORK /**

*General and Diagnostic Options:*
< **GRAPH_DIRECTION=DIRECTED | UNDIRECTED** >
< **INCLUDE_SELFLINK** >
< **LOGFREQ=***number* >
< **LOGLEVEL=***number* | *string* >
< **MAXTIME=***number* >
< **TIMETYPE=***number* | *string* >

*Data Input and Output Options:*
< **LINKS=(** *suboptions* **)** >
< **NODES=(** *suboptions* **)** >
< **OUT=(** *suboptions* **)** >
< **SUBGRAPH=(** *suboptions* **)** >

*Algorithm Options:*
< **BICONCOMP**< **=()** > >
< **CLIQUE**< **=(** *suboption* **)** > >
< **CONCOMP**< **=(** *suboption* **)** > >
< **CYCLE**< **=(** *suboptions* **)** > >
< **LINEAR_ASSIGNMENT**< **=()** > >
< **MINCOSTFLOW**< **=()** > >
< **MINCUT**< **=(** *suboptions* **)** > >
< **MINSPANTREE**< **=()** > >
< **SHORTPATH**< **=(** *suboptions* **)** > >
< **TRANSITIVE_CLOSURE**< **=()** > >
< **TSP**< **=(** *suboptions* **)** > > **;**

There are three types of SOLVE WITH NETWORK statement options:

- General and diagnostic options have the same meaning for multiple algorithms.
- Data input and output options, such as the LINKS=, NODES=, and OUT= options, control the names of the sets and variables that the network solver uses to build the graph and that the algorithms use for output.
- Algorithm options select an algorithm to run, and where available, provide further algorithm-specific configuration directives.

The section "Functional Summary" on page 381 provides a quick reference for each of the suboptions for each option. Each option is then described in more detail in its own section, in alphabetical order.

## Functional Summary

Table 9.2 summarizes the options and suboptions available in the SOLVE WITH NETWORK statement.

**Table 9.2** Functional Summary of SOLVE WITH NETWORK Options

| Description | Option<br>Suboption |
|---|---|
| **General Options** | |
| Specifies directed or undirected graphs | GRAPH_DIRECTION= |
| Includes self links in the graph definition | INCLUDE_SELFLINK= |
| Specifies the iteration log frequency | LOGFREQ= |
| Controls the amount of information that is displayed in the SAS log | LOGLEVEL= |
| Specifies the maximum time spent calculating results | MAXTIME= |
| Specifies whether time units are in CPU time or real time | TIMETYPE= |
| **Input and Output Options** | |
| Groups link-indexed data | LINKS=() |
| Names a set of links to include in the graph definition even if no weights or bounds are available for them | INCLUDE= |
| Specifies the flow lower bound for each link | LOWER= |
| Specifies the flow upper bound for each link | UPPER= |
| Specifies link weights | WEIGHT= |
| Groups node-indexed data | NODES=() |
| Names a set of nodes to include in the graph definition even if no weights are available for them | INCLUDE= |
| Specifies node weights | WEIGHT= |
| Specifies node supply upper bounds in the minimum-cost network flow problem | WEIGHT2= |
| Specifies the input sets that enable you to solve a problem over a subgraph | SUBGRAPH=() |
| Specifies the subset of links to use | LINKS= |
| Specifies the subset of nodes to use | NODES= |
| Specifies the output sets or arrays for each algorithm (see Table 9.4 for which OUT= suboptions you can specify for each algorithm) | OUT=() |
| Specifies the output set for articulation points | ARTPOINTS= |
| Specifies the output set for linear assignment | ASSIGNMENTS= |
| Specifies the array to contain the biconnected component of each link | BICONCOMP= |
| Specifies the output set for cliques | CLIQUES= |
| Specifies the output array for connected components | CONCOMP= |
| Specifies the output set for the cut-sets for minimum cuts | CUTSETS= |
| Specifies the output set for cycles | CYCLES= |
| Specifies the output array for the flow on each link | FLOW= |
| Specifies the output set for the minimum spanning tree (forest) | FOREST= |
| Specifies the output set for the links that remain after the SUBGRAPH= option is applied | LINKS= |
| Specifies the output set for the nodes that remain after the SUBGRAPH= option is applied | NODES= |
| Specifies the output array for the node order in the traveling salesman problem | ORDER= |

**Table 9.2** *(continued)*

| Description | Option<br>Suboption |
|---|---|
| Specifies the output set for the partitions for minimum cuts | PARTITIONS= |
| Specifies the set to contain the link sequence for each path | SPPATHS= |
| Specifies the numeric array to contain the path weight for each source and sink node pair | SPWEIGHTS= |
| Specifies the output set for the tour in the traveling salesman problem | TOUR= |
| Specifies the set to contain the pairs $(u, v)$ of nodes where $v$ is reachable from $u$ | TRANSCL= |
| **Algorithm Options and Suboptions** | |
| Finds biconnected components and articulation points of an undirected input graph | BICONCOMP |
| Finds maximal cliques in the input graph | CLIQUE= |
| Specifies the maximum number of cliques to return | MAXCLIQUES= |
| Finds the connected components of the input graph | CONCOMP= |
| Specifies the algorithm to use for calculating connected components | ALGORITHM= |
| Finds the cycles (or the existence of a cycle) in the input graph | CYCLE= |
| Specifies the maximum number of cycles to return | MAXCYCLES= |
| Specifies the maximum link count for the cycles to return | MAXLENGTH= |
| Specifies the maximum link weight for the cycles to return | MAXLINKWEIGHT= |
| Specifies the maximum sum of node weights to allow in a cycle | MAXNODEWEIGHT= |
| Specifies the minimum link count for the cycles to return | MINLENGTH= |
| Specifies the minimum link weight for the cycles to return | MINLINKWEIGHT= |
| Specifies the minimum node weight for the cycles to return | MINNODEWEIGHT= |
| Specifies whether to stop after finding the first cycle | MODE= |
| Solves the minimal-cost linear assignment problem | LINEAR_ASSIGNMENT |
| Solves the minimum-cost network flow problem | MINCOSTFLOW |
| Finds the minimum link-weighted cut of an input graph | MINCUT= |
| Specifies the maximum number of cuts to return from the algorithm | MAXNUMCUTS= |
| Specifies the maximum weight of each cut to return from the algorithm | MAXWEIGHT= |
| Solves the minimum link-weighted spanning tree problem on an input graph | MINSPANTREE |
| Calculates shortest paths between sets of nodes on the input graph | SHORTPATH= |
| Specifies the type of output for shortest paths results | PATHS= |
| Specifies the set of sink nodes | SINK= |
| Specifies the set of source nodes | SOURCE= |
| Specifies whether to use weights in calculating shortest paths | USEWEIGHT= |
| Calculates the transitive closure of an input graph | TRANSITIVE_CLOSURE |
| Solves the traveling salesman problem | TSP= |
| Requests that the stopping criterion be based on the absolute objective gap | ABSOBJGAP= |
| Specifies the level of conflict search | CONFLICTSEARCH= |

**Table 9.2** *(continued)*

| Description | Option<br>Suboption |
|---|---|
| Specifies the cutoff value for branch-and-bound node removal | CUTOFF= |
| Specifies the level of cutting planes to be generated by the network solver | CUTSTRATEGY= |
| Emphasizes feasibility or optimality | EMPHASIS= |
| Specifies the initial and primal heuristics level | HEURISTICS= |
| Specifies the maximum number of branch-and-bound nodes to be processed | MAXNODES= |
| Specifies the maximum number of feasible tours to be identified | MAXSOLS= |
| Specifies whether to use a mixed integer linear programming solver | MILP= |
| Specifies the branch-and-bound node selection strategy | NODESEL= |
| Specifies the probing level | PROBE= |
| Requests that the stopping criterion be based on relative objective gap | RELOBJGAP= |
| Specifies the number of simplex iterations to be performed on each variable in the strong branching strategy | STRONGITER= |
| Specifies the number of candidates for the strong branching strategy | STRONGLEN= |
| Requests that the stopping criterion be based on the target objective value | TARGET= |
| Specifies the rule for selecting branching variable | VARSEL= |

Table 9.3 lists the valid GRAPH_DIRECTION= values for each algorithm option in the SOLVE WITH NETWORK statement.

**Table 9.3** Supported Graph Directions by Algorithm

| | Direction | |
|---|---|---|
| **Algorithm** | Undirected | Directed |
| BICONCOMP | x | |
| CLIQUE | x | |
| CONCOMP | x | x |
| CYCLE | x | x |
| LINEAR_ASSIGNMENT | | x |
| MINCOSTFLOW | | x |
| MINCUT | x | |
| MINSPANTREE | x | |
| SHORTPATH | x | x |
| TRANSITIVE_CLOSURE | x | x |
| TSP | x | x |

Table 9.4 indicates, for each algorithm option in the SOLVE WITH NETWORK statement, which output options you can specify, and what their types can be. The types vary depending on whether nodes are of type STRING or NUMBER.

**Table 9.4** Output Suboptions and Types by Algorithm

| Algorithm Option<br>OUT= Suboption | OPTMODEL Type |
|---|---|
| BICONCOMP | |
| ARTPOINTS= | SET<STRING> or SET<NUMBER> |
| BICONCOMP= | NUMBER indexed over links (<NUMBER,NUMBER> or <STRING,STRING>) |
| CLIQUE | |
| CLIQUES= | SET<NUMBER,NUMBER> or SET<NUMBER,STRING> |
| CONCOMP | |
| CONCOMP= | NUMBER indexed over nodes (NUMBER or STRING) |
| CYCLE | |
| CYCLES= | SET<NUMBER,NUMBER,NUMBER> or SET<NUMBER,NUMBER,STRING> |
| LINEAR_ASSIGNMENT | |
| ASSIGNMENTS= | SET<NUMBER,NUMBER> or SET<STRING,STRING> |
| MINCOSTFLOW | |
| FLOW= | NUMBER indexed over links (<NUMBER,NUMBER> or <STRING,STRING>) |
| MINCUT | |
| CUTSETS= | SET<NUMBER,NUMBER,NUMBER> or SET<NUMBER,STRING,STRING> |
| PARTITIONS= | SET<NUMBER,NUMBER> or SET<NUMBER,STRING> |
| MINSPANTREE | |
| FOREST= | SET<NUMBER,NUMBER> or SET<STRING,STRING> |
| SHORTPATH | |
| SPPATHS= | SET<NUMBER,NUMBER,NUMBER,NUMBER,NUMBER> or SET<STRING,STRING,NUMBER,STRING,STRING> |
| SPWEIGHTS= | NUMBER indexed over sink and source node pairs (<NUMBER,NUMBER> or <STRING,STRING>) |
| TRANSITIVE_CLOSURE | |
| CLOSURE= | SET<NUMBER,NUMBER> or SET<STRING,STRING> |
| TSP | |
| ORDER= | NUMBER indexed over nodes (NUMBER or STRING) |
| TOUR= | SET<NUMBER,NUMBER> or SET<STRING,STRING> |

## SOLVE WITH NETWORK Statement

**SOLVE WITH NETWORK** */ < options >* **;**

The SOLVE WITH NETWORK statement invokes the network solver. You can specify the following *options* to define various processing and diagnostic controls, the graph input and output, and the algorithm to run:

## General Options

You can specify the following general options, which have the same meaning for multiple algorithms.

**GRAPH_DIRECTION=DIRECTED | UNDIRECTED**
**DIRECTION=DIRECTED | UNDIRECTED**
specifies directed or undirected graphs.

**Table 9.5** Values for the GRAPH_DIRECTION= Option

| Option Value | Description |
|---|---|
| DIRECTED | Requests a directed graph. In a directed graph, each link $(i, j)$ has a direction that defines how something (for example, information) might flow over that link. In link $(i, j)$, information flows from node $i$ to node $j$ ($i \rightarrow j$). The node $i$ is called the *source* (*tail*) node, and node $j$ is called the *sink* (*head*) node. |
| UNDIRECTED | Requests an undirected graph. In an undirected graph, each link $\{i, j\}$ has no direction and information can flow in either direction. That is, $\{i, j\} = \{j, i\}$. |

By default, GRAPH_DIRECTION=UNDIRECTED.

**INCLUDE_SELFLINK**
includes self links in the graph definition—for example, $(i, i)$—when an input graph is read. By default, when the network solver reads the LINKS= data, it removes all self links.

**LOGFREQ=***number*
controls the frequency with which an algorithm reports progress from its underlying solver. This setting is recognized by the traveling salesman problem and minimum-cost flow algorithms. You can set *number* to 0 to turn off log updates from underlying algorithms.

**LOGLEVEL=***number* | *string*
controls the amount of information that is displayed in the SAS log. This setting sets the log level for all algorithms. Table 9.6 describes the valid values for this option.

**Table 9.6** Values for LOGLEVEL= Option

| *number* | *string* | Description |
|---|---|---|
| 0 | NONE | Turns off all procedure-related messages in the SAS log |
| 1 | BASIC | Displays a basic summary of the input, output, and algorithmic processing |
| 2 | MODERATE | Displays a summary of the input, output, and algorithmic processing |
| 3 | AGGRESSIVE | Displays a detailed summary of the input, output, and algorithmic processing |

By default, LOGLEVEL=BASIC.

**MAXTIME=***number*
specifies the maximum time spent calculating results. The type of time (either CPU time or real time) is determined by the value of the TIMETYPE= option. The value of *number* can be any positive number; the default value is the positive number that has the largest absolute value that can be represented in your operating environment. The clique, cycle, minimum-cost network flow, and traveling salesman problem algorithms recognize the MAXTIME= option.

**TIMETYPE=***number* | *string*
specifies whether CPU time or real time is used for measuring solution times. This affects the MAXTIME= option for each applicable algorithm. Table 9.7 describes the valid values of the TIMETYPE= option.

**Table 9.7** Values for TIMETYPE= Option

| *number* | *string* | **Description** |
|---|---|---|
| 0 | CPU | Specifies units of CPU time |
| 1 | REAL | Specifies units of real time |

By default, TIMETYPE=REAL if the solver is invoked in an OPTMODEL COFOR loop or executes remotely. Otherwise, TIMETYPE=CPU.

## Input and Output Options

The following options enable you to specify the graph to run algorithms on. These options take array and set names. They are known as *identifier expressions* in Chapter 5, "The OPTMODEL Procedure." Also see Table 9.4 for semantic requirements and the section "Input Data for the Network Solver" on page 398 for use cases.

**LINKS=(** *suboptions* **)**
groups link-indexed data. For more information, see the section "Input Data for the Network Solver" on page 398.

You can specify the following *suboptions*:

**INCLUDE=***set-name*
names a set of links to include in the graph definition even if no weights or bounds are available for them. For more information, see "Example 9.1: Articulation Points in a Terrorist Network" on page 463. The set must be numeric, and it must be indexed over a subset of the links of the graph.

**LOWER=***array-name*
specifies the flow lower bound for each link. The array must be numeric, and it must be indexed over a subset of the inks of the graph.

**UPPER=***array-name*
specifies the flow upper bound for each link. The array must be numeric, and it must be indexed over a subset of the links of the graph.

**WEIGHT=***array-name*
specifies link weights. The array must be numeric, and it must be indexed over a subset of the links of the graph. If you specify this suboption, then any link that does not appear in the index set of the WEIGHT= array has weight 0. If you do not specify this suboption, then every link has weight 1.

**NODES=(** *suboptions* **)**
groups node-indexed data. For more information, see the section "Input Data for the Network Solver" on page 398.

You can specify the following *suboptions*:

**INCLUDE=***set-name*
names a set of nodes to include in the graph definition even if no weights are available for them. For more information, see the section "Connected Components" on page 413.

**WEIGHT=***array-name*
specifies node weights. The array must be numeric, and it must be indexed over a subset of the nodes of the graph.

**WEIGHT2=***array-name*
specifies node supply upper bounds in the minimum-cost network flow problem. The array must be numeric, and it must be indexed over a subset of the nodes of the graph. For more information, see the section "Minimum-Cost Network Flow" on page 424.

**OUT=(** *suboptions* **)**
specifies the output sets or arrays for each algorithm (see Table 9.4 for which OUT= suboptions you can specify for each algorithm). You can use some of these options (even if you do not invoke any algorithm) to see the filtering outcome that is produced by the SUBGRAPH= option.

If you do not specify a *suboption* that matches the algorithm option in the statement, the algorithm runs and only updates the objective.

If you specify a suboption that does not match the algorithm option in the statement, OPTMODEL issues a warning.

When you declare arrays that are indexed over nodes, over links, or over sets of nodes or links, you must use the same type you used in your node definition.

See the various algorithm sections for examples of the use of these OUT= *suboptions*.

**ARTPOINTS=***set-name*
specifies the output set for articulation points. Each element of the set represents a node ID. This suboption is used with the BICONCOMP algorithm option.

**ASSIGNMENTS=***set-name*
specifies the output set for linear assignment. This suboption is used with the LINEAR_ASSIGNMENT algorithm option.

**BICONCOMP=***array-name*
specifies the array to contain the biconnected component of each link. This suboption is used with the BICONCOMP algorithm option.

**CLIQUES=***set-name*
specifies the output set for cliques. Each tuple of the set represents clique ID and node ID. This suboption is used with the CLIQUE algorithm option.

**CONCOMP=***array-name*
specifies the output array for connected components. This suboption is used with the CONCOMP algorithm option.

**CUTSETS=***set-name*
specifies the output set for the cut-sets for minimum cuts. Each tuple of the set represents the cut ID, the tail node ID, and the head node ID. This suboption is used with the MINCUT algorithm option.

**CYCLES=***set-name*
specifies the output set for cycles. Each tuple of the set represents a cycle ID, the order within that cycle, and the node ID. This suboption is used with the CYCLE algorithm option.

**FLOW=***array-name*
specifies the output array for the flow on each link. This suboption is used with the MINCOST-FLOW algorithm option.

**FOREST=***set-name*
specifies the output set for the minimum spanning tree (forest). This suboption is used with the MINSPANTREE algorithm option.

**LINKS=***set-name*
specifies the output set for the links that remain after the SUBGRAPH= option is applied. Each tuple of the set represents tail and head nodes, followed by a sequence of numbers which correspond to the attributes you provide in the LINKS= suboptions. The length of the tuples must be the number of attributes you specify plus two (for the tail and head node information). The options you specify in the LINKS= option will appear in the output in the order: WEIGHT=, LOWER=, and UPPER=. For an example, see Figure 9.12 in "Solving over Subsets of Nodes and Links (Filters)" on page 401.

**NODES=***set-name*
specifies the output set for the nodes that remain after the SUBGRAPH= option is applied. Each tuple of the set represents a node, followed by a sequence of numbers which correspond to the attributes you provide in the NODES= suboptions. The length of the tuples must be the number of attributes you specify plus one (for node information). The options you specify in the NODES= option will appear in the output in the order: WEIGHT= and WEIGHT2=. For an example, see the section "Minimum-Cost Network Flow with Flexible Supply and Demand" on page 428.

**ORDER=***array-name*
specifies the numeric array to contain the position of each node within the optimal tour. This suboption is used with the TSP algorithm option.

**PARTITIONS=***set-name*
specifies the output set for the partitions for minimum cuts. The set contains, for each partition, the node IDs in the smaller of the two subsets. Each tuple of the set represents a cut ID and a node ID. This suboption is used with the MINCUT algorithm option.

**SPPATHS=***set-name*
specifies the set to contain the link sequence for each path. Each tuple of the set represents a source node ID, a sink node ID, a sequence number, a tail node ID, and a head node ID. This suboption is used with the SHORTPATH algorithm option.

**SPWEIGHTS=***array-name*
specifies the numeric array to contain the path weight for each source and sink node pair. This suboption is used with the SHORTPATH algorithm option.

**TOUR=***set-name*
specifies the output set for the tour in the traveling salesman problem. This suboption is used with the TSP algorithm option.

**TRANSCL=***set-name*
specifies the set to contain the pairs $(u, v)$ of nodes where $v$ is reachable from $u$. This suboption is used with the TRANSITIVE_CLOSURE algorithm option.

**SUBGRAPH=***( suboptions )*
specifies the input sets that enable you to solve a problem over a subgraph. For more information, see the section "Input Data for the Network Solver" on page 398.

You can specify the following *suboptions*:

**LINKS=***set-name*
specifies the subset of links to use. If you specify a node pair that is not referenced in any of the suboptions of the LINKS= option, then the network solver returns an error.

**NODES=***set-name*
specifies the subset of nodes to use. If you specify a node that is not referenced in any of the suboptions of the LINKS= option or the NODES= option, then the network solver returns an error.

## Algorithm Options

**BICONCOMP< =() >**
finds biconnected components and articulation points of an undirected input graph. For more information, see the section "Biconnected Components and Articulation Points" on page 406.

**CLIQUE< =(** *suboption* **) >**
finds maximal cliques in the input graph. For more information, see the section "Clique" on page 410.

You can specify the following *suboption*:

**MAXCLIQUES=***number*
specifies the maximum number of cliques to return. The default is the positive number that has the largest absolute value that can be represented in your operating environment.

**CONCOMP< =(** *suboption* **) >**
finds the connected components of the input graph. For more information, see the section "Connected Components" on page 413.

You can specify the following *suboption*:

**ALGORITHM=DFS | UNION_FIND**
specifies the algorithm to use for calculating connected components. Table 9.8 describes the valid values for this option.

**Table 9.8** Values for the ALGORITHM= Option

| Option Value | Description |
|---|---|
| DFS | Uses the depth-first search algorithm for connected components. |
| UNION_FIND | Uses the union-find algorithm for connected components. You can use ALGORITHM=UNION_FIND only with undirected graphs. |

By default, ALGORITHM=UNION_FIND for undirected graphs, and ALGORITHM=DFS for directed graphs.

**CYCLE< =( *suboptions* ) >**
finds the cycles (or the existence of a cycle) in the input graph. For more information, see the section "Cycle" on page 417.

You can specify the following *suboptions* in the CYCLE= option:

**MAXCYCLES=*number***
specifies the maximum number of cycles to return. The default is the positive number that has the largest absolute value that can be represented in your operating environment. This option works only when you also specify MODE=ALL_CYCLES.

**MAXLENGTH=*number***
specifies the maximum number of links to allow in a cycle. Any cycle whose length is greater than *number* is removed from the results. The default is the positive number that has the largest absolute value that can be represented in your operating environment. By default, nothing is removed from the results. This option works only when you also specify MODE=ALL_CYCLES.

**MAXLINKWEIGHT=*number***
specifies the maximum sum of link weights to allow in a cycle. Any cycle whose sum of link weights is greater than *number* is removed from the results. The default is the positive number that has the largest absolute value that can be represented in your operating environment. By default, nothing is filtered. This option works only when you also specify MODE=ALL_CYCLES.

**MAXNODEWEIGHT=*number***
specifies the maximum sum of node weights to allow in a cycle. Any cycle whose sum of node weights is greater than *number* is removed from the results. The default is the positive number that has the largest absolute value that can be represented in your operating environment. By default, nothing is filtered. This option works only when you also specify MODE=ALL_CYCLES.

**MINLENGTH=*number***
specifies the minimum number of links to allow in a cycle. Any cycle that has fewer links than *number* is removed from the results. The default is 1. By default, only self-loops are filtered. This option works only when you also specify MODE=ALL_CYCLES.

**MINLINKWEIGHT=***number*
specifies the minimum sum of link weights to allow in a cycle. Any cycle whose sum of link weights is less than *number* is removed from the results. The default the negative number that has the largest absolute value that can be represented in your operating environment. By default, nothing is filtered. This option works only when you also specify MODE=ALL_CYCLES.

**MINNODEWEIGHT=***number*
specifies the minimum sum of node weights to allow in a cycle. Any cycle whose sum of node weights is less than *number* is removed from the results. The default is the negative number that has the largest absolute value that can be represented in your operating environment. By default, nothing is filtered. This option works only when you also specify MODE=ALL_CYCLES.

**MODE=ALL_CYCLES | FIRST_CYCLE**
specifies whether to stop after finding the first cycle. Table 9.9 describes the valid values for this option.

**Table 9.9** Values for the MODE= Option

| Option Value | Description |
|---|---|
| ALL_CYCLES | Returns all (unique, elementary) cycles found. |
| FIRST_CYCLE | Returns the first cycle found. |

By default, MODE=FIRST_CYCLE.

**LINEAR_ASSIGNMENT< =() >**
**LAP< =() >**
solves the minimal-cost linear assignment problem. In graph terms, this problem is also known as the minimum link-weighted matching problem on a bipartite directed graph. The input data (the cost matrix) is defined as a directed graph by specifying the LINKS= option in the SOLVE WITH NETWORK statement, where the costs are defined as link weights. Internally, the graph is treated as a bipartite directed graph.

For more information, see the section "Linear Assignment (Matching)" on page 423.

**MINCOSTFLOW< =() >**
**MCF< =() >**
solves the minimum-cost network flow problem.

For more information, see the section "Minimum-Cost Network Flow" on page 424.

**MINCUT< =(** *suboptions* **) >**
finds the minimum link-weighted cut of an input graph. For more information, see the section "Minimum Cut" on page 430. You can specify the following *suboptions* in the MINCUT= option:

**MAXNUMCUTS=***number*
specifies the maximum number of cuts to return from the algorithm. The minimal cut and any others found during the search, up to *number*, are returned. By default, MAXNUMCUTS=1.

**MAXWEIGHT=***number*
specifies the maximum weight of the cuts to return from the algorithm. Only cuts that have weight less than or equal to *number* are returned. The default is the positive number that has the largest absolute value that can be represented in your operating environment.

**MINSPANTREE< =() >**

**MST< =() >**
solves the minimum link-weighted spanning tree problem on an input graph. For more information, see the section "Minimum Spanning Tree" on page 434.

**SHORTPATH< =(** *suboptions* **) >**
calculates shortest paths between sets of nodes on the input graph. For more information, see the section "Shortest Path" on page 436.

You can specify the following suboptions:

**PATHS=ALL | LONGEST | SHORTEST**
specifies the type of output for shortest paths results.

Table 9.10 lists the valid values for this suboption.

**Table 9.10** Values for the PATHS= Option

| Option Value | Description |
|---|---|
| ALL | Outputs shortest paths for all pairs of source-sinks. |
| LONGEST | Outputs shortest paths for the source-sink pair that has the longest (finite) length. If other source-sink pairs (up to 100) have equally long length, they are also output. |
| SHORTEST | Outputs shortest paths for the source-sink pair that has the shortest length. If other source-sink pairs (up to 100) have equally short length, they are also output. |

By default, SHORTPATH=ALL.

**SINK=***set-name*
specifies the set of sink nodes.

**SOURCE=***set-name*
specifies the set of source nodes.

**USEWEIGHT=YES | NO**
specifies whether to use weights in calculating shortest paths as listed in Table 9.11.

**Table 9.11** Values for USEWEIGHT= Option

| Option Value | Description |
|---|---|
| YES | Uses weights (if they exist) in shortest path calculations. |
| NO | Does not use weights in shortest path calculations. |

By default, USEWEIGHT=YES.

**TRANSITIVE_CLOSURE< =() >**

**TRANSCL< =() >**

calculates the transitive closure of an input graph. For more information, see the section "Transitive Closure" on page 450.

**TSP< =(** *suboptions* **) >**

solves the traveling salesman problem. For more information, see the section "Traveling Salesman Problem" on page 453.

The algorithm that is used to solve this problem is built around the same method as is used in PROC OPTMILP: a branch-and-cut algorithm. Many of the following *suboptions* are the same as those described for the OPTMILP procedure in the *SAS/OR User's Guide: Mathematical Programming*.

You can specify the following *suboptions*:

**ABSOBJGAP=***number*

specifies a stopping criterion. When the absolute difference between the best integer objective and the objective of the best remaining branch-and-bound node becomes less than the value of *number*, the solver stops. The value of *number* can be any nonnegative number. By default, ABSOBJGAP=1E–6.

**CONFLICTSEARCH=***number* | *string*

specifies the level of conflict search that the network solver performs. The solver performs a conflict search to find clauses that result from infeasible subproblems that arise in the search tree. Table 9.12 describes the valid values for this option.

**Table 9.12** Values for CONFLICTSEARCH= Option

| *number* | *string* | **Description** |
|---|---|---|
| –1 | AUTOMATIC | Performs a conflict search based on a strategy that is determined by the network solver |
| 0 | NONE | Disables conflict search |
| 1 | MODERATE | Performs a moderate conflict search |
| 2 | AGGRESSIVE | Performs an aggressive conflict search |

By default, CONFLICTSEARCH=AUTOMATIC.

**CUTOFF=***number*

cuts off any branch-and-bound nodes in a minimization problem that has an objective value that is greater than *number*. The value of *number* can be any number.

The default value is the positive number that has the largest absolute value that can be represented in your operating environment.

**CUTSTRATEGY=***number* | *string*

specifies the level of cutting planes to be generated by the network solver. TSP-specific cutting planes are always generated. Table 9.13 describes the valid values for this option.

**Table 9.13** Values for CUTSTRATEGY= Option

| *number* | *string* | **Description** |
|---|---|---|
| –1 | AUTOMATIC | Generates cutting planes based on a strategy determined by the mixed integer linear programming solver |
| 0 | NONE | Disables generation of mixed integer programming cutting planes (some TSP-specific cutting planes are still active for validity) |
| 1 | MODERATE | Uses a moderate cut strategy |
| 2 | AGGRESSIVE | Uses an aggressive cut strategy |

By default, CUTSTRATEGY=NONE.

**EMPHASIS=***number* | *string*
specifies a search emphasis option. Table 9.14 describes the valid values for this option.

**Table 9.14** Values for EMPHASIS= Option

| *number* | *string* | **Description** |
|---|---|---|
| 0 | BALANCE | Performs a balanced search |
| 1 | OPTIMAL | Emphasizes optimality over feasibility |
| 2 | FEASIBLE | Emphasizes feasibility over optimality |

By default, EMPHASIS=BALANCE.

**HEURISTICS=***number* | *string*
controls the level of initial and primal heuristics that the network solver applies. This level determines how frequently the network solver applies primal heuristics during the branch-and-bound tree search. It also affects the maximum number of iterations that are allowed in iterative heuristics. Some computationally expensive heuristics might be disabled by the solver at less aggressive levels. Table 9.15 lists the valid values for this option.

**Table 9.15** Values for HEURISTICS= Option

| *number* | *string* | **Description** |
|---|---|---|
| –1 | AUTOMATIC | Applies the default level of heuristics |
| 0 | NONE | Disables all initial and primal heuristics |
| 1 | BASIC | Applies basic initial and primal heuristics at low frequency |
| 2 | MODERATE | Applies most initial and primal heuristics at moderate frequency |
| 3 | AGGRESSIVE | Applies all initial primal heuristics at high frequency |

By default, HEURISTICS=AUTOMATIC.

**MAXNODES=***number*

specifies the maximum number of branch-and-bound nodes to be processed. The value of *number* can be any nonnegative integer up to the largest four-byte signed integer, which is $2^{31} - 1$.

By default, MAXNODES=$2^{31} - 1$.

**MAXSOLS=***number*

specifies the maximum number of feasible tours to be identified. If *number* solutions have been found, then the solver stops. The value of *number* can be any positive integer up to the largest four-byte signed integer, which is $2^{31} - 1$.

By default, MAXSOLS=$2^{31} - 1$.

**MILP=***number* | *string*

specifies whether to use a mixed integer linear programming (MILP) solver for solving the traveling salesman problem. The MILP solver attempts to find the overall best TSP tour by using a branch-and-bound based algorithm. This algorithm can be expensive for large-scale problems. If MILP=OFF, then the network solver uses its initial heuristics to find a feasible, but not necessarily optimal, tour as quickly as possible. Table 9.16 describes the valid values for this option.

**Table 9.16** Values for MILP= Option

| *number* | *string* | **Description** |
|---|---|---|
| 1 | ON | Uses a mixed integer linear programming solver |
| 0 | OFF | Does not use a mixed integer linear programming solver |

By default, MILP=ON

**NODESEL=***number* | *string*

specifies the branch-and-bound node selection strategy option. For more information about node selection, see Chapter 13, "The OPTMILP Procedure." Table 9.17 describes the valid values for this option.

**Table 9.17** Values for NODESEL= Option

| *number* | *string* | **Description** |
|---|---|---|
| –1 | AUTOMATIC | Uses automatic node selection |
| 0 | BESTBOUND | Chooses the node that has the best relaxed objective (best-bound-first strategy) |
| 1 | BESTESTIMATE | Chooses the node that has the best estimate of the integer objective value (best-estimate-first strategy) |
| 2 | DEPTH | Chooses the most recently created node (depth-first strategy) |

By default, NODESEL=AUTOMATIC.

**PROBE=***number* | *string*
specifies a probing option. Table 9.18 describes the valid values for this option.

**Table 9.18** Values for PROBE= Option

| *number* | *string* | **Description** |
|---|---|---|
| –1 | AUTOMATIC | Uses an automatic probing strategy |
| 0 | NONE | Disables probing |
| 1 | MODERATE | Uses the probing moderately |
| 2 | AGGRESSIVE | Uses the probing aggressively |

By default, PROBE=NONE.

**RELOBJGAP=***number*
specifies a stopping criterion that is based on the best integer objective (BestInteger) and the objective of the best remaining node (BestBound). The relative objective gap is equal to

$$|\text{BestInteger} - \text{BestBound}| / (1\text{E}{-}10 + |\text{BestBound}|)$$

When this value becomes less than the specified gap size *number*, the solver stops. The value of *number* can be any nonnegative number.

By default, RELOBJGAP=1E–4.

**STRONGITER=***number* | **AUTOMATIC**
specifies the number of simplex iterations that the network solver performs for each variable in the candidate list when it uses the strong branching variable selection strategy. The value of *number* can be any positive integer up to the largest four-byte signed integer, which is $2^{31} - 1$. If you specify the keyword AUTOMATIC or the value –1, the network solver uses the default value, which it calculates automatically.

**STRONGLEN=***number* | **AUTOMATIC**
specifies the number of candidates that the network solver considers when it uses the strong branching variable selection strategy. The value of *number* can be any positive integer up to the largest four-byte signed integer, which is $2^{31} - 1$. If you specify the keyword AUTOMATIC or the value –1, the network solver uses the default value, which it calculates automatically.

**TARGET=***number*
specifies a stopping criterion for minimization problems. If the best integer objective is better than or equal to *number*, the solver stops. The value of *number* can be any number.

By default, TARGET is the negative number that has the largest absolute value that can be represented in your operating environment.

**VARSEL=***number* | *string*
specifies the rule for selecting the branching variable. For more information about variable selection, see Chapter 13, "The OPTMILP Procedure." Table 9.19 describes the valid values for this option.

**Table 9.19** Values for VARSEL= Option

| *number* | *string* | **Description** |
|---|---|---|
| –1 | AUTOMATIC | Uses automatic branching variable selection |
| 0 | MAXINFEAS | Chooses the variable that has maximum infeasibility |
| 1 | MININFEAS | Chooses the variable that has minimum infeasibility |
| 2 | PSEUDO | Chooses a branching variable based on pseudocost |
| 3 | STRONG | Uses the strong branching variable selection strategy |

By default, VARSEL=AUTOMATIC.

# Details: Network Solver

The network solver uses a collection of specialized algorithms that optimize specific types of common problems. When you use the network solver, you specify variable arrays, numeric arrays, and sets, both to define an instance and to get solutions, without explicitly formulating objectives and constraints.

## Input Data for the Network Solver

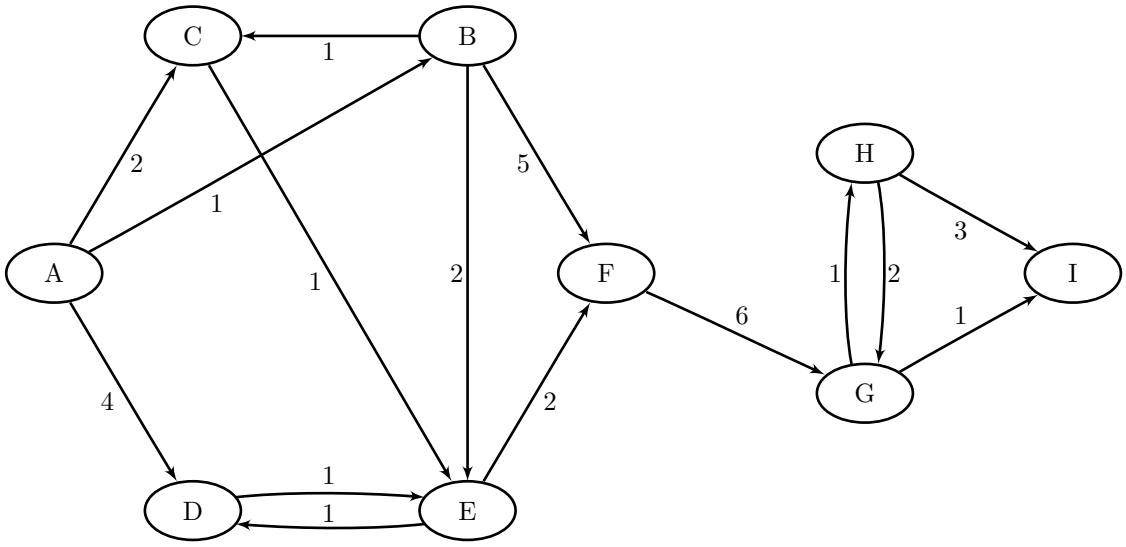
This section describes how you can import and export node, link, and problem data from and to SAS data sets and how you can solve problems over a subgraph without changing your original sets. The section "Graph Input Data" on page 398 describes how to load node and link data in some common formats. The section "Solving over Subsets of Nodes and Links (Filters)" on page 401 describes subgraphs and how to access the objective value of a network problem.

### Graph Input Data

This section describes how to input a graph for analysis by the network solver. Because PROC OPTMODEL uses node and link attributes that are indexed over the sets of nodes and links, you need to provide only node and link attributes. PROC OPTMODEL infers the graph from the attributes you provide. When a documented default value exists for the attribute of a link or a node, you need to provide only the values that differ from the default. For example, the section "Minimum-Cost Network Flow" on page 424 assumes that the link flow upper bound is $\infty$. You need to specify only the finite upper bounds.

Consider the directed graph shown in Figure 9.5.

**Figure 9.5** A Simple Directed Graph




Each node and link has associated attributes: a node label and a link weight.

None of the algorithms in PROC OPTMODEL support Null Graphs, i.e., graphs with 0 nodes. PROC OPTMODEL will usually raise a semantic error and stop processing any remaining statements in a block if after processing its inputs it determines that the graph is null. If the graph definition itself is not null, but the graph to be passed to the solver after applying the SUBGRAPH= option is null, then the predeclared parameter _SOLUTION_STATUS_ will be set to NULL_GRAPH. For more information, see "Solving over Subsets of Nodes and Links (Filters)" on page 401.

## Data Indexed by Nodes or Links

Nodes often represent entities, and links represent relationships between these entities. Therefore, it is common to store a graph as a link-indexed table. When nodes have attributes beyond their name (label), these attributes are stored in a node-indexed table. This section covers the more complex link-indexed case. The node-indexed case is essentially identical to this one, except that the PROC OPTMODEL set has tuple length of one when node-indexed data are read, whereas the PROC OPTMODEL set has tuple length two when link-indexed data are read.

Let $G = (N, A)$ define a graph with a set $N$ of nodes and a set $A$ of links. A link is an ordered pair of nodes. Each node is defined by using either numeric or string labels.

The directed graph $G$ shown in Figure 9.5 can be represented by the following links data set LinkSetIn:

```
data LinkSetIn;
   input from $ to $ weight @@;
   datalines;
A B 1   A C 2   A D 4   B C 1   B E 2
B F 5   C E 1   D E 1   E D 1   E F 2
F G 6   G H 1   G I 1   H G 2   H I 3
;
```

The following statements read in this graph and print the resulting links and nodes sets. These statements do not run any algorithms, so the resulting output contains only the input graph.

```
proc optmodel;
   set<str,str> LINKS;
   set NODES = union{<ni,nj> in LINKS} {ni,nj};
   num weight{LINKS};

   read data LinkSetIn into LINKS=[from to] weight;

   print weight;
   put NODES=; /* computed automatically by OPTMODEL */
quit;
```

The network solver preserves the node order of each link that you provide, even in cases where the link is traversed in the opposite order, such as in paths or tours. For an example, see the tour $(1, 4, 2, 3, 5)$ in Figure 9.8.

The log output in Figure 9.6 shows the nodes that are read from the input link data set. In this example PROC OPTMODEL computed the node set *N* (NODES) from its definition when it was needed. The ODS output in Figure 9.7 shows the weights that are read from the input link data set, which is indexed by link. PUT is used for NODES because PROC OPTMODEL sets are basic types such as number and string. Thus, you use PUT to quickly inspect a set value. In contrast, you use PRINT to inspect an array, such as weight.

**Figure 9.6** Node Set Printout of a Simple Directed Graph

```
NOTE: There were 15 observations read from the data set WORK.LINKSETIN.
NODES={'A','B','C','D','E','F','G','H','I'}
```

**Figure 9.7** Link Set of a Simple Directed Graph That Includes Weights

**The OPTMODEL Procedure**

| [1] | [2] | weight |
|---|---|---|
| A | B | 1 |
| A | C | 2 |
| A | D | 4 |
| B | C | 1 |
| B | E | 2 |
| B | F | 5 |
| C | E | 1 |
| D | E | 1 |
| E | D | 1 |
| E | F | 2 |
| F | G | 6 |
| G | H | 1 |
| G | I | 1 |
| H | G | 2 |
| H | I | 3 |

As described in the GRAPH_DIRECTION= option, if the graph is undirected, the *from* and *to* labels are interchangeable. If you define this graph as undirected, then reciprocal links (for example, $D \rightarrow E$ and $E \rightarrow D$) are treated as the same link, and duplicates are removed. The network solver takes the first

occurrence of the link and ignores the others. By default, GRAPH_DIRECTION=UNDIRECTED, so to declare the graph as undirected you can just omit this option.

After you read the data into PROC OPTMODEL sets, you pass link information to the solver by using the LINKS= option. Node input is analogous to link input. You pass node information to the solver by using the NODES= option.

The INCLUDE= suboption is especially useful for algorithms that depend only on the graph topology, (such as the connected components algorithm). If an algorithm requires a node or link property and that property is not defined for a node or link that is added by the INCLUDE= suboption, the algorithm will not run.

### Matrix Input Data

The contents of a table can be represented as a graph. The relationships between two sets of nodes, $N_1$ and $N_2$, can be represented by a $|N_1|$ by $|N_2|$ incidence matrix $A$, in which $N_1$ is the set of rows and $N_2$ is the set of columns.

To read a matrix that is stored in a data set into PROC OPTMODEL, you need to take two extra steps:

1. Determine the name of each numeric variable that you want to use. PROC CONTENTS can be useful for this task.
2. Use an iterated READ DATA statement.

For more information, see "Example 9.3: Linear Assignment Problem for Minimizing Swim Times" on page 469.

## Solving over Subsets of Nodes and Links (Filters)

You can solve a problem over a subgraph without declaring new link and node sets. You can specify the LINKS= and NODES= suboptions of the SUBGRAPH= option to filter nodes and links before PROC OPTMODEL builds and solves the instance. If you want to see the resulting subgraph, you can specify the LINKS= and NODES= suboptions of the OUT= option. If you just want to produce a subgraph, you do not need to invoke an algorithm.

You can keep all the input and output arrays defined over the original graph and define a subgraph by providing any combination of the LINKS= and NODES= suboptions of the SUBGRAPH= option. If you specify either of the suboptions of the SUBGRAPH= option, then union semantics apply. PROC OPTMODEL uses the following rules:

- Only the links that are included in the set named in the LINKS= option are used to create the instance.
- Only the nodes that appear either in the NODES= suboption of the SUBGRAPH= option or that appear as the head or tail of a link in the LINKS= suboption are used to create the instance.
- A node or a link that appears only in the SUBGRAPH= option, but not in the original graph, is discarded. To add nodes or links that do not have attributes, see the INCLUDE= suboption of the LINKS= and NODE= options.

If the value of the LOGLEVEL= suboption is equal to or greater than 3, PROC OPTMODEL issues a message for each of the nodes and links that it discards until the number of messages issued during problem generation reaches the value of the MSGLIMIT= option in the PROC OPTMODEL statement. If the value of the LOGLEVEL= suboption is greater than 0, PROC OPTMODEL also issues a summary that shows the total count of discarded nodes and links from each input array or set.

The following statements call PROC OPTMODEL and declare a five-node complete undirected graph; a subset of links that contains all links between nodes 1, 2, 3, and 4; and a subset of nodes that contains nodes 3, 4, and 5:

```
proc optmodel;
   set NODES = 1..5;
   set LINKS = {vi in NODES, vj in NODES: vi < vj};
   num distance {<vi,vj> in LINKS} = 10*vi + vj;

   set <num,num> TOUR;

   /* Build a link set using only nodes 1..4 nodes */
   set <num,num> LINKS_BETWEEN_1234 = {vi in 1..3, vj in (vi+1)..4};
   /* Build a node subset consisting of nodes 3..5 */
   set NODES_345 = 3..5;
```

After the sets are declared, the statements in the following steps solve several traveling salesman problems (TSPs) on subgraphs. For more information about TSPs, see the section "Traveling Salesman Problem" on page 453.

1. The first SOLVE statement solves a TSP on the original graph. Note that the links in the tour (see Figure 9.8) are returned with the same orientation that you provide in the input. For example, the second step on the tour goes from node 4 to node 2 using link $(2, 4)$. This guarantees that you do not need to do extra processing of output to check for link orientation. You can just use the output directly.

   ```
   /* Implicit network 1: solve over nodes 1..5 -- The original network*/
   solve with NETWORK /
      links=( weight=distance )
      out=( tour=TOUR )
      tsp
   ;
   put TOUR=;
   ```

   As shown in Figure 9.8, all links implied by the WEIGHT= suboption of the LINKS= option become part of the graph.

**Figure 9.8** SOLVE WITH NETWORK Log: Traveling Salesman Tour of an Unfiltered Graph

```
NOTE: The number of nodes in the input graph is 5.
NOTE: The number of links in the input graph is 10.
NOTE: Processing the traveling salesman problem.
NOTE: The initial TSP heuristics found a tour with cost 111 using 0.01 (cpu:
      0.00) seconds.
NOTE: The MILP presolver value NONE is applied.
NOTE: The MILP solver is called.
NOTE: The Branch and Cut algorithm is used.
NOTE: Optimal.
NOTE: Objective = 111.
NOTE: Processing the traveling salesman problem used 0.03 (cpu: 0.00) seconds.
TOUR={<1,4>,<2,4>,<2,3>,<3,5>,<1,5>}
```

To access the objective value of a network problem, use the _OROPTMODEL_NUM_ predefined array. The network solver ignores the _OBJ_ predefined symbol, which is part of the current named problem. The current named problem is independent of the network solver, because the network solver uses sets and numeric arrays for input and output. For more information, see the sections "Multiple Subproblems" on page 148 and "Solver Status Parameters" on page 159 in Chapter 5, "The OPTMODEL Procedure."

```
put _OROPTMODEL_NUM_['OBJECTIVE'];
```

2. The next SOLVE statement solves a TSP on the subgraph that is defined by the link set LINKS_BETWEEN_1234.

```
/* Filter on LINKS: solve over nodes 1..4 */
solve with NETWORK /
   links=( weight=distance )
   subgraph=( links=LINKS_BETWEEN_1234 )
   out=( tour=TOUR )
   tsp
;
put TOUR=;
```

As shown in Figure 9.9, the network solver now ignores node 5.

**Figure 9.9** SOLVE WITH NETWORK Log: Traveling Salesman Tour over Nodes $N = \{1, 2, 3, 4\}$

```
111
NOTE: The SUBGRAPH= option filtered 4 elements from 'distance.'
NOTE: The number of nodes in the input graph is 4.
NOTE: The number of links in the input graph is 6.
NOTE: Processing the traveling salesman problem.
NOTE: The initial TSP heuristics found a tour with cost 74 using 0.00 (cpu:
      0.00) seconds.
NOTE: The MILP presolver value NONE is applied.
NOTE: The MILP solver is called.
NOTE: The Branch and Cut algorithm is used.
NOTE: Optimal.
NOTE: Objective = 74.
NOTE: Processing the traveling salesman problem used 0.00 (cpu: 0.00) seconds.
TOUR={<1,3>,<2,3>,<2,4>,<1,4>}
```

3. The next SOLVE statement solves a TSP on the subgraph that is defined by the node set NODES_345.

```
/* Filter on NODES: solve over nodes 3..5 */
solve with NETWORK /
   links=( weight=distance )
   subgraph=( nodes=NODES_345 )
   out=( tour=TOUR )
   tsp
;
put TOUR=;
```

As shown in Figure 9.10, the network solver now ignores nodes 1 and 2, along with any links incident to them.

**Figure 9.10** SOLVE WITH NETWORK Log: Traveling Salesman Tour over Nodes $N = \{3, 4, 5\}$

```
NOTE: The SUBGRAPH= option filtered 7 elements from 'distance.'
NOTE: The number of nodes in the input graph is 3.
NOTE: The number of links in the input graph is 3.
NOTE: Processing the traveling salesman problem.
NOTE: The initial TSP heuristics found a tour with cost 114 using 0.00 (cpu:
      0.00) seconds.
NOTE: Optimal.
NOTE: Objective = 114.
NOTE: Processing the traveling salesman problem used 0.00 (cpu: 0.00) seconds.
TOUR={<3,4>,<4,5>,<3,5>}
```

4. The next SOLVE statement attempts to solve a TSP on the subgraph that is defined by the node set NODES_345 and the link set that is defined by the links on the nodes $\{1, 2, 3, 4\}$. This subgraph creates an infeasible instance because the links $\{(1, 5), (2, 5), (3, 5), (4, 5)\}$ that were defined in the original graph have been filtered out. Thus, node 5 is disconnected and no tour can exist.

```
/* Explicit nodes and links: semantic error over nodes 1..5
 * Links <u,5> are undefined and no documented default exists. */
solve with NETWORK /
   links=( weight=distance )
   subgraph=( nodes=NODES_345 links=LINKS_BETWEEN_1234 )
   out=( tour=TOUR )
   tsp
;
```

As shown in Figure 9.11, the network solver identifies that no tour exists over the surviving nodes and links.

**Figure 9.11** SOLVE WITH NETWORK Log: Infeasible Traveling Salesman Problem after Filtering

```
NOTE: The SUBGRAPH= option filtered 4 elements from 'distance.'
NOTE: The number of nodes in the input graph is 5.
NOTE: The number of links in the input graph is 6.
NOTE: The number of singleton nodes in the input graph is 1.
NOTE: Processing the traveling salesman problem.
NOTE: Infeasible.
NOTE: Processing the traveling salesman problem used 0.00 (cpu: 0.00) seconds.
```

5. The last SOLVE statement uses the LINKS= suboption of the OUT= option to capture exactly which nodes and links were generated and with which attributes. In this case, because the only attribute defined is link weight, the set LINKS_OUT has tuples of length three.

```
   /* make room for tail, head, and weight */
   set<num,num,num> LINKS_OUT;
   solve with NETWORK /
      links=( weight=distance )
      subgraph=( nodes=NODES_345 links=LINKS_BETWEEN_1234 )
      out=( tour=TOUR links=LINKS_OUT )
      tsp
   ;
   put LINKS_OUT=;
quit;
```

As shown in Figure 9.12, the network solver can return the graph after filtering. This feature can sometimes help you identify why you might get counterintuitive results.

**Figure 9.12** SOLVE WITH NETWORK Log: Remaining Links after Filtering

```
NOTE: The SUBGRAPH= option filtered 4 elements from 'distance.'
NOTE: The number of nodes in the input graph is 5.
NOTE: The number of links in the input graph is 6.
NOTE: The number of singleton nodes in the input graph is 1.
NOTE: Processing the traveling salesman problem.
NOTE: Infeasible.
NOTE: Processing the traveling salesman problem used 0.00 (cpu: 0.00) seconds.
LINKS_OUT={<1,2,12>,<1,3,13>,<1,4,14>,<2,3,23>,<2,4,24>,<3,4,34>}
```

## Numeric Limitations

Extremely large or extremely small numerical values might cause computational difficulties for some of the algorithms in the network solver. For this reason, each algorithm restricts the magnitude of the data values to a particular threshold number. If the user data values exceed this threshold, the network solver issues an error message. The value of the threshold limit is different for each algorithm and depends on the operating environment. The threshold limits are listed in Table 9.20, where $M$ is defined as the largest absolute value representable in your operating environment.

**Table 9.20** Threshold Limits by Algorithm

| | Graph Links | | | | Graph Nodes | |
|---|---|---|---|---|---|---|
| **Algorithm** | weight | weight2 | lower | upper | weight | weight2 |
| CYCLE | $\sqrt{M}$ | | | | $\sqrt{M}$ | |
| LINEAR_ASSIGNMENT | $\sqrt{M}$ | | | | | |
| MINCOSTFLOW | 1e15 | | 1e15 | 1e15 | 1e15 | 1e15 |
| MINCUT | $\sqrt{M}$ | | | | | |
| MINSPANTREE | $\sqrt{M}$ | | | | | |
| SHORTPATH | $\sqrt{M}$ | $\sqrt{M}$ | | | | |
| TSP | 1e20 | | | | | |

To obtain these limits, use the SAS CONSTANT function. For example, the following PROC OPTMODEL code assigns $\sqrt{M}$ to a variable x and prints that value to the log:

```
proc optmodel;
   num c = constant('SQRTBIG');
   put c=;
quit;
```

### Missing Values

A missing value has no valid interpretation for most of the algorithms in the network solver. If the user data contain a missing value, the network solver issues an error message. There is only one exception: the minimum-cost network flow algorithm interprets a missing value in the lower or upper bound option as the default bound value. For more information about this algorithm, see the section "Minimum-Cost Network Flow" on page 424.

### Negative Link Weights

For certain algorithms in the network solver, a negative link weight is not allowed. The following algorithms issue an error message if a negative link weight is provided:

- MINCUT

## Biconnected Components and Articulation Points

A *biconnected component* of a graph $G = (N, A)$ is a connected subgraph that cannot be broken into disconnected pieces by deleting any single node (and its incident links). An *articulation point* is a node of a graph whose removal would cause an increase in the number of connected components. Articulation points can be important when you analyze any graph that represents a communications network. Consider an articulation point $i \in N$ which, if removed, disconnects the graph into two components $C^1$ and $C^2$. All paths in $G$ between some nodes in $C^1$ and some nodes in $C^2$ must pass through node $i$. In this sense, articulation points are critical to communication. Examples of where articulation points are important are airline hubs, electric circuits, network wires, protein bonds, traffic routers, and numerous other industrial applications.

In the network solver, you can find biconnected components and articulation points of an input graph by invoking the BICONCOMP option. This algorithm works only with undirected graphs.

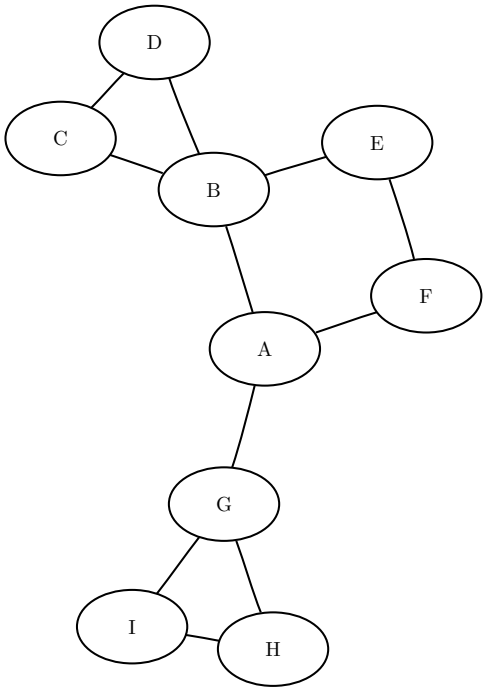
The results for the biconnected components algorithm are written to the link-indexed numeric array that is specified in the BICONCOMP= suboption of the OUT= option. For each link in the links array, the value in this array identifies its component. The component identifiers are numbered sequentially starting from 1. The articulation points are written to the set that is specified in the ARTPOINTS= suboption of the OUT= option.

The algorithm that the network solver uses to compute biconnected components is a variant of depth-first search (Tarjan 1972). This algorithm runs in time $O(|N| + |A|)$ and therefore should scale to very large graphs.

## Biconnected Components of a Simple Undirected Graph

This section illustrates the use of the biconnected components algorithm on the simple undirected graph $G$ that is shown in Figure 9.13.

**Figure 9.13** A Simple Undirected Graph $G$




The undirected graph $G$ can be represented by the links data set LinkSetInBiCC as follows:

```
data LinkSetInBiCC;
   input from $ to $ @@;
   datalines;
A B  A F  A G  B C  B D
B E  C D  E F  G I  G H
H I
;
```

The following statements calculate the biconnected components and articulation points and output the results in the data sets LinkSetOut and NodeSetOut:

```
proc optmodel;
   set<str,str> LINKS;
   read data LinkSetInBiCC into LINKS=[from to];
   set NODES = union{<i,j> in LINKS} {i,j};
   num bicomponent{LINKS};
   set<str> ARTPOINTS;

   solve with NETWORK /
      loglevel  = moderate
      links     = (include=LINKS)
      biconcomp
      out       = (biconcomp=bicomponent artpoints=ARTPOINTS)
   ;

   print bicomponent;
   put ARTPOINTS;
   create data LinkSetOut from [from to] biconcomp=bicomponent;
   create data NodeSetOut from [node]=ARTPOINTS artpoint=1;
quit;
```

The data set LinkSetOut now contains the biconnected components of the input graph, as shown in Figure 9.14.

**Figure 9.14** Biconnected Components of a Simple Undirected Graph

| from | to | biconcomp |
|---|---|---|
| A | B | 2 |
| A | F | 2 |
| A | G | 4 |
| B | C | 1 |
| B | D | 1 |
| B | E | 2 |
| C | D | 1 |
| E | F | 2 |
| G | I | 3 |
| G | H | 3 |
| H | I | 3 |

In addition, the data set NodeSetOut contains the articulation points of the input graph, as shown in Figure 9.15.

**Figure 9.15** Articulation Points of a Simple Undirected Graph

| node | artpoint |
|---|---|
| A | 1 |
| B | 1 |
| G | 1 |

The biconnected components are shown graphically in Figure 9.16 and Figure 9.17.

**Figure 9.16** Biconnected Components $C^1$ and $C^2$

$C^1 = \{B, C, D\}$ $C^2 = \{A, B, E, F\}$

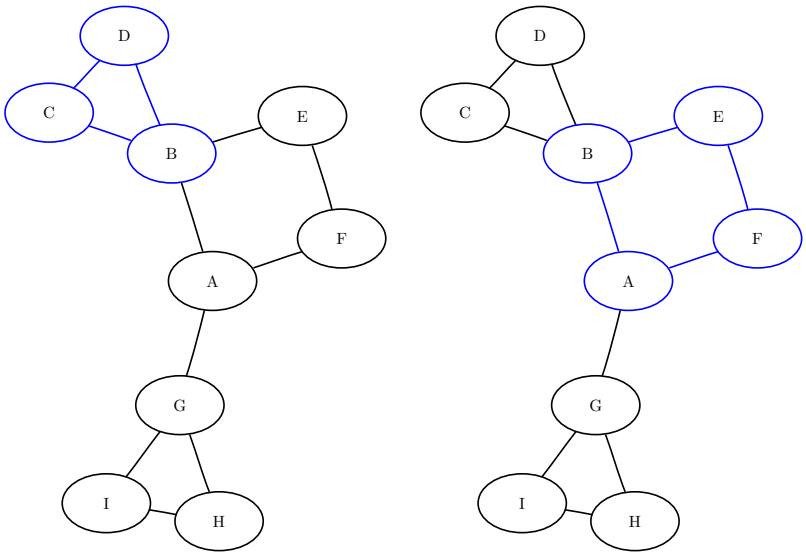
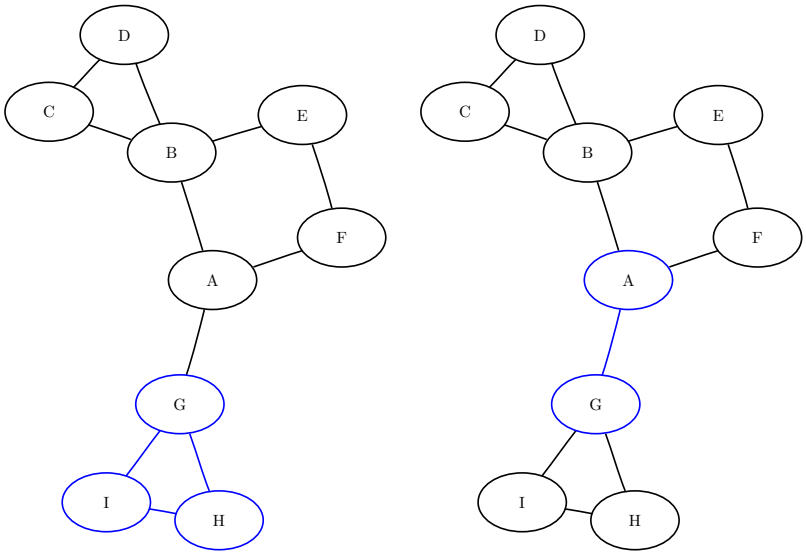



**Figure 9.17** Biconnected Components $C^3$ and $C^4$

$C^3 = \{G, H, I\}$ $C^4 = \{A, G\}$




For a more detailed example, see "Example 9.1: Articulation Points in a Terrorist Network" on page 463.

## Clique

A *clique* of a graph $G = (N, A)$ is an induced subgraph that is a complete graph. Every node in a clique is connected to every other node in that clique. A *maximal clique* is a clique that is not a subset of the nodes of any larger clique. That is, it is a set $C$ of nodes such that every pair of nodes in $C$ is connected by a link and every node not in $C$ is missing a link to at least one node in $C$. The number of maximal cliques in a particular graph can be very large and can grow exponentially with every node added. Finding cliques in graphs has applications in numerous industries including bioinformatics, social networks, electrical engineering, and chemistry.

You can find the maximal cliques of an input graph by invoking the CLIQUE= option. The clique algorithm works only with undirected graphs.

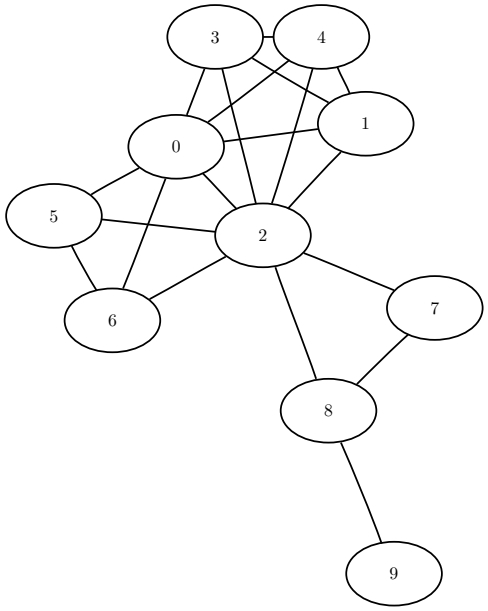
The results for the clique algorithm are written to the set that is specified in the CLIQUES= suboption of the OUT= option. Each node of each clique is listed in the set along with a clique ID (the first argument of the tuple) to identify the clique to which it belongs. A node can appear multiple times in this set if it belongs to multiple cliques.

The algorithm that the network solver uses to compute maximal cliques is a variant of the Bron-Kerbosch algorithm (Bron and Kerbosch 1973; Harley 2003). Enumerating all maximal cliques is NP-hard, so this algorithm typically does not scale to very large graphs.

### Maximal Cliques of a Simple Undirected Graph

This section illustrates the use of the clique algorithm on the simple undirected graph $G$ that is shown in Figure 9.18.

**Figure 9.18** A Simple Undirected Graph $G$




The undirected graph $G$ can be represented by the following links data set LinkSetIn:

```
data LinkSetIn;
   input from to @@;
   datalines;
0 1  0 2  0 3  0 4  0 5
0 6  1 2  1 3  1 4  2 3
2 4  2 5  2 6  2 7  2 8
3 4  5 6  7 8  8 9
;
```

The following statements calculate the maximal cliques, output the results in the data set Cliques, and use the CARD function and SLICE operator as a convenient way to compute the clique sizes, which are output to a data set called CliqueSizes:

```
proc optmodel;
   set<num,num> LINKS;
   read data LinkSetIn into LINKS=[from to];
   set<num,num> CLIQUES;

   solve with NETWORK /
      links  = (include=LINKS)
      clique
      out    = (cliques=CLIQUES)
   ;

   put CLIQUES;
   create data Cliques from [clique node]=CLIQUES;
   num num_cliques = card(setof {<cid,node> in CLIQUES} cid);
   set CLIQUE_IDS = 1..num_cliques;
   num size {cid in CLIQUE_IDS} = card(slice(<cid,*>, CLIQUES));
   create data CliqueSizes from [clique] size;
quit;
```

The data set Cliques now contains the maximal cliques of the input graph; it is shown in Figure 9.19.

**Figure 9.19** Maximal Cliques of a Simple Undirected Graph

| clique | node |
|---|---|
| 1 | 0 |
| 1 | 2 |
| 1 | 1 |
| 1 | 3 |
| 1 | 4 |
| 2 | 0 |
| 2 | 2 |
| 2 | 5 |
| 2 | 6 |
| 3 | 2 |
| 3 | 8 |
| 3 | 7 |
| 4 | 8 |
| 4 | 9 |

In addition, the data set CliqueSizes contains the number of nodes in each clique; it is shown in Figure 9.20.

**Figure 9.20** Sizes of Maximal Cliques of a Simple Undirected Graph

| clique | size |
|---|---|
| 1 | 5 |
| 2 | 4 |
| 3 | 3 |
| 4 | 2 |

The maximal cliques are shown graphically in Figure 9.21 and Figure 9.22.

**Figure 9.21** Maximal Cliques $C^1$ and $C^2$

$C^1 = \{0, 1, 2, 3, 4\}$ $C^2 = \{0, 2, 5, 6\}$

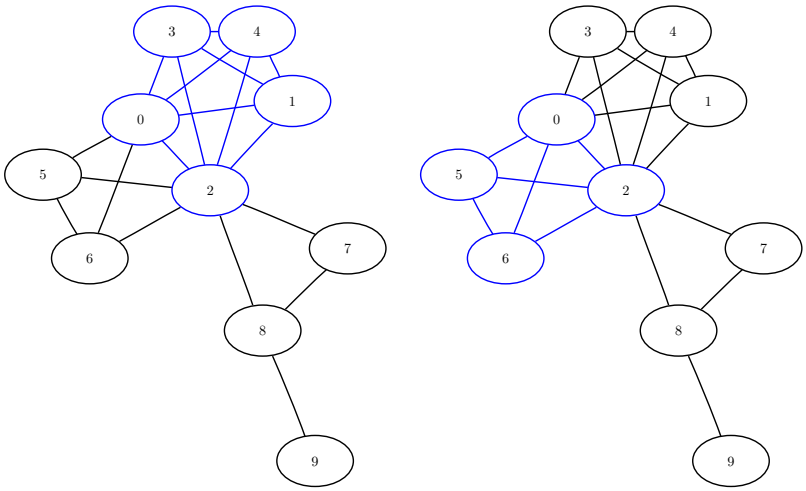
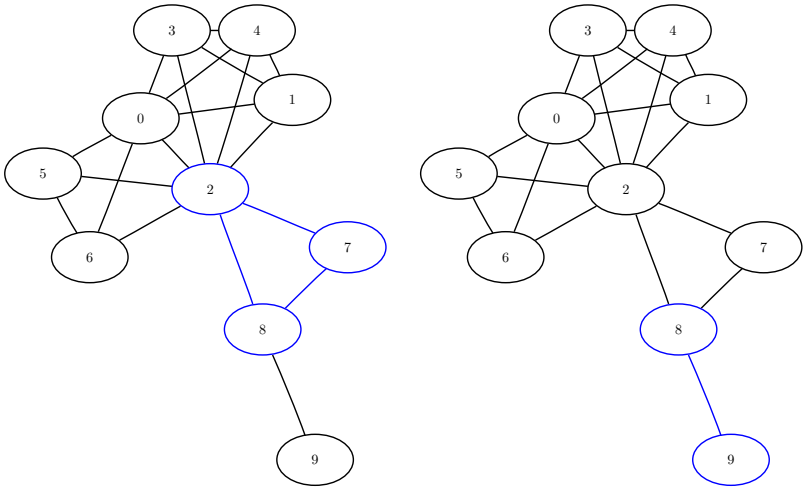



**Figure 9.22** Maximal Cliques $C^3$ and $C^4$

$C^3 = \{2, 7, 8\}$ $C^4 = \{8, 9\}$



## Connected Components

A *connected component* of a graph is a set of nodes that are all reachable from each other. That is, if two nodes are in the same component, then there exists a path between them. For a directed graph, there are two types of components: a *strongly connected component* has a directed path between any two nodes, and a *weakly connected component* ignores direction and requires only that a path exist between any two nodes.

In the network solver, you can invoke connected components by using the CONCOMP= option.

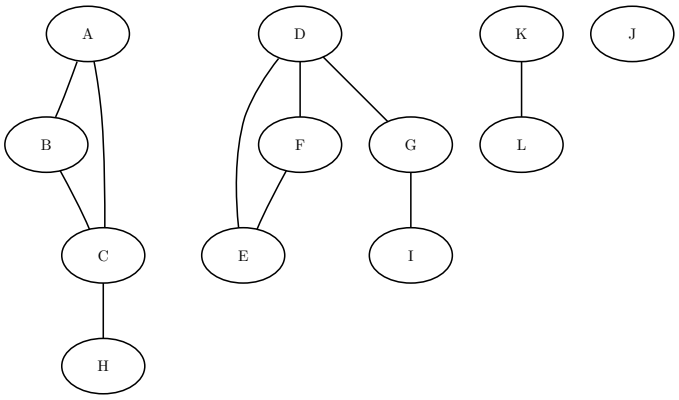
There are two main algorithms for finding connected components in an undirected graph: a depth-first search algorithm (ALGORITHM=DFS) and a union-find algorithm (ALGORITHM=UNION_FIND). For a graph $G = (N, A)$, both algorithms run in time $O(|N| + |A|)$ and can usually scale to very large graphs. The default is the union-find algorithm. For directed graphs, only the depth-first search algorithm is available.

The results of the connected components algorithm are written to the node-indexed numeric array that you specify in the CONCOMP= suboption of the OUT= option. For each node in the set, the value of this array identifies its component. The component identifiers are numbered sequentially starting from 1.

### Connected Components of a Simple Undirected Graph

This section illustrates the use of the connected components algorithm on the simple undirected graph $G$ that is shown in Figure 9.23.

**Figure 9.23** A Simple Undirected Graph $G$




The undirected graph $G$ can be represented by the following links data set, LinkSetIn:

```
data LinkSetIn;
   input from $ to $ @@;
   datalines;
A B  A C  B C  C H  D E  D F  D G  F E  G I  K L
;
```

The following statements calculate the connected components and output the results in the data set NodeSetOut:

```
proc optmodel;
   set<str,str> LINKS;
   read data LinkSetIn into LINKS=[from to];
   set NODES = union {<i,j> in LINKS} {i,j};
   num component{NODES};

   solve with NETWORK /
      links   = (include=LINKS)
      concomp
      out     = (concomp=component)
   ;

   print component;
   create data NodeSetOut from [node] concomp=component;
quit;
```

The data set NodeSetOut contains the connected components of the input graph and is shown in Figure 9.24.

**Figure 9.24** Connected Components of a Simple Undirected Graph

| node | concomp |
|---|---|
| A | 1 |
| B | 1 |
| C | 1 |
| H | 1 |
| D | 2 |
| E | 2 |
| F | 2 |
| G | 2 |
| I | 2 |
| K | 3 |
| L | 3 |

Notice that the graph is defined by using only the links array. As seen in Figure 9.23, this graph also contains a singleton node labeled J, which has no associated links. By definition, this node defines its own component. But because the input graph was defined by using only the links array, node J did not show up in the results data set. To define a graph by using nodes that have no associated links, you should also define the input nodes set. In this case, you can define a nodes data set NodeSetIn as follows:

```
data NodeSetIn;
   input node $ @@;
   datalines;
A  B  C  D  E  F  G  H  I  J  K  L
;
```

You could also have defined the set directly in PROC OPTMODEL, but in this case, a separate data set nicely preserves the independence between the model and the data.

Now, when you calculate the connected components, you define the input graph by using both the nodes input data set and the links input data set:

```
proc optmodel;
   set<str,str> LINKS;
   read data LinkSetIn into LINKS=[from to];
   set<str> NODES;
   read data NodeSetIn into NODES=[node];
   num component{NODES};

   solve with NETWORK /
      links   = (include=LINKS)
      nodes   = (include=NODES)
      concomp
      out     = (concomp=component)
   ;

   print component;
   create data NodeSetOut from [node] concomp=component;
quit;
```

The resulting data set, NodeSetOut, includes the singleton node J as its own component, as shown in Figure 9.25.

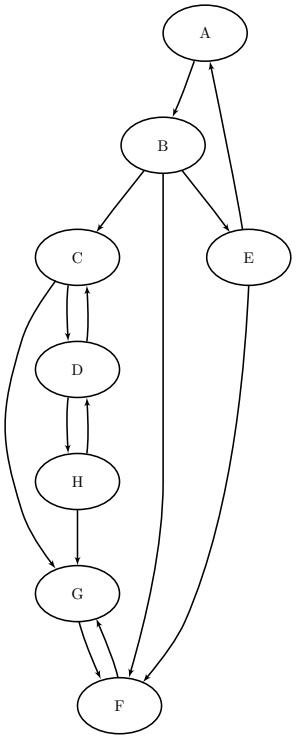
**Figure 9.25** Connected Components of a Simple Undirected Graph

| node | concomp |
|---|---|
| A | 1 |
| B | 1 |
| C | 1 |
| D | 2 |
| E | 2 |
| F | 2 |
| G | 2 |
| H | 1 |
| I | 2 |
| J | 3 |
| K | 4 |
| L | 4 |

## Connected Components of a Simple Directed Graph

This section illustrates the use of the connected components algorithm on the simple directed graph *G* that is shown in Figure 9.26.

**Figure 9.26** A Simple Directed Graph $G$




The directed graph $G$ can be represented by the following links data set, LinkSetIn:

```
data LinkSetIn;
   input from $ to $ @@;
   datalines;
A B  B C  B E  B F  C G
C D  D C  D H  E A  E F
F G  G F  H G  H D
;
```

The following statements calculate the connected components and output the results in the data set NodeSetOut:

```
proc optmodel;
   set<str,str> LINKS;
   read data LinkSetIn into LINKS=[from to];
   set NODES = union {<i,j> in LINKS} {i,j};
   num component{NODES};

   solve with NETWORK /
      graph_direction = directed
      links           = (include=LINKS)
      concomp
      out             = (concomp=component)
   ;

   print component;
   create data NodeSetOut from [node] concomp=component;
quit;
```

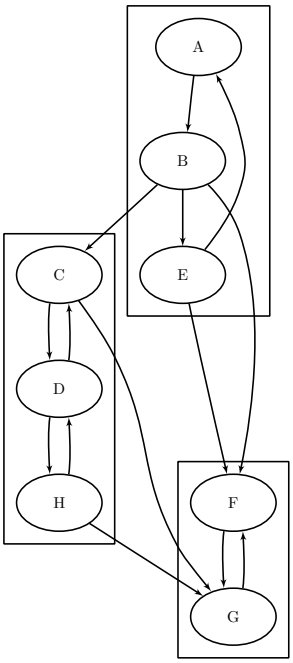
The data set NodeSetOut, shown in Figure 9.27, now contains the connected components of the input graph.

**Figure 9.27** Connected Components of a Simple Directed Graph

| node | concomp |
|---|---|
| A | 3 |
| B | 3 |
| C | 2 |
| E | 3 |
| F | 1 |
| G | 1 |
| D | 2 |
| H | 2 |

The connected components are represented graphically in Figure 9.28.

**Figure 9.28** Strongly Connected Components of Graph *G*




## Cycle

A *path* in a graph is a sequence of nodes, each of which has a link to the next node in the sequence. An *elementary cycle* is a path in which the start node and end node are the same and otherwise no node appears more than once in the sequence.

In the network solver, you can find you can find (or just count) the elementary cycles of an input graph by invoking the CYCLE= algorithm option. To find the cycles and report them in a set, use the CYCLES= suboption in the OUT= option. You do not need to use the CYCLES= suboption to simply count the cycles.

For undirected graphs, each link represents two directed links. For this reason, the following cycles are filtered out: trivial cycles ($A \rightarrow B \rightarrow A$) and duplicate cycles that are found by traversing a cycle in both directions ($A \rightarrow B \rightarrow C \rightarrow A$ and $A \rightarrow C \rightarrow B \rightarrow A$).

The results of the cycle detection algorithm are written to the set that you specify in the CYCLES= suboption in the OUT= option. Each node of each cycle is listed in the CYCLES= set along with a cycle ID (the first argument of the tuple) to identify the cycle to which it belongs. The second argument of the tuple defines the order (sequence) of the node in the cycle.

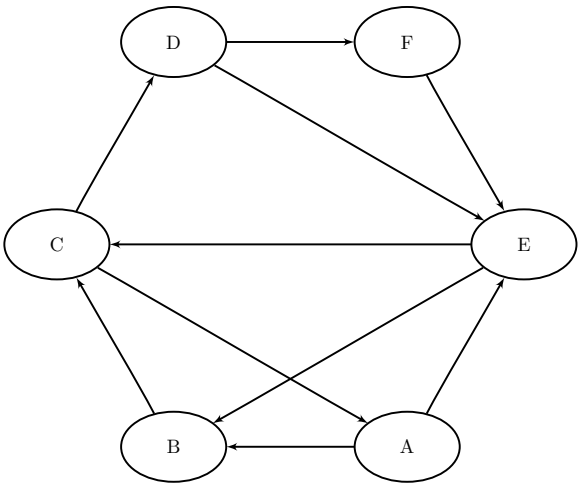
The algorithm that the network solver uses to compute all cycles is a variant of the algorithm in Johnson (1975). This algorithm runs in time $O((|N| + |A|)(c + 1))$, where $c$ is the number of elementary cycles in the graph. So the algorithm should scale to large graphs that contain few cycles. However, some graphs can have a very large number of cycles, so the algorithm might not scale.

If MODE=ALL_CYCLES and there are many cycles, the CYCLES= set can become very large. It might be beneficial to check the number of cycles before you try to create the CYCLES= set. When you specify MODE=FIRST_CYCLE, the algorithm returns the first cycle that it finds and stops processing. This should run relatively quickly. For large-scale graphs, the MINLINKWEIGHT= and MAXLINKWEIGHT= suboptions might increase the computation time.

## Cycle Detection of a Simple Directed Graph

This section provides a simple example of using the cycle detection algorithm on the simple directed graph $G$ that is shown in Figure 9.29. Two other examples are "Example 9.2: Cycle Detection for Kidney Donor Exchange" on page 465, which shows the use of cycle detection for optimizing a kidney donor exchange, and "Example 9.6: Transitive Closure for Identification of Circular Dependencies in a Bug Tracking System" on page 477, which shows an application of cycle detection to dependencies between bug reports.

**Figure 9.29** A Simple Directed Graph $G$




The directed graph $G$ can be represented by the following links data set, LinkSetIn:

```
data LinkSetIn;
   input from $ to $ @@;
   datalines;
A B  A E  B C  C A  C D
D E  D F  E B  E C  F E
;
```

The following statements check whether the graph has a cycle:

```
proc optmodel;
   set<str,str> LINKS;
   read data LinkSetIn into LINKS=[from to];
   set<num,num,str> CYCLES;

   solve with NETWORK /
      graph_direction = directed
      links           = (include=LINKS)
      cycle           = (mode=first_cycle)
   ;
quit;
```

The result is written to the log of the procedure, as shown in Figure 9.30.

**Figure 9.30** Network Solver Log: Check the Existence of a Cycle in a Simple Directed Graph

```
NOTE: There were 10 observations read from the data set WORK.LINKSETIN.
NOTE: The number of nodes in the input graph is 6.
NOTE: The number of links in the input graph is 10.
NOTE: Processing cycle detection.
NOTE: The graph does have a cycle.
NOTE: Processing cycle detection used 0.00 (cpu: 0.00) seconds.
```

The following statements count the number of cycles in the graph:

```
proc optmodel;
   set<str,str> LINKS;
   read data LinkSetIn into LINKS=[from to];
   set<num,num,str> CYCLES;

   solve with NETWORK /
      graph_direction = directed
      links           = (include=LINKS)
      cycle           = (mode=all_cycles)
   ;
quit;
```

The result is written to the log of the procedure, as shown in Figure 9.31.

**Figure 9.31** Network Solver Log: Count the Number of Cycles in a Simple Directed Graph

```
NOTE: There were 10 observations read from the data set WORK.LINKSETIN.
NOTE: The number of nodes in the input graph is 6.
NOTE: The number of links in the input graph is 10.
NOTE: Processing cycle detection.
NOTE: The graph has 7 cycles.
NOTE: Processing cycle detection used 0.00 (cpu: 0.00) seconds.
```

The following statements return the first cycle found in the graph:

```
proc optmodel;
   set<str,str> LINKS;
   read data LinkSetIn into LINKS=[from to];
   set<num,num,str> CYCLES;

   solve with NETWORK /
      graph_direction = directed
      links           = (include=LINKS)
      cycle           = (mode=first_cycle)
      out             = (cycles=CYCLES)
   ;

   put CYCLES;
   create data Cycles from [cycle order node]=CYCLES;
quit;
```

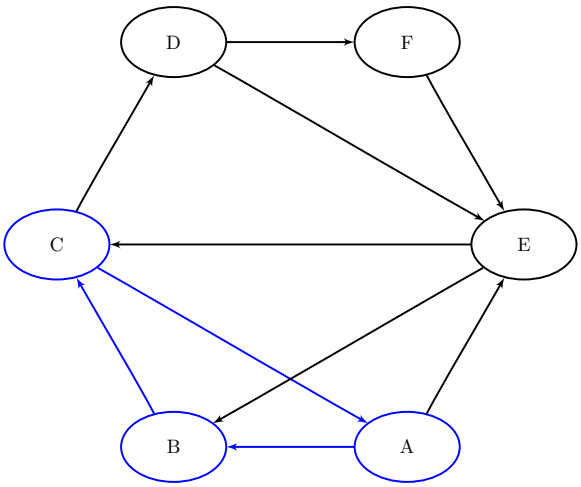
The data set Cycles now contains the first cycle found in the input graph; it is shown in Figure 9.32.

**Figure 9.32** First Cycle Found in a Simple Directed Graph

| cycle | order | node |
|---|---|---|
| 1 | 1 | A |
| 1 | 2 | B |
| 1 | 3 | C |
| 1 | 4 | A |

The first cycle that is found in the input graph is shown graphically in Figure 9.33.

**Figure 9.33** $A \rightarrow B \rightarrow C \rightarrow A$



The following statements return all the cycles in the graph:

```
proc optmodel;
   set<str,str> LINKS;
   read data LinkSetIn into LINKS=[from to];
   set<num,num,str> CYCLES;

   solve with NETWORK /
      graph_direction = directed
      links           = (include=LINKS)
      cycle           = (mode=all_cycles)
      out             = (cycles=CYCLES)
   ;

   put CYCLES;
   create data Cycles from [cycle order node]=CYCLES;
quit;
```

The data set Cycles now contains all the cycles in the input graph; it is shown in Figure 9.34.

**Figure 9.34** All Cycles in a Simple Directed Graph

| cycle | order | node |
|---|---|---|
| 1 | 1 | A |
| 1 | 2 | B |
| 1 | 3 | C |
| 1 | 4 | A |
| 2 | 1 | A |
| 2 | 2 | E |
| 2 | 3 | B |
| 2 | 4 | C |
| 2 | 5 | A |

| cycle | order | node |
|---|---|---|
| 3 | 1 | A |
| 3 | 2 | E |
| 3 | 3 | C |
| 3 | 4 | A |
| 4 | 1 | B |
| 4 | 2 | C |
| 4 | 3 | D |
| 4 | 4 | E |
| 4 | 5 | B |

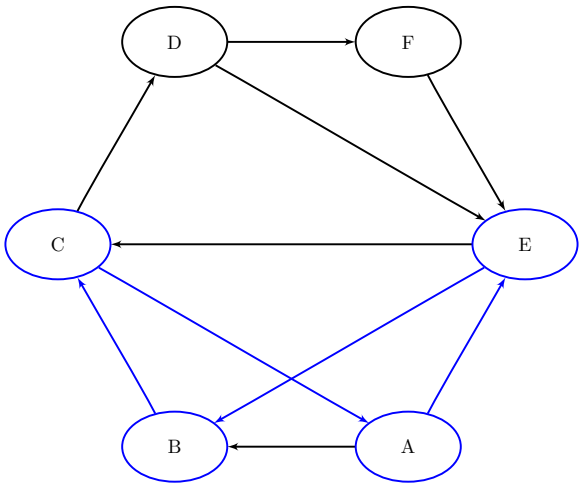
| cycle | order | node |
|---|---|---|
| 5 | 1 | B |
| 5 | 2 | C |
| 5 | 3 | D |
| 5 | 4 | F |
| 5 | 5 | E |
| 5 | 6 | B |
| 6 | 1 | E |
| 6 | 2 | C |
| 6 | 3 | D |

| cycle | order | node |
|---|---|---|
| 6 | 4 | E |
| 7 | 1 | E |
| 7 | 2 | C |
| 7 | 3 | D |
| 7 | 4 | F |
| 7 | 5 | E |

The six additional cycles are shown graphically in Figure 9.35 through Figure 9.37.

**Figure 9.35** Cycles

$A \to E \to B \to C \to A$




$A \to E \to C \to A$

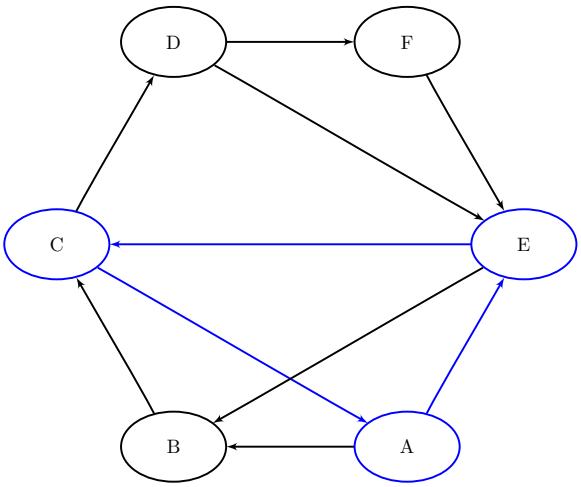
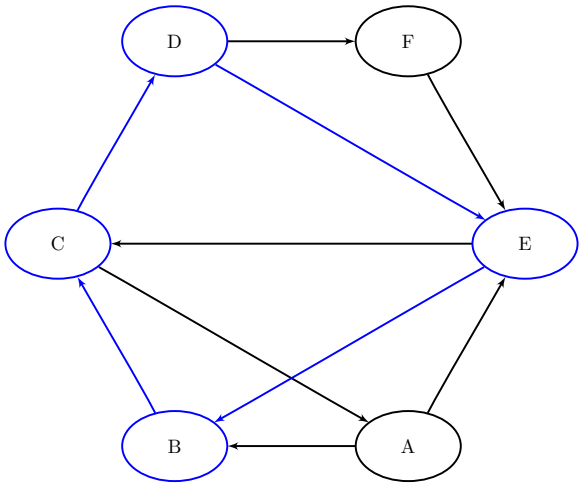



**Figure 9.36** Cycles

$B \to C \to D \to E \to B$




$B \to C \to D \to F \to E \to B$

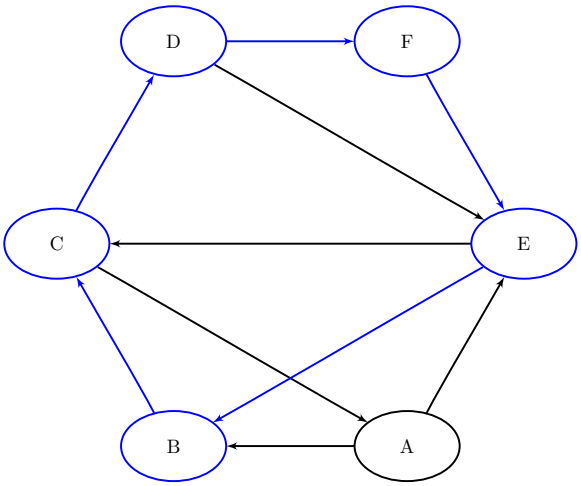
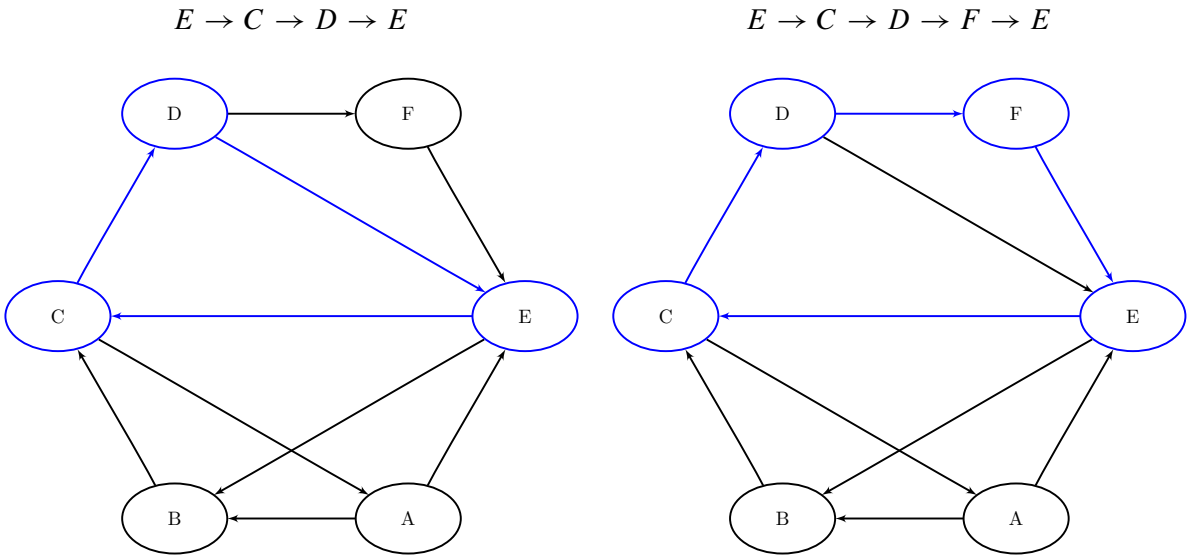


**Figure 9.37** Cycles




## Linear Assignment (Matching)

The *linear assignment problem* (LAP) is a fundamental problem in combinatorial optimization that involves assigning workers to tasks at minimal costs. In graph theoretic terms, the LAP is equivalent to finding a minimum-weight matching in a weighted bipartite directed graph. In a *bipartite graph*, the nodes can be divided into two disjoint sets $S$ (workers) and $T$ (tasks) such that every link connects a node in $S$ to a node in $T$. That is, the node sets $S$ and $T$ are independent. The concept of assigning workers to tasks can be generalized to the assignment of any abstract object from one group to some abstract object from a second group.

The linear assignment problem can be formulated as an integer programming optimization problem. The form of the problem depends on the sizes of the two input sets, $S$ and $T$. Let $A$ represent the set of possible assignments between sets $S$ and $T$. In the bipartite graph, these assignments are the links. If $|S| \geq |T|$, then the following optimization problem is solved:

$$\begin{aligned}
\text{minimize} \quad & \sum_{(i,j)\in A} c_{ij} x_{ij} \\
\text{subject to} \quad & \sum_{(i,j)\in A} x_{ij} \leq 1 \quad i \in S \\
& \sum_{(i,j)\in A} x_{ij} = 1 \quad j \in T \\
& x_{ij} \in \{0,1\} \qquad (i,j) \in A
\end{aligned}$$

This model allows for some elements of set $S$ (workers) to go unassigned (if $|S| > |T|$).

If $|S| < |T|$, then the following optimization problem is solved:

$$\begin{aligned}
\text{minimize} \quad & \sum_{(i,j)\in A} c_{ij} x_{ij} \\
\text{subject to} \quad & \sum_{(i,j)\in A} x_{ij} = 1 \quad i \in S \\
& \sum_{(i,j)\in A} x_{ij} \leq 1 \quad j \in T \\
& x_{ij} \in \{0,1\} \qquad (i,j) \in A
\end{aligned}$$

This model allows for some elements of set $T$ (tasks) to go unassigned.

In the network solver, you can invoke the linear assignment problem solver by using the LINEAR_ASSIGNMENT option. The algorithm that the network solver uses for solving a LAP is based on augmentation of shortest paths (Jonker and Volgenant 1987). This algorithm can be applied as long as the graph is bipartite.

The resulting assignment (or matching) is contained in the set that is specified in the ASSIGNMENTS= suboption of the OUT= option.

For a detailed example, see "Example 9.3: Linear Assignment Problem for Minimizing Swim Times" on page 469.

## Minimum-Cost Network Flow

The *minimum-cost network flow problem* (MCF) is a fundamental problem in network analysis that involves sending flow over a network at minimal cost. Let $G = (N, A)$ be a directed graph. For each link $(i, j) \in A$, associate a cost per unit of flow, designated by $c_{ij}$. The demand (or supply) at each node $i \in N$ is designated as $b_i$, where $b_i \geq 0$ denotes a supply node and $b_i < 0$ denotes a demand node. These values must be within $[b_i^l, b_i^u]$. Define decision variables $x_{ij}$ that denote the amount of flow sent from node $i$ to node $j$. The amount of flow that can be sent across each link is bounded to be within $[l_{ij}, u_{ij}]$. The problem can be modeled as a linear programming problem as follows:

$$\begin{aligned}
\text{minimize} \quad & \sum_{(i,j)\in A} c_{ij} x_{ij} \\
\text{subject to} \quad & b_i^l \leq \sum_{(i,j)\in A} x_{ij} - \sum_{(j,i)\in A} x_{ji} \leq b_i^u \quad i \in N \\
& l_{ij} \leq x_{ij} \leq u_{ij} \qquad\qquad (i,j) \in A
\end{aligned}$$

When $b_i = b_i^l = b_i^u$ for all nodes $i \in N$, the problem is called a *pure network flow problem*. For these problems, the sum of the supplies and demands must be equal to 0 to ensure that a feasible solution exists.

In the network solver, you can invoke the minimum-cost network flow solver by using the MINCOSTFLOW option.

The algorithm that the network solver uses for solving MCF is a variant of the primal network simplex algorithm (Ahuja, Magnanti, and Orlin 1993). Sometimes the directed graph $G$ is disconnected. In this case, the problem is first decomposed into its weakly connected components, and then each minimum-cost flow problem is solved separately.

The input for the network is the standard graph input, which is described in the section "Input Data for the Network Solver" on page 398. The MCF option uses the following suboptions of the LINKS= input option that specify link-indexed numeric arrays:

- The WEIGHT= suboption defines the link cost $c_{ij}$ per unit of flow. (The default is 0, but if the WEIGHT= suboption is not specified, then the default is 1.)
- The LOWER= suboption defines the link flow lower bound $l_{ij}$. (The default is 0.)
- The UPPER= suboption defines the link flow upper bound $u_{ij}$. (The default is $\infty$.)

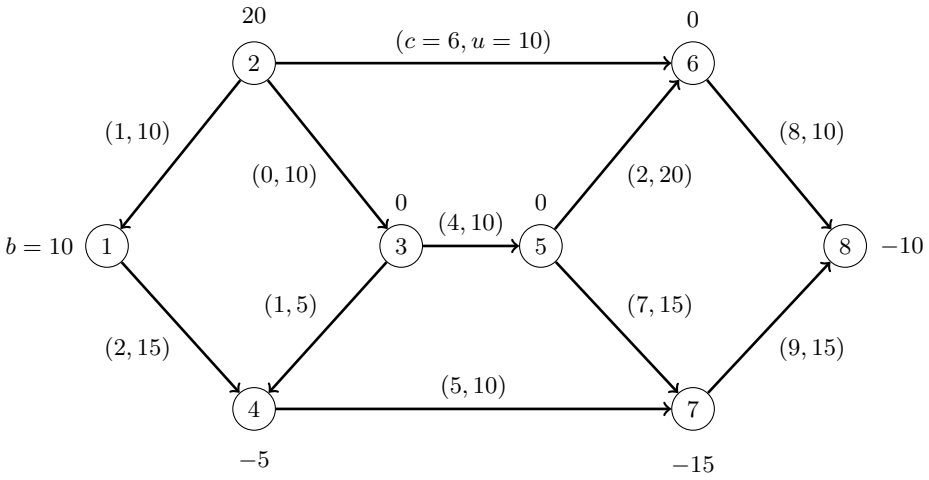
The MCF option uses the WEIGHT= suboption of the NODES= option to specify supply. The parameter is a numeric array that is positive for supply nodes and negative for demand nodes.

The resulting optimal flow through the network is written to the link-indexed numeric array that is specified in the FLOW= suboption of the OUT= option in the SOLVE WITH NETWORK statement.

## Minimum Cost Network Flow for a Simple Directed Graph

The following example demonstrates how to use the network simplex algorithm to find a minimum-cost flow in a directed graph. Consider the directed graph in Figure 9.38, which appears in Ahuja, Magnanti, and Orlin (1993).

**Figure 9.38** Minimum-Cost Network Flow Problem: Data




The directed graph $G$ can be represented by the following links data set LinkSetIn and nodes data set NodeSetIn:

```
data LinkSetIn;
   input from to weight upper;
   datalines;
1 4 2 15
2 1 1 10
2 3 0 10
2 6 6 10
3 4 1  5
```

```
3 5 4 10
4 7 5 10
5 6 2 20
5 7 7 15
6 8 8 10
7 8 9 15
;

data NodeSetIn;
   input node weight;
   datalines;
1  10
2  20
4  -5
7 -15
8 -10
;
```

You can use the following call to the network solver to find a minimum-cost flow:

```
proc optmodel;
   set <num,num> LINKS;
   num cost{LINKS};
   num upper{LINKS};
   read data LinkSetIn into LINKS=[from to] cost=weight upper;
   set NODES = union {<i,j> in LINKS} {i,j};
   num supply{NODES} init 0;
   read data NodeSetIn into [node] supply=weight;
   num flow{LINKS};

   solve with network /
      loglevel        = moderate
      logfreq         = 1
      graph_direction = directed
      links           = (upper=upper weight=cost)
      nodes           = (weight=supply)
      mcf
      out             = (flow=flow)
   ;

   print flow;
   create data LinkSetOut from [from to] upper cost flow;
quit;
```

The progress of the procedure is shown in Figure 9.39.

**Figure 9.39** Network Solver Log for Minimum-Cost Network Flow

```
NOTE: There were 11 observations read from the data set WORK.LINKSETIN.
NOTE: There were 5 observations read from the data set WORK.NODESETIN.
NOTE: The number of nodes in the input graph is 8.
NOTE: The number of links in the input graph is 11.
NOTE: Processing the minimum-cost network flow problem.
NOTE: The network has 1 connected component.
                      Primal          Primal           Dual
      Iteration      Objective  Infeasibility  Infeasibility     Time
              1   0.000000E+00   2.000000E+01   8.900000E+01     0.00
              2   0.000000E+00   2.000000E+01   8.900000E+01     0.00
              3   5.000000E+00   1.500000E+01   8.400000E+01     0.00
              4   5.000000E+00   1.500000E+01   8.300000E+01     0.00
              5   7.500000E+01   1.500000E+01   8.300000E+01     0.00
              6   7.500000E+01   1.500000E+01   7.900000E+01     0.00
              7   1.300000E+02   1.000000E+01   7.600000E+01     0.00
              8   2.700000E+02   0.000000E+00   0.000000E+00     0.00
NOTE: The Network Simplex solve time is 0.00 seconds.
NOTE: Objective = 270.
NOTE: Processing the minimum-cost network flow problem used 0.00 (cpu: 0.00)
      seconds.
NOTE: The data set WORK.LINKSETOUT has 11 observations and 5 variables.
```

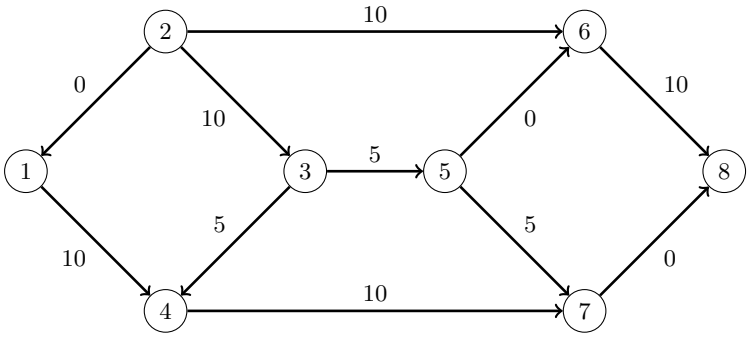
The optimal solution is displayed in Figure 9.40.

**Figure 9.40** Minimum-Cost Network Flow Problem: Optimal Solution

| Obs | from | to | upper | cost | flow |
|---|---|---|---|---|---|
| **1** | 1 | 4 | 15 | 2 | 10 |
| **2** | 2 | 1 | 10 | 1 | 0 |
| **3** | 2 | 3 | 10 | 0 | 10 |
| **4** | 2 | 6 | 10 | 6 | 10 |
| **5** | 3 | 4 | 5 | 1 | 5 |
| **6** | 3 | 5 | 10 | 4 | 5 |
| **7** | 4 | 7 | 10 | 5 | 10 |
| **8** | 5 | 6 | 20 | 2 | 0 |
| **9** | 5 | 7 | 15 | 7 | 5 |
| **10** | 6 | 8 | 10 | 8 | 10 |
| **11** | 7 | 8 | 15 | 9 | 0 |

The optimal solution is represented graphically in Figure 9.41.

**Figure 9.41** Minimum-Cost Network Flow Problem: Optimal Solution




## Minimum-Cost Network Flow with Flexible Supply and Demand

Using the same directed graph shown in Figure 9.38, this example demonstrates a network that has a flexible supply and demand. Consider the following adjustments to the node bounds:

- Node 1 has an infinite supply, but it still requires at least 10 units to be sent.
- Node 4 is a throughput node that can now handle an infinite amount of demand.
- Node 8 has a flexible demand. It requires between 6 and 10 units.

You use the special missing values (.I) to represent infinity and (.M) to represent minus infinity. The adjusted node bounds can be represented by the following nodes data set:

```
data NodeSetIn;
   input node weight weight2;
   datalines;
1  10  .I
2  20  20
4  .M  -5
7 -15 -15
8 -10  -6
;
```

You can use the following call to PROC OPTMODEL to find a minimum-cost flow:

```
proc optmodel;
   set <num,num> LINKS;
   num cost{LINKS};
   num upper{LINKS};
      read data LinkSetIn into LINKS=[from to] cost=weight upper;
   set <num> NODES;
   num supply{NODES};
   num supplyUB{NODES}; /* also demand lower bound, if negative */
      read data NodeSetIn into NODES=[node] supply=weight supplyUB=weight2;
   num flow{LINKS};
```

```
   solve with NETWORK /
      direction = directed
      links     = ( upper = upper weight = cost )
      nodes     = ( weight = supply weight2 = supplyUB )
      mcf
      out       = ( flow = flow )
   ;
   print flow;
   create data LinkSetOut from [from to] upper cost flow;
quit;
```

The progress of the procedure is shown in Figure 9.42.

**Figure 9.42** PROC OPTMODEL Log for Minimum-Cost Network Flow

```
NOTE: There were 11 observations read from the data set WORK.LINKSETIN.
NOTE: There were 5 observations read from the data set WORK.NODESETIN.
NOTE: The number of nodes in the input graph is 8.
NOTE: The number of links in the input graph is 11.
NOTE: Processing the minimum-cost network flow problem.
NOTE: Objective = 226.
NOTE: Processing the minimum-cost network flow problem used 0.00 (cpu: 0.00)
      seconds.
NOTE: The data set WORK.LINKSETOUT has 11 observations and 5 variables.
```

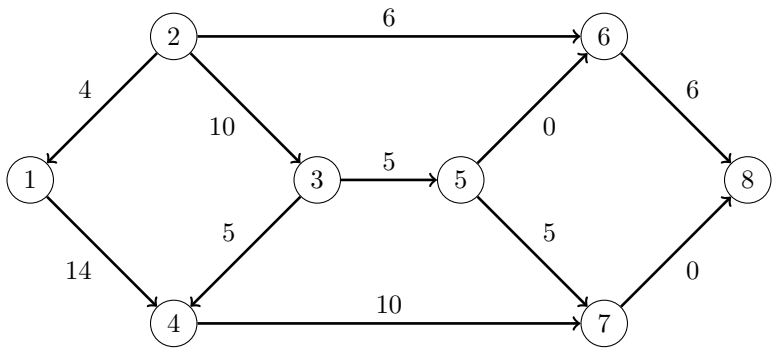
The optimal solution is displayed in Figure 9.43.

**Figure 9.43** Minimum-Cost Network Flow Problem: Optimal Solution

| Obs | from | to | upper | cost | flow |
|---|---|---|---|---|---|
| **1** | 1 | 4 | 15 | 2 | 14 |
| **2** | 2 | 1 | 10 | 1 | 4 |
| **3** | 2 | 3 | 10 | 0 | 10 |
| **4** | 2 | 6 | 10 | 6 | 6 |
| **5** | 3 | 4 | 5 | 1 | 5 |
| **6** | 3 | 5 | 10 | 4 | 5 |
| **7** | 4 | 7 | 10 | 5 | 10 |
| **8** | 5 | 6 | 20 | 2 | 0 |
| **9** | 5 | 7 | 15 | 7 | 5 |
| **10** | 6 | 8 | 10 | 8 | 6 |
| **11** | 7 | 8 | 15 | 9 | 0 |

The optimal solution is represented graphically in Figure 9.44.

**Figure 9.44** Minimum-Cost Network Flow Problem: Optimal Solution




## Minimum Cut

A *cut* is a partition of the nodes of a graph into two disjoint subsets. The *cut-set* is the set of links whose *from* and *to* nodes are in different subsets of the partition. A *minimum cut* of an undirected graph is a cut whose cut-set has the smallest link metric, which is measured as follows: For an unweighted graph, the link metric is the number of links in the cut-set. For a weighted graph, the link metric is the sum of the link weights in the cut-set.

In the network solver, you can invoke the minimum cut algorithm by using the MINCUT= option. This algorithm can be used only on undirected graphs.

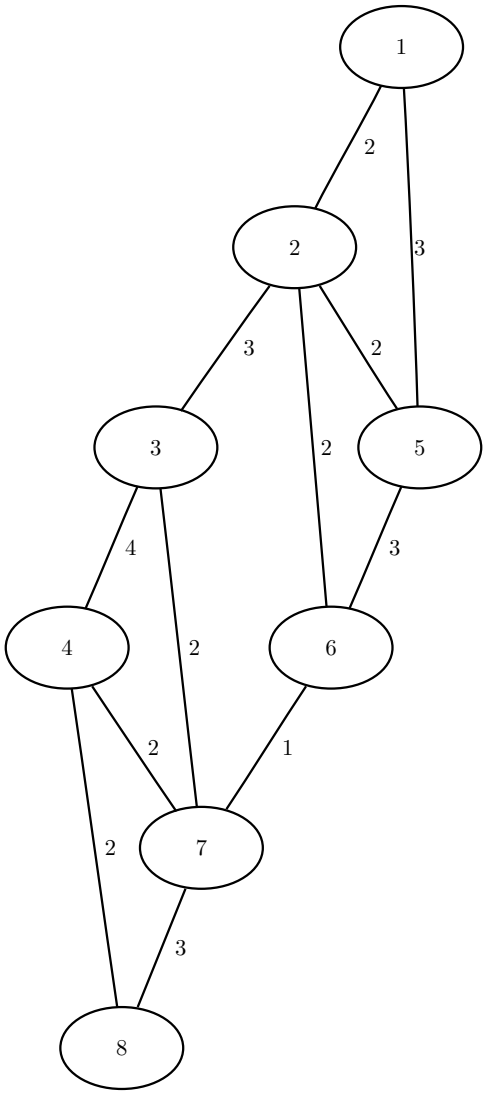
If the value of the MAXNUMCUTS= suboption is greater than 1, then the algorithm can return more than one set of cuts. The resulting cuts can be described in terms of partitions of the nodes of the graph or the links in the cut-sets. The example in the next section illustrates several ways to manipulate the output from the minimum cut algorithm. The node partition is specified in the PARTITIONS= suboption of the OUT= option in the SOLVE WITH NETWORK statement. Each tuple in this set has a cut ID and a node. PROC OPTMODEL provides only the smaller of the two subsets that form each partition. You can use the DIFF set operator to get the complement of each partition. The cut-set is specified in the CUTSETS= suboption of the OUT= option. This set contains the cut ID and the corresponding list of links.

The network solver uses the Stoer-Wagner algorithm (Stoer and Wagner 1997) to compute the minimum cuts. This algorithm runs in time $O(|N||A| + |N|^2 \log |N|)$.

### Minimum Cut for a Simple Undirected Graph

As a simple example, consider the weighted undirected graph in Figure 9.45.

**Figure 9.45** A Simple Undirected Graph




The links data set can be represented as follows:

```
data LinkSetIn;
   input from to weight @@;
   datalines;
1 2 2  1 5 3  2 3 3  2 5 2  2 6 2
3 4 4  3 7 2  4 7 2  4 8 2  5 6 3
6 7 1  7 8 3
;
```

The following statements calculate minimum cuts in the graph and output the results in the data set MinCut:

```
proc optmodel;
   set<num,num> LINKS;
   num weight{LINKS};
   read data LinkSetIn into LINKS=[from to] weight;
   set<num> NODES = union {<i,j> in LINKS} {i,j};
   set<num,num> PARTITIONS;
   set<num,num,num> CUTSETS;

   solve with NETWORK /
      loglevel = moderate
      links    = (weight=weight)
      mincut   = (maxnumcuts=3)
      out      = (partitions=PARTITIONS cutsets=CUTSETS)
   ;
   set CUTS = setof {<cut,i,j> in CUTSETS} cut;
   num minCutWeight {cut in CUTS} = sum {<(cut),i,j> in CUTSETS} weight[i,j];
   print minCutWeight;

   for { cut in CUTS }
      put "Cut ID: "    cut
          "Partition: " ( slice( <cut,*>, PARTITIONS ) )
          "and "        ( NODES diff slice( <cut,*>, PARTITIONS ) )
          "Cut: "       ( slice( <cut,*,*>, CUTSETS ) )
          "Weight: "    minCutWeight[cut];

   create data MinCut from [mincut from to]=CUTSETS weight[from,to];
   num mincut {cut in CUTS, node in NODES} =
      if <cut,node> in PARTITIONS then 0 else 1;
   print mincut;
   create data NodeSetOut from [node]=NODES
      {cut in CUTS} <col('mincut_'||cut)=mincut[cut,node]>;
quit;
```

The progress of the procedure is shown in Figure 9.46.

**Figure 9.46** Network Solver Log for Minimum Cut

```
NOTE: There were 12 observations read from the data set WORK.LINKSETIN.
NOTE: The number of nodes in the input graph is 8.
NOTE: The number of links in the input graph is 12.
NOTE: Processing the minimum-cut problem.
NOTE: The minimum-cut algorithm found 3 cuts.
NOTE: The cut 1 has weight 4.
NOTE: The cut 2 has weight 5.
NOTE: The cut 3 has weight 5.
NOTE: Processing the minimum-cut problem used 0.00 (cpu: 0.00) seconds.
Cut ID: 1 Partition: {3,4,7,8} and {1,2,5,6} Cut: {<2,3>,<6,7>} Weight: 4
Cut ID: 2 Partition: {8} and {1,2,5,3,6,4,7} Cut: {<4,8>,<7,8>} Weight: 5
Cut ID: 3 Partition: {1} and {2,5,3,6,4,7,8} Cut: {<1,2>,<1,5>} Weight: 5
NOTE: The data set WORK.MINCUT has 6 observations and 4 variables.
NOTE: The data set WORK.NODESETOUT has 8 observations and 4 variables.
```

The data set NodeSetOut now contains the partition of the nodes for each cut, shown in Figure 9.47.

**Figure 9.47** Minimum Cut Node Partition

| node | mincut_1 | mincut_2 | mincut_3 |
|---|---|---|---|
| 1 | 1 | 1 | 0 |
| 2 | 1 | 1 | 1 |
| 5 | 1 | 1 | 1 |
| 3 | 0 | 1 | 1 |
| 6 | 1 | 1 | 1 |
| 4 | 0 | 1 | 1 |
| 7 | 0 | 1 | 1 |
| 8 | 0 | 0 | 1 |

The data set MinCut contains the links in the cut-sets for each cut. This data set is shown in Figure 9.48, which also shows each cut separately.

**Figure 9.48** Minimum Cut-sets

| mincut | from | to | weight |
|---|---|---|---|
| 1 | 2 | 3 | 3 |
| 1 | 6 | 7 | 1 |
| 2 | 4 | 8 | 2 |
| 2 | 7 | 8 | 3 |
| 3 | 1 | 2 | 2 |
| 3 | 1 | 5 | 3 |

**mincut=1**

| from | to | weight |
|---|---|---|
| 2 | 3 | 3 |
| 6 | 7 | 1 |
| **mincut** | | **4** |

**mincut=2**

| from | to | weight |
|---|---|---|
| 4 | 8 | 2 |
| 7 | 8 | 3 |
| **mincut** | | **5** |

**mincut=3**

| from | to | weight |
|---|---|---|
| 1 | 2 | 2 |
| 1 | 5 | 3 |
| **mincut** | | **5** |
| | | **14** |

## Minimum Spanning Tree

A *spanning tree* of a connected undirected graph is a subgraph that is a tree that connects all the nodes together. When weights have been assigned to the links, a *minimum spanning tree* (MST) is a spanning tree whose sum of link weights is less than or equal to the sum of link weights of every other spanning tree. More generally, any undirected graph (not necessarily connected) has a *minimum spanning forest*, which is a union of minimum spanning trees of its connected components.

In the network solver, you can invoke the minimum spanning tree algorithm by using the MINSPANTREE option. This algorithm can be used only on undirected graphs.

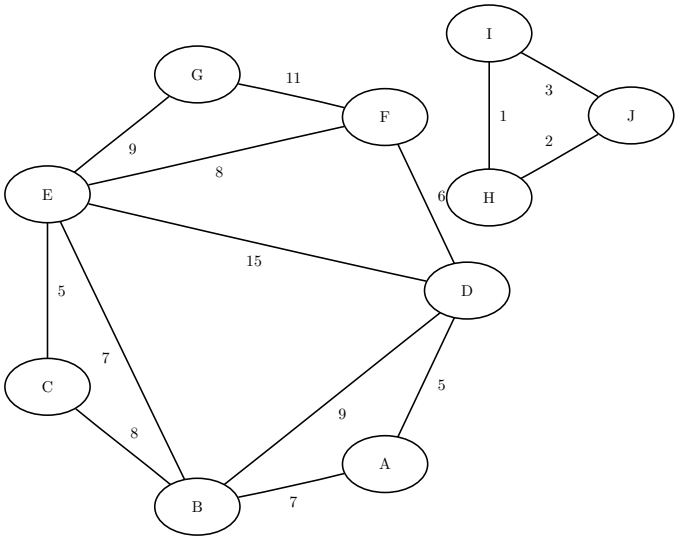
The resulting minimum spanning tree is contained in the set that is specified in the FOREST= suboption of the OUT= option in the SOLVE WITH NETWORK statement.

The network solver uses Kruskal's algorithm (Kruskal 1956) to compute the minimum spanning tree. This algorithm runs in time $O(|A| \log |N|)$ and therefore should scale to very large graphs.

### Minimum Spanning Tree for a Simple Undirected Graph

As a simple example, consider the weighted undirected graph in Figure 9.49.

**Figure 9.49** A Simple Undirected Graph



The links data set can be represented as follows:

```
data LinkSetIn;
   input from $ to $ weight @@;
   datalines;
A B  7  A D  5  B C 8  B D 9  B E 7
C E  5  D E 15  D F 6  E F 8  E G 9
F G 11  H I  1  I J 3  H J 2
;
```

The following statements calculate a minimum spanning forest and output the results in the data set MinSpanForest:

```
proc optmodel;
   set<str,str> LINKS;
   num weight{LINKS};
   read data LinkSetIn into LINKS=[from to] weight;
   set<str,str> FOREST;

   solve with NETWORK /
      links       = (weight=weight)
      minspantree
      out         = (forest=FOREST)
   ;

   put FOREST;
   create data MinSpanForest from [from to]=FOREST weight;
quit;
```

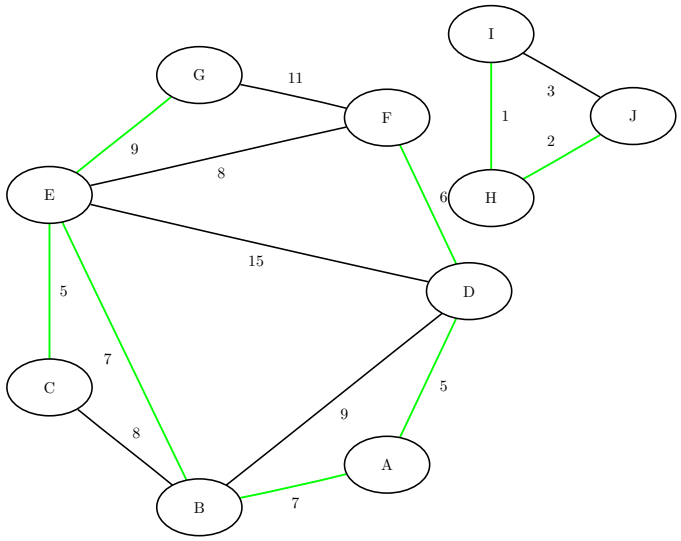
The data set MinSpanForest now contains the links that belong to a minimum spanning forest, which is shown in Figure 9.50.

**Figure 9.50** Minimum Spanning Forest

| from | to | weight |
|---|---|---|
| H | I | 1 |
| H | J | 2 |
| A | D | 5 |
| C | E | 5 |
| D | F | 6 |
| B | E | 7 |
| A | B | 7 |
| E | G | 9 |
| | | **42** |

The minimal cost links are shown in green in Figure 9.51.

**Figure 9.51** Minimum Spanning Forest




For a more detailed example, see "Example 9.5: Minimum Spanning Tree for Computer Network Topology" on page 475.

## Shortest Path

A *shortest path* between two nodes *u* and *v* in a graph is a path that starts at *u* and ends at *v* and has the lowest total link weight. The starting node is called the *source node*, and the ending node is called the *sink node*.

In the network solver, you can calculate shortest paths by using the SHORTPATH= option.

By default, the network solver finds shortest paths for all pairs. That is, it finds a shortest path for each possible combination of source and sink nodes. Alternatively, you can use the SOURCE= suboption to fix a particular source node and find shortest paths from the fixed source node to all possible sink nodes. Conversely, by using the SINK= suboption, you can fix a sink node and find shortest paths from all possible source nodes to the fixed sink node. By using both suboptions together, you can request one particular shortest path for a specific source-sink pair. In addition, you can use the SOURCE= and SINK= suboptions to define a list of source-sink pairs to process. The following sections show examples of these suboptions.

Which algorithm the network solver uses to find shortest paths depends on the data. The algorithm and run-time complexity for each graph type is shown in Table 9.21.

**Table 9.21** Algorithms for Shortest Paths

| Graph Type | Algorithm | Complexity (per Source Node) |
|---|---|---|
| Unweighted | Breadth-first search | $O(\|N\| + \|A\|)$ |
| Weighted (nonnegative) | Dijkstra's algorithm | $O(\|N\| \log \|N\| + \|A\|)$ |
| Weighted (positive and negative allowed) | Bellman-Ford algorithm | $O(\|N\|\|A\|)$ |

Details for each algorithm can be found in Ahuja, Magnanti, and Orlin (1993).

For weighted graphs, the algorithm uses the parameter that is specified in the WEIGHT= suboption of the SHORTPATH= option to evaluate a path's total weight (cost).

## Outputs

The shortest path algorithm produces up to two outputs. The output set that you specify in the SPPATHS= suboption contains the links of a shortest path for each source-sink pair combination. The output parameter that you specify in the SPWEIGHTS= suboption contains the total weight for the shortest path for each source-sink pair combination.

### *SPPATHS= Set*

The SPPATHS= set contains the links present in the shortest path for each source-sink pair.

The individual links in this set always appear in the order that you provide. If you provide link $(u, v)$ and solve a shortest path problem on an undirected graph, and that path visits node $v$ before node $u$, then this set will contain link $(u, v)$, not link $(v, u)$, which is not part of the network. This approach simplifies indexing throughout your model. In certain use cases, especially for producing output, you might prefer to see the links in the order in which the nodes are visited. For one way to do that, see "Example 9.7: Traveling Salesman Tour through US Capital Cities" on page 480.

For large graphs and a large requested number of source-sink pairs, this set can be extremely large. For extremely large graphs, generating the output can sometimes take longer than computing the shortest paths. For example, using the US road network data for the state of New York, the data contain a directed graph that has 264,346 nodes. Finding the shortest path for all pairs from only one source node results in 140,969,120 observations, which is a set of size 11 GB. Finding shortest paths for all pairs from all nodes would produce an enormous set.

The SPPATHS= set contains the following tuple members:

1. the source node of this shortest path
2. the sink node of this shortest path
3. for this source-sink pair, the order of this link in a shortest path
4. the *tail* node of this link in a shortest path
5. the *head* node of this link in a shortest path

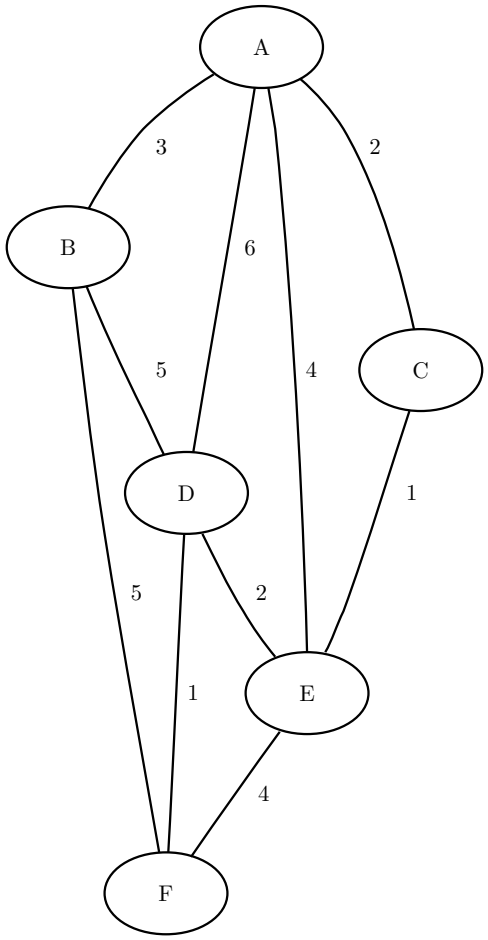
### *SPWEIGHTS= Array*

This array contains the total weight for the shortest path for each of the source-sink pairs.

## Shortest Paths for All Pairs

This example illustrates the use of the shortest path algorithm for all source-sink pairs on the simple undirected graph $G$ that is shown in Figure 9.52.

**Figure 9.52** A Simple Undirected Graph $G$




The undirected graph $G$ can be represented by the following links data set, LinkSetIn:

```
data LinkSetIn;
   input from $ to $ weight @@;
   datalines;
A B 3  A C 2  A D 6  A E 4  B D 5
B F 5  C E 1  D E 2  D F 1  E F 4
;
```

The following statements calculate shortest paths for all source-sink pairs:

```
proc optmodel;
   set <str,str> LINKS;
   num weight{LINKS};
   read data LinkSetIn into LINKS=[from to] weight;
   set <str,str,num,str,str> PATHS; /* source, sink, order, from, to */
   set NODES = union{<i,j> in LINKS} {i,j};
   num path_length{NODES, NODES};

   solve with NETWORK /
      links     = (weight=weight)
      shortpath
      out       = (sppaths=PATHS spweights=path_length)
   ;

   put PATHS;
   print path_length;
   create data ShortPathP from [source sink order from to]=PATHS
      weight[from,to];
   create data ShortPathW from [source sink]
      path_weight=path_length;
quit;
```

The data set ShortPathP contains the shortest paths and is shown in Figure 9.53.

**Figure 9.53** All-Pairs Shortest Paths

**ShortPathP**

| source | sink | order | from | to | weight |
|---|---|---|---|---|---|
| A | B | 1 | A | B | 3 |
| A | C | 1 | A | C | 2 |
| A | D | 1 | A | C | 2 |
| A | D | 2 | C | E | 1 |
| A | D | 3 | D | E | 2 |
| A | E | 1 | A | C | 2 |
| A | E | 2 | C | E | 1 |
| A | F | 1 | A | C | 2 |
| A | F | 2 | C | E | 1 |
| A | F | 3 | D | E | 2 |
| A | F | 4 | D | F | 1 |
| B | A | 1 | A | B | 3 |
| B | C | 1 | A | B | 3 |
| B | C | 2 | A | C | 2 |
| B | D | 1 | B | D | 5 |
| B | E | 1 | A | B | 3 |
| B | E | 2 | A | C | 2 |
| B | E | 3 | C | E | 1 |
| B | F | 1 | B | F | 5 |
| C | A | 1 | A | C | 2 |
| C | B | 1 | A | C | 2 |
| C | B | 2 | A | B | 3 |
| C | D | 1 | C | E | 1 |
| C | D | 2 | D | E | 2 |
| C | E | 1 | C | E | 1 |
| C | F | 1 | C | E | 1 |
| C | F | 2 | D | E | 2 |
| C | F | 3 | D | F | 1 |
| D | A | 1 | D | E | 2 |
| D | A | 2 | C | E | 1 |
| D | A | 3 | A | C | 2 |
| D | B | 1 | B | D | 5 |
| D | C | 1 | D | E | 2 |
| D | C | 2 | C | E | 1 |
| D | E | 1 | D | E | 2 |
| D | F | 1 | D | F | 1 |
| E | A | 1 | C | E | 1 |
| E | A | 2 | A | C | 2 |
| E | B | 1 | C | E | 1 |
| E | B | 2 | A | C | 2 |
| E | B | 3 | A | B | 3 |
| E | C | 1 | C | E | 1 |
| E | D | 1 | D | E | 2 |
| E | F | 1 | D | E | 2 |
| E | F | 2 | D | F | 1 |
| F | A | 1 | D | F | 1 |
| F | A | 2 | D | E | 2 |
| F | A | 3 | C | E | 1 |
| F | A | 4 | A | C | 2 |
| F | B | 1 | B | F | 5 |
| F | C | 1 | D | F | 1 |
| F | C | 2 | D | E | 2 |
| F | C | 3 | C | E | 1 |
| F | D | 1 | D | F | 1 |
| F | E | 1 | D | F | 1 |
| F | E | 2 | D | E | 2 |

The data set ShortPathW contains the path weight for the shortest paths of each source-sink pair and is shown in Figure 9.54.

**Figure 9.54** All-Pairs Shortest Paths Summary

**ShortPathW**

| source | sink | path_weight |
|---|---|---|
| A | A | . |
| A | B | 3 |
| A | C | 2 |
| A | D | 5 |
| A | E | 3 |
| A | F | 6 |
| B | A | 3 |
| B | B | . |
| B | C | 5 |
| B | D | 5 |
| B | E | 6 |
| B | F | 5 |
| C | A | 2 |
| C | B | 5 |
| C | C | . |
| C | D | 3 |
| C | E | 1 |
| C | F | 4 |
| D | A | 5 |
| D | B | 5 |
| D | C | 3 |
| D | D | . |
| D | E | 2 |
| D | F | 1 |
| E | A | 3 |
| E | B | 6 |
| E | C | 1 |
| E | D | 2 |
| E | E | . |
| E | F | 3 |
| F | A | 6 |
| F | B | 5 |
| F | C | 4 |
| F | D | 1 |
| F | E | 3 |
| F | F | . |

When you are interested only in the source-sink pair that has the longest shortest path, you can use the PATHS= suboption. This suboption affects only the output processing; it does not affect the computation. All the designated source-sink shortest paths are calculated, but only the longest ones are written to the output set.

The following statements display only the longest shortest paths:

```
proc optmodel;
   set <str,str> LINKS;
   num weight{LINKS};
   read data LinkSetIn into LINKS=[from to] weight;
   set <str,str,num,str,str> PATHS; /* source, sink, order, from, to */

   solve with NETWORK /
      links     = ( weight = weight )
      shortpath = ( paths = longest )
      out       = ( sppaths = PATHS )
   ;

   put PATHS;
   create data ShortPathLong from [source sink order from to]=PATHS
      weight[from,to];
quit;
```

The data set ShortPathLong now contains the longest shortest paths and is shown in Figure 9.55.

**Figure 9.55** Longest Shortest Paths

**ShortPathLong**

| source | sink | order | from | to | weight |
|---|---|---|---|---|---|
| A | F | 1 | A | C | 2 |
| A | F | 2 | C | E | 1 |
| A | F | 3 | D | E | 2 |
| A | F | 4 | D | F | 1 |
| B | E | 1 | A | B | 3 |
| B | E | 2 | A | C | 2 |
| B | E | 3 | C | E | 1 |
| E | B | 1 | C | E | 1 |
| E | B | 2 | A | C | 2 |
| E | B | 3 | A | B | 3 |
| F | A | 1 | D | F | 1 |
| F | A | 2 | D | E | 2 |
| F | A | 3 | C | E | 1 |
| F | A | 4 | A | C | 2 |

## Shortest Paths for a Subset of Source-Sink Pairs

This section illustrates the use of the SOURCE= and SINK= suboptions and the shortest path algorithm to calculate shortest paths for a subset of source-sink pairs. If $S$ denotes the nodes in the SOURCE= set and $T$ denotes the nodes in the SINK= set, the network solver calculates all the source-sink pairs in the crossproduct of these two sets.

For example, the following statements calculate a shortest path for the four combinations of source-sink pairs in $S \times T = \{A, C\} \times \{B, F\}$:

```
proc optmodel;
   set <str,str> LINKS;
   num weight{LINKS};
   read data LinkSetIn into LINKS=[from to] weight;
   set <str,str,num,str,str> PATHS; /* source, sink, order, from, to */
   set SOURCES = / A C /;
   set SINKS   = / B F /;

   solve with NETWORK /
      links     = (weight=weight)
      shortpath = (source=SOURCES sink=SINKS)
      out       = (sppaths=PATHS)
   ;

   put PATHS;
   create data ShortPath from [source sink order from to]=PATHS weight[from,to];
quit;
```

The data set ShortPath contains the shortest paths and is shown in Figure 9.56.

**Figure 9.56** Shortest Paths for a Subset of Source-Sink Pairs

**ShortPath**

| source | sink | order | from | to | weight |
|---|---|---|---|---|---|
| A | B | 1 | A | B | 3 |
| A | F | 1 | A | C | 2 |
| A | F | 2 | C | E | 1 |
| A | F | 3 | D | E | 2 |
| A | F | 4 | D | F | 1 |
| C | B | 1 | A | C | 2 |
| C | B | 2 | A | B | 3 |
| C | F | 1 | C | E | 1 |
| C | F | 2 | D | E | 2 |
| C | F | 3 | D | F | 1 |

## Shortest Paths for a Subset of Source or Sink Pairs

This section illustrates the use of the shortest path algorithm to calculate shortest paths between a subset of source (or sink) nodes and all other sink (or source) nodes.

In this case, you designate the subset of source (or sink) nodes in the node set by specifying the SOURCE= (or SINK=) suboption. By specifying only one of the suboptions, you indicate that you want the network solver to calculate all pairs from a subset of source nodes (or to calculate all pairs to a subset of sink nodes).

For example, the following statements calculate all the shortest paths from nodes B and E.:

```
proc optmodel;
   set <str,str> LINKS;
   num weight{LINKS};
   read data LinkSetIn into LINKS=[from to] weight;
   set <str,str,num,str,str> PATHS; /* source, sink, order, from, to */
   set SOURCES = / B E /;

   solve with NETWORK /
      links     = (weight=weight)
      shortpath = (source=SOURCES)
      out       = (sppaths=PATHS)
   ;

   put PATHS;
   create data ShortPath from [source sink order from to]=PATHS weight[from,to];
quit;
```

The data set ShortPath contains the shortest paths and is shown in Figure 9.57.

**Figure 9.57** Shortest Paths for a Subset of Source Pairs

**ShortPath**

| source | sink | order | from | to | weight |
|---|---|---|---|---|---|
| B | A | 1 | A | B | 3 |
| B | C | 1 | A | B | 3 |
| B | C | 2 | A | C | 2 |
| B | D | 1 | B | D | 5 |
| B | E | 1 | A | B | 3 |
| B | E | 2 | A | C | 2 |
| B | E | 3 | C | E | 1 |
| B | F | 1 | B | F | 5 |
| E | A | 1 | C | E | 1 |
| E | A | 2 | A | C | 2 |
| E | B | 1 | C | E | 1 |
| E | B | 2 | A | C | 2 |
| E | B | 3 | A | B | 3 |
| E | C | 1 | C | E | 1 |
| E | D | 1 | D | E | 2 |
| E | F | 1 | D | E | 2 |
| E | F | 2 | D | F | 1 |

Conversely, the following statements calculate all the shortest paths to nodes B and E.:

```
proc optmodel;
   set <str,str> LINKS;
   num weight{LINKS};
   read data LinkSetIn into LINKS=[from to] weight;
   set <str,str,num,str,str> PATHS; /* source, sink, order, from, to */
   set SINKS = / B E /;
```

```
   solve with NETWORK /
      links     = (weight=weight)
      shortpath = (sink=SINKS)
      out       = (sppaths=PATHS)
   ;

   put PATHS;
   create data ShortPath from [source sink order from to]=PATHS weight[from,to];
quit;
```

The data set ShortPath contains the shortest paths and is shown in Figure 9.58.

**Figure 9.58** Shortest Paths for a Subset of Sink Pairs

**ShortPath**

| source | sink | order | from | to | weight |
|---|---|---|---|---|---|
| A | B | 1 | A | B | 3 |
| A | E | 1 | A | C | 2 |
| A | E | 2 | C | E | 1 |
| B | E | 1 | A | B | 3 |
| B | E | 2 | A | C | 2 |
| B | E | 3 | C | E | 1 |
| C | B | 1 | A | C | 2 |
| C | B | 2 | A | B | 3 |
| C | E | 1 | C | E | 1 |
| D | B | 1 | B | D | 5 |
| D | E | 1 | D | E | 2 |
| E | B | 1 | C | E | 1 |
| E | B | 2 | A | C | 2 |
| E | B | 3 | A | B | 3 |
| F | B | 1 | B | F | 5 |
| F | E | 1 | D | F | 1 |
| F | E | 2 | D | E | 2 |

## Shortest Paths for One Source-Sink Pair

This section illustrates the use of the shortest path algorithm to calculate shortest paths between one source-sink pair by using the SOURCE= and SINK= suboptions.

The following statements calculate a shortest path between node *C* and node *F*:

```
proc optmodel;
   set <str,str> LINKS;
   num weight{LINKS};
   read data LinkSetIn into LINKS=[from to] weight;
   set <str,str,num,str,str> PATHS; /* source, sink, order, from, to */
   set SOURCES = / C /;
   set SINKS   = / F /;

   solve with NETWORK /
      links     = (weight=weight)
      shortpath = (source=SOURCES sink=SINKS)
      out       = (sppaths=PATHS)
   ;
```

```
    put PATHS;
    create data ShortPath from [source sink order from to]=PATHS weight[from,to];
quit;
```

The data set ShortPath contains this shortest path and is shown in Figure 9.59.

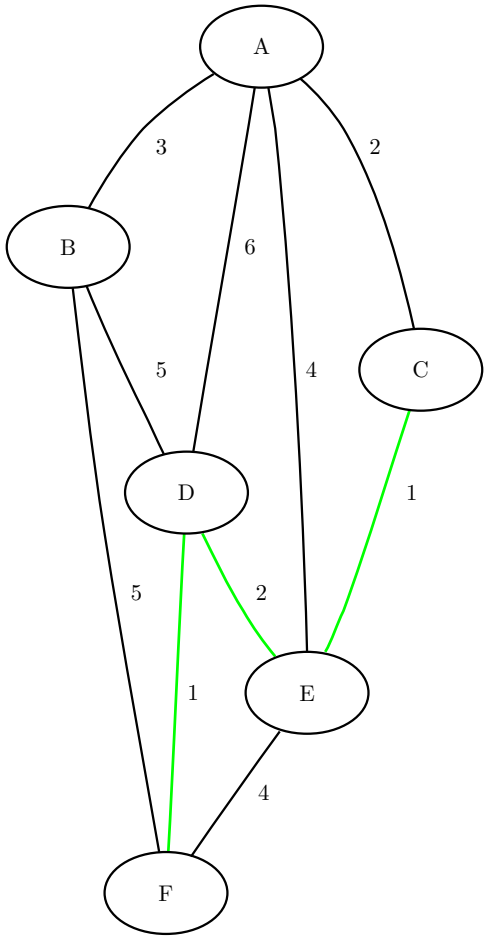
**Figure 9.59** Shortest Paths for One Source-Sink Pair

**ShortPath**

| source | sink | order | from | to | weight |
|---|---|---|---|---|---|
| C | F | 1 | C | E | 1 |
| C | F | 2 | D | E | 2 |
| C | F | 3 | D | F | 1 |

The shortest path is shown graphically in Figure 9.60.

**Figure 9.60** Shortest Path between Nodes *C* and *F*



## Shortest Paths with Auxiliary Weight Calculation

This section illustrates the use of the shortest path algorithm, where auxiliary weights are used to calculate the shortest paths between all source-sink pairs.

Consider a links data set in which the auxiliary weight is a counter for each link:

```
data LinkSetIn;
   input from $ to $ weight count @@;
   datalines;
A B 3 1  A C 2 1  A D 6 1  A E 4 1  B D 5 1
B F 5 1  C E 1 1  D E 2 1  D F 1 1  E F 4 1
;
```

The following statements calculate shortest paths for all source-sink pairs:

```
proc optmodel;
   set <str,str> LINKS;
   num weight{LINKS};
   num count{LINKS};
   read data LinkSetIn into LINKS=[from to] weight count;
   set <str,str,num,str,str> PATHS; /* source, sink, order, from, to */
   set NODES = union{<i,j> in LINKS} {i,j};
   num path_length{i in NODES, j in NODES: i ~= j};

   solve with NETWORK /
      links     = (weight=weight)
      shortpath
      out       = (sppaths=PATHS spweights=path_length)
   ;

   put PATHS;
   num path_weight2{source in NODES, sink in NODES: source ~= sink} =
      sum {<(source),(sink),order,from,to> in PATHS} count[from,to];
   print path_length path_weight2;
   create data ShortPathW from [source sink]
      path_weight=path_length path_weight2;
quit;
```

The data set ShortPathW contains the total path weight for shortest paths in each source-sink pair and is shown in Figure 9.61. Because the variable count in LinkSetIn has a value of 1 for all links, the value in the output data set variable path_weights2 contains the number of links in each shortest path.

**Figure 9.61** Shortest Paths Including Auxiliary Weights in Calculation

**ShortPathW**

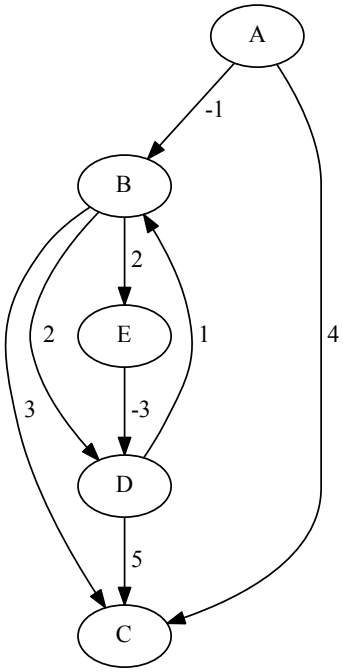
| source | sink | path_weight | path_weight2 |
|---|---|---|---|
| A | B | 3 | 1 |
| A | C | 2 | 1 |
| A | D | 5 | 3 |
| A | E | 3 | 2 |
| A | F | 6 | 4 |
| B | A | 3 | 1 |
| B | C | 5 | 2 |
| B | D | 5 | 1 |
| B | E | 6 | 3 |
| B | F | 5 | 1 |
| C | A | 2 | 1 |
| C | B | 5 | 2 |
| C | D | 3 | 2 |
| C | E | 1 | 1 |
| C | F | 4 | 3 |
| D | A | 5 | 3 |
| D | B | 5 | 1 |
| D | C | 3 | 2 |
| D | E | 2 | 1 |
| D | F | 1 | 1 |
| E | A | 3 | 2 |
| E | B | 6 | 3 |
| E | C | 1 | 1 |
| E | D | 2 | 1 |
| E | F | 3 | 2 |
| F | A | 6 | 4 |
| F | B | 5 | 1 |
| F | C | 4 | 3 |
| F | D | 1 | 1 |
| F | E | 3 | 2 |

The section "Getting Started: Network Solver" on page 376 shows an example of using the shortest path algorithm to minimize travel to and from work based on traffic conditions.

## Shortest Paths with Negative Link Weights

This section illustrates the use of the shortest path algorithm on a simple directed graph *G* with negative link weights, shown in Figure 9.62.

**Figure 9.62** A Simple Directed Graph *G* with Negative Link Weights




The following statements call PROC OPTMODEL and declare the directed graph *G* by using set and array literals. For more information about literals, see the section "NUMBER, STRING, and SET Parameter Declarations" on page 45 in Chapter 5, "The OPTMODEL Procedure."

```
proc optmodel;
   set LINKS            = / <A B> <A C> <B C> <B D> <B E> <D B> <D C> <E D> /;
   num weight{LINKS} init [   -1     4     3     2     2     1     5    -3  ];
```

The next statements declare a set of the correct type for path output and calculate the shortest paths between source node *E* and sink node *B*:

```
   set NODES = union{<i,j> in LINKS} {i,j};
   /* Use the type (in this case, STRING) of NODES but leave PATHS empty */
   set PATHS init {NODES,NODES,/0/,NODES,NODES:0};
   set SOURCE = /E/, SINK = /B/;
   solve with NETWORK /
      links     = ( weight = weight )
      direction = directed
      shortpath = ( source = SOURCE sink = SINK )
      out       = ( sppaths = PATHS )
   ;
   put "Path and Weight: " (setof{<s,t,i,u,v> in PATHS} <u,v,weight[u,v]> );
```

As shown in Figure 9.63, the network solver identifies a shortest path that has negative weights.

**Figure 9.63** SOLVE WITH NETWORK Log: Shortest Paths with Negative Link Weights

```
NOTE: The number of nodes in the input graph is 5.
NOTE: The number of links in the input graph is 8.
NOTE: Processing the shortest paths problem.
NOTE: Processing the shortest paths problem used 0.00 (cpu: 0.00) seconds.
Path and Weight: {<'E','D',-3>,<'D','B',1>}
```

If you reduce the weight on link $(B, E)$ from 2 units to 1 unit, there is a negative weight cycle $(E \rightarrow D \rightarrow B \rightarrow E)$. The Bellman-Ford algorithm catches this and produces an error, as shown in Figure 9.64.

```
   weight['B','E'] = 1;
   solve with NETWORK /
      links     = (weight=weight)
      direction = directed
      shortpath = ( source = SOURCE sink = SINK )
      out       = ( sppaths = PATHS )
   ;
   put _SOLUTION_STATUS_=;
quit;
```

**Figure 9.64** SOLVE WITH NETWORK Log: Negative Weight Cycle

```
NOTE: The number of nodes in the input graph is 5.
NOTE: The number of links in the input graph is 8.
NOTE: Processing the shortest paths problem.
ERROR: The graph contains a negative weight cycle.
NOTE: Processing the shortest paths problem used 0.00 (cpu: 0.00) seconds.
_SOLUTION_STATUS_=BAD_PROBLEM_TYPE
```

## Transitive Closure

The *transitive closure* of a graph $G$ is a graph $G^T = (N, A^T)$ such that for all $i, j \in N$ there is a link $(i, j) \in A^T$ if and only if there exists a path from $i$ to $j$ in $G$.

The transitive closure of a graph can help you efficiently answer questions about reachability. Suppose you want to answer the question of whether you can get from node $i$ to node $j$ in the original graph $G$. Given the transitive closure $G^T$ of $G$, you can simply check for the existence of link $(i, j)$ to answer the question. Transitive closure has many applications, including speeding up the processing of structured query languages, which are often used in databases.

In the network solver, you can invoke the transitive closure algorithm by using the TRANSITIVE_CLOSURE option.

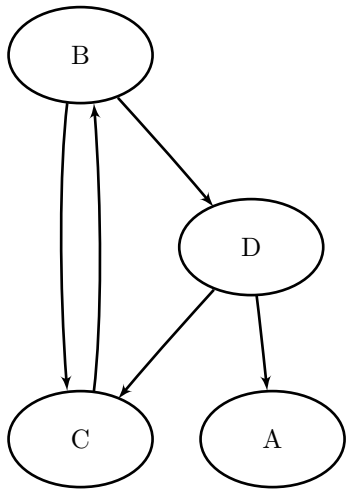
The results for the transitive closure algorithm are written to the set that is specified in the CLOSURE= suboption of the OUT= option.

The algorithm that the network solver uses to compute transitive closure is a sparse version of the Floyd-Warshall algorithm (Cormen, Leiserson, and Rivest 1990). This algorithm runs in time $O(|N|^3)$ and therefore might not scale to very large graphs.

## Transitive Closure of a Simple Directed Graph

This example illustrates the use of the transitive closure algorithm on the simple directed graph $G$ that is shown in Figure 9.65.

**Figure 9.65** A Simple Directed Graph $G$




The directed graph $G$ can be represented by the links data set LinkSetIn as follows:

```
data LinkSetIn;
   input from $ to $ @@;
   datalines;
B C  B D  C B  D A  D C
;
```

The following statements calculate the transitive closure and output the results in the data set TransClosure:

```
proc optmodel;
   set<str,str> LINKS;
   read data LinkSetIn into LINKS=[from to];
   set<str,str> CAN_REACH;

   solve with NETWORK /
      graph_direction = directed
      links = ( include = LINKS )
      transcl
      out = ( closure = CAN_REACH )
   ;

   put CAN_REACH;
   create data TransClosure from [from to]=CAN_REACH;
quit;
```

The data set TransClosure contains the transitive closure of $G$ and is shown in Figure 9.66.

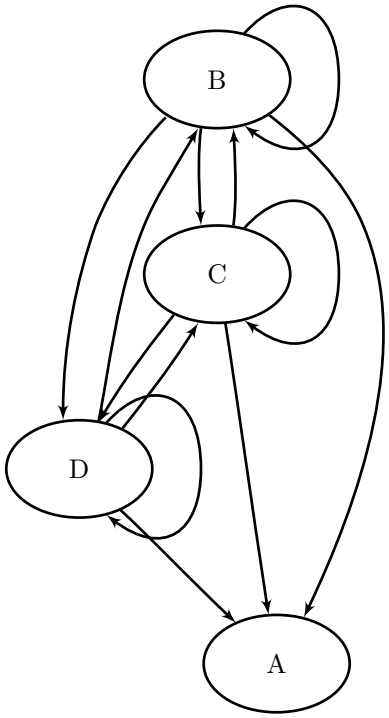
**Figure 9.66** Transitive Closure of a Simple Directed Graph

**Transitive Closure**

| from | to |
|---|---|
| B | C |
| B | D |
| C | B |
| D | A |
| D | C |
| B | B |
| D | B |
| C | C |
| C | D |
| D | D |
| B | A |
| C | A |

The transitive closure of $G$ is shown graphically in Figure 9.67.

**Figure 9.67** Transitive Closure of $G$




For a more detailed example, see Example 9.6.

## Traveling Salesman Problem

The *traveling salesman problem* (TSP) finds a minimum-cost tour in an undirected graph, $G$, that has a node set, $N$, and a link set, $A$. A *path* in a graph is a sequence of nodes, each of which has a link to the next node in the sequence. An *elementary cycle* is a path in which the start node and end node are the same and otherwise no node appears more than once in the sequence. A *Hamiltonian cycle* (or *tour*) is an elementary cycle that visits every node. In solving the TSP, then, the goal is to find a Hamiltonian cycle of minimum total cost, where the total cost is the sum of the costs of the links in the tour. Associated with each link $(i, j) \in A$ are a binary variable $x_{ij}$, which indicates whether link $x_{ij}$ is part of the tour, and a cost $c_{ij}$. Let $\delta(S) = \{(i, j) \in A \mid i \in S,\ j \notin S\}$. Then an integer linear programming formulation of the TSP (for an undirected graph $G$) is as follows:

$$
\begin{array}{lllll}
\text{minimize} & \displaystyle\sum_{(i,j)\in A} c_{ij}x_{ij} & & & \\
\text{subject to} & \displaystyle\sum_{(i,j)\in\delta(i)} x_{i,j} = 2 & i \in N & & \text{(two\_match)} \\
& \displaystyle\sum_{(i,j)\in\delta(S)} x_{ij} \geq 2 & S \subset N,\ 2 \leq |S| \leq |N| - 1 & & \text{(subtour\_elim)} \\
& x_{ij} \in \{0, 1\} & (i, j) \in A & &
\end{array}
$$

The equations (two_match) are the *matching constraints*, which ensure that each node has degree two in the subgraph. The inequalities (subtour_elim) are the *subtour elimination constraints* (SECs), which enforce connectivity.

For a directed graph, $G$, the same formulation and solution approach is used on an expanded graph $G'$, as described in Kumar and Li (1994). The network solver takes care of the construction of the expanded graph and returns the solution in terms of the original input graph.

In practical terms, you can think of the TSP in the context of a routing problem in which each node is a city and the links are roads that connect those cities. If you know the distance between each pair of cities, the goal is to find the shortest possible route that visits each city exactly once and returns to the starting city. The TSP has applications in planning, logistics, manufacturing, genomics, and many other areas.

In the network solver, you can invoke the traveling salesman problem solver by using the TSP= option.

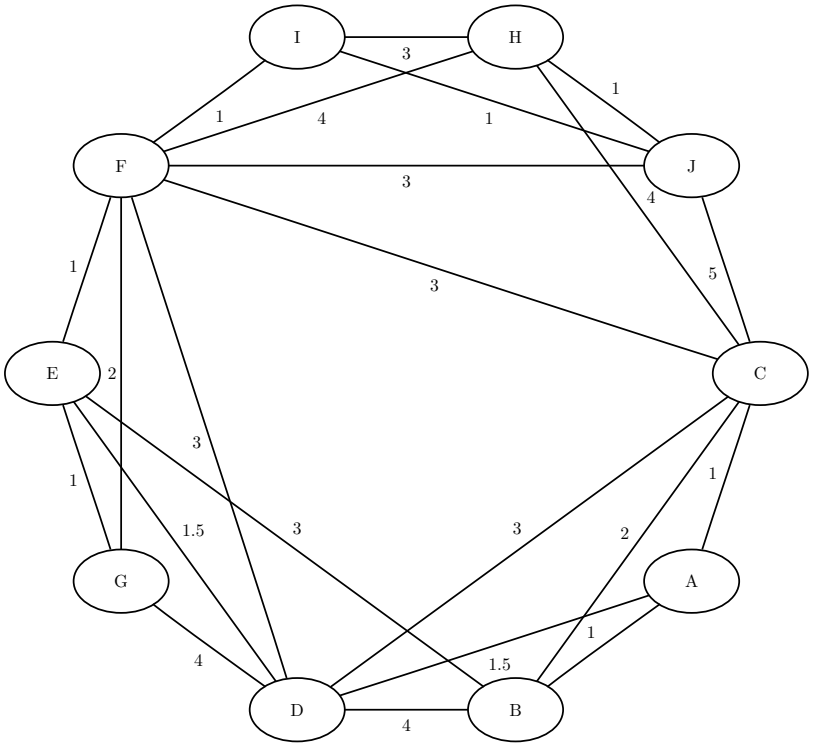
The algorithm that the network solver uses for solving a TSP is based on a variant of the branch-and-cut process described in Applegate et al. (2006).

The resulting tour is represented in two ways: in the numeric array that is specified in the ORDER= suboption in the SOLVE WITH NETWORK statement, the tour is specified as a sequence of nodes; in the set that is specified in the TOUR= suboption of the TSP option, the tour is specified as a list of links in the optimal tour.

### Traveling Salesman Problem Applied to a Simple Undirected Graph

As a simple example, consider the weighted undirected graph in Figure 9.68.

**Figure 9.68** A Simple Undirected Graph




You can represent the links data set as follows:

```
data LinkSetIn;
   input from $ to $ weight @@;
   datalines;
A B 1.0   A C 1.0   A D 1.5   B C 2.0   B D 4.0
B E 3.0   C D 3.0   C F 3.0   C H 4.0   D E 1.5
D F 3.0   D G 4.0   E F 1.0   E G 1.0   F G 2.0
F H 4.0   H I 3.0   I J 1.0   C J 5.0   F J 3.0
F I 1.0   H J 1.0
;
```

The following statements calculate an optimal traveling salesman tour and output the results in the data sets TSPTour and NodeSetOut:

```
proc optmodel;
   set<str,str> EDGES;
   set<str> NODES = union{<i,j> in EDGES} {i,j};
   num weight{EDGES};
   read data LinkSetIn into EDGES=[from to] weight;
   num tsp_order{NODES};
   set<str,str> TOUR;

   solve with NETWORK /
      loglevel = moderate
      links    = (weight=weight)
      tsp
      out      = (order=tsp_order tour=TOUR)
   ;
```

```
   put TOUR;
   print {<i,j> in TOUR} weight;
   print tsp_order;
   create data NodeSetOut from [node]          tsp_order;
   create data TSPTour    from [from to]=TOUR weight;
quit;
```

The progress of the procedure is shown in Figure 9.69.

**Figure 9.69** Network Solver Log: Optimal Traveling Salesman Tour of a Simple Undirected Graph

```
NOTE: There were 22 observations read from the data set WORK.LINKSETIN.
NOTE: The number of nodes in the input graph is 10.
NOTE: The number of links in the input graph is 22.
NOTE: Processing the traveling salesman problem.
NOTE: The initial TSP heuristics found a tour with cost 16 using 0.00 (cpu:
      0.00) seconds.
NOTE: The MILP presolver value NONE is applied.
NOTE: The MILP solver is called.
NOTE: The Branch and Cut algorithm is used.
          Node  Active    Sols    BestInteger      BestBound      Gap    Time
             0       1       1     16.0000000     15.5005000    3.22%       0
             0       0       1     16.0000000     16.0000000    0.00%       0
NOTE: Optimal.
NOTE: Objective = 16.
NOTE: Processing the traveling salesman problem used 0.00 (cpu: 0.00) seconds.
{<'A','B'>,<'B','C'>,<'C','H'>,<'H','J'>,<'I','J'>,<'F','I'>,<'F','G'>,<'E','G'>
,<'D','E'>,<'A','D'>}
NOTE: The data set WORK.NODESETOUT has 10 observations and 2 variables.
NOTE: The data set WORK.TSPTOUR has 10 observations and 3 variables.
```

The data set NodeSetOut now contains a sequence of nodes in the optimal tour and is shown in Figure 9.70.

**Figure 9.70** Nodes in the Optimal Traveling Salesman Tour

**Traveling Salesman Problem**

| node | tsp_order |
|---|---|
| A | 1 |
| B | 2 |
| C | 3 |
| H | 4 |
| J | 5 |
| I | 6 |
| F | 7 |
| G | 8 |
| E | 9 |
| D | 10 |

The data set TSPTour now contains the links in the optimal tour and is shown in Figure 9.71.

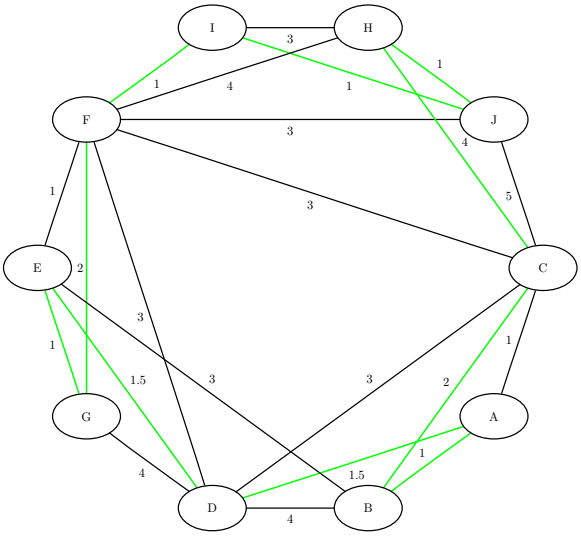
**Figure 9.71** Links in the Optimal Traveling Salesman Tour

**Traveling Salesman Problem**

| from | to | weight |
|---|---|---|
| A | B | 1.0 |
| B | C | 2.0 |
| C | H | 4.0 |
| H | J | 1.0 |
| I | J | 1.0 |
| F | I | 1.0 |
| F | G | 2.0 |
| E | G | 1.0 |
| D | E | 1.5 |
| A | D | 1.5 |
| | | **16.0** |

The minimum-cost links are shown in green in Figure 9.72.

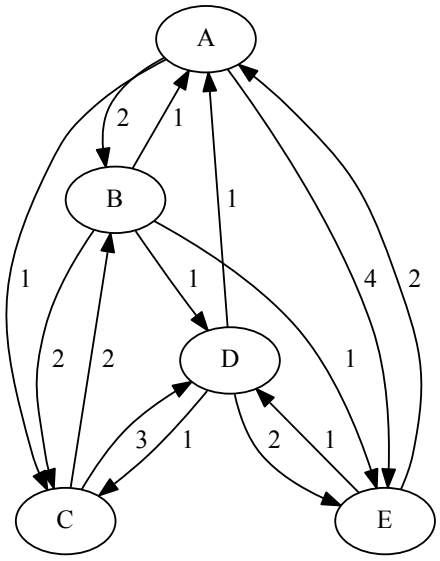
**Figure 9.72** Optimal Traveling Salesman Tour




## Traveling Salesman Problem Applied to a Simple Directed Graph

As another simple example, consider the weighted directed graph in Figure 9.73.

**Figure 9.73** A Simple Directed Graph




You can represent the links data set as follows:

```
data LinkSetIn;
   input from $ to $ weight @@;
   datalines;
A B 2.0   A C 1.0   A E 4.0
B A 1.0   B C 2.0   B D 1.0   B E 1.0
C B 2.0   C D 3.0
D A 1.0   D C 1.0   D E 2.0
E A 2.0   E D 1.0
;
```

The following statements, which are identical to those in the undirected example above except for the SOLVE statement clause DIRECTION=DIRECTED, calculate an optimal traveling salesman tour (on a directed graph) and output the results in the data sets TSPTour and NodeSetOut:

```
proc optmodel;
   set<str,str> EDGES;
   set<str> NODES = union{<i,j> in EDGES} {i,j};
   num weight{EDGES};
   read data LinkSetIn into EDGES=[from to] weight;
   num tsp_order{NODES};
   set<str,str> TOUR;

   solve with NETWORK /
      loglevel  = moderate
      links     = (weight=weight)
      direction = directed
      tsp
```

```
      out       = (order=tsp_order tour=TOUR)
   ;

   put TOUR;
   print {<i,j> in TOUR} weight;
   print tsp_order;
   create data NodeSetOut from [node]         tsp_order;
   create data TSPTour    from [from to]=TOUR weight;
quit;
```

The progress of the procedure is shown in Figure 9.74.

**Figure 9.74** Network Solver Log: Optimal Traveling Salesman Tour of a Simple Directed Graph

```
NOTE: There were 14 observations read from the data set WORK.LINKSETIN.
NOTE: The number of nodes in the input graph is 5.
NOTE: The number of links in the input graph is 14.
NOTE: The TSP solver is starting using an augmented symmetric graph with 10
      nodes and 19 links.
NOTE: Processing the traveling salesman problem.
NOTE: The initial TSP heuristics found a tour with cost 6 using 0.00 (cpu:
      0.00) seconds.
NOTE: The MILP presolver value NONE is applied.
NOTE: The MILP solver is called.
NOTE: The Branch and Cut algorithm is used.
          Node  Active    Sols    BestInteger      BestBound       Gap    Time
             0       1       1      6.0000000      5.9001000    1.69%       0
             0       0       1      6.0000000      6.0000000    0.00%       0
NOTE: Optimal.
NOTE: Objective = 6.
NOTE: Processing the traveling salesman problem used 0.00 (cpu: 0.00) seconds.
{<'A','C'>,<'C','B'>,<'B','E'>,<'E','D'>,<'D','A'>}
NOTE: The data set WORK.NODESETOUT has 5 observations and 2 variables.
NOTE: The data set WORK.TSPTOUR has 5 observations and 3 variables.
```

The data set NodeSetOut now contains a sequence of nodes in the optimal tour and is shown in Figure 9.75.

**Figure 9.75** Nodes in the Optimal Traveling Salesman Tour

| node | tsp_order |
|---|---|
| A | 1 |
| C | 2 |
| B | 3 |
| E | 4 |
| D | 5 |

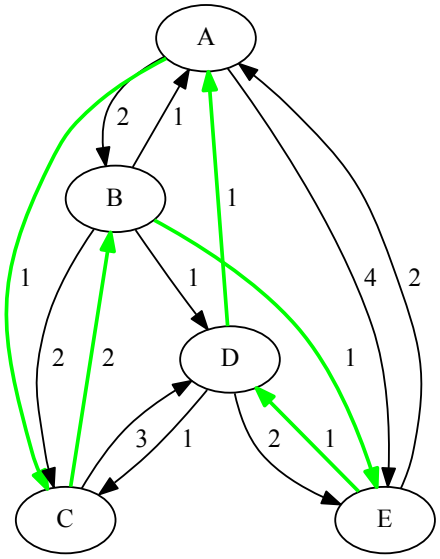
The data set TSPTour now contains the links in the optimal tour and is shown in Figure 9.76.

**Figure 9.76** Links in the Optimal Traveling Salesman Tour

| from | to | weight |
|---|---|---|
| A | C | 1 |
| C | B | 2 |
| B | E | 1 |
| E | D | 1 |
| D | A | 1 |
| | | **6** |

The minimum-cost links are shown in green in Figure 9.77.

**Figure 9.77** Optimal Traveling Salesman Tour




## Macro Variable _OROPTMODEL_

PROC OPTMODEL always creates and initializes a SAS macro variable called _OROPTMODEL_, which contains a character string. After each PROC OPTMODEL run, you can examine this macro variable by specifying `%put &_OROPTMODEL_;` and check the execution of the most recently invoked solver from the value of the macro variable.

Each keyword and value pair in _OROPTMODEL_ also appears in two other places: the PROC OPTMODEL automatic arrays _OROPTMODEL_NUM_ and _OROPTMODEL_STR_; and the ODS tables ProblemSummary and SolutionSummary, which appear after a SOLVE statement, unless you set the PRINTLEVEL= option to NONE. You can use these variables to obtain details about the solution even if you do not specify an output destination in the OUT= option.

After the solver is called, the various keywords in the variable are interpreted as follows:

**STATUS**
indicates the solver status at termination. It can take one of the following values:

| | |
|---|---|
| OK | The solver terminated normally. |
| SYNTAX_ERROR | The use of syntax is incorrect. |
| DATA_ERROR | The input data is inconsistent. |
| OUT_OF_MEMORY | Insufficient memory was allocated to the procedure. |
| IO_ERROR | A problem in reading or writing of data has occurred. |
| SEMANTIC_ERROR | An evaluation error, such as an invalid operand type, has occurred. |
| ERROR | The status cannot be classified into any of the preceding categories. |

**SOLUTION_STATUS**
indicates the solution status at termination. It can take one of the following values:

| | |
|---|---|
| ABORT_NOSOL | The solver was stopped by the user and did not find a solution. |
| ABORT_SOL | The solver was stopped by the user but still found a solution. |
| BAD_PROBLEM_TYPE | The problem type is not supported by the solver. |
| CONDITIONAL_OPTIMAL | The optimality of the solution cannot be proven. |
| ERROR | The algorithm encountered an error. |
| FAIL_NOSOL | The solver stopped due to errors and did not find a solution. |
| FAIL_SOL | The solver stopped due to errors but still found a solution. |
| FAILED | The solver failed to converge, possibly due to numerical issues. |
| HEURISTIC_NOSOL | The solver used only heuristics and did not find a solution. |
| HEURISTIC_SOL | The solver used only heuristics and found a solution. |
| INFEASIBLE | The problem is infeasible. |
| INFEASIBLE_OR_UNBOUNDED | The problem is infeasible or unbounded. |
| INTERRUPTED | The solver was interrupted by the system or the user before completing its work. |
| ITERATION_LIMIT_REACHED | The solver reached the maximum number of iterations that is specified in the MAXITER= option. |
| NODE_LIM_NOSOL | The solver reached the maximum number of nodes specified in the MAXNODES= option and did not find a solution. |
| NODE_LIM_SOL | The solver reached the maximum number of nodes specified in the MAXNODES= option and found a solution. |
| NULL_GRAPH | The graph was null (it had 0 nodes) after PROC OPTMODEL processed the SUBGRAPH= option. |
| OK | The algorithm terminated normally. |

| | |
|---|---|
| OPTIMAL | The solution is optimal. |
| OPTIMAL_AGAP | The solution is optimal within the absolute gap that is specified in the ABSOBJGAP= option. |
| OPTIMAL_COND | The solution is optimal, but some infeasibilities (primal, bound, or integer) exceed tolerances because of scaling. |
| OPTIMAL_RGAP | The solution is optimal within the relative gap that is specified in the RELOBJGAP= option. |
| OUTMEM_NOSOL | The solver ran out of memory and either did not find a solution or failed to output the solution due to insufficient memory. |
| OUTMEM_SOL | The solver ran out of memory but still found a solution. |
| SOLUTION_LIM | The solver reached the maximum number of solutions specified in the MAXCLIQUES=, MAXCYCLES=, or MAXSOLS= option. |
| TARGET | The solution is not worse than the target that is specified in the TARGET= option. |
| TIME_LIM_NOSOL | The solver reached the execution time limit specified in the MAXTIME= option and did not find a solution. |
| TIME_LIM_SOL | The solver reached the execution time limit specified in the MAXTIME= option and found a solution. |
| TIME_LIMIT_REACHED | The solver reached its execution time limit. |
| UNBOUNDED | The problem is unbounded. |

**PROBLEM_TYPE**

indicates the type of problem solved. It can take one of the following values:

| | |
|---|---|
| BICONCOMP | Biconnected components |
| CLIQUE | Maximal cliques |
| CONCOMP | Connected components |
| CYCLE | Cycle detection |
| LAP | Linear assignment (matching) |
| MCF | Minimum-cost network flow |
| MINCUT | Minimum cut |
| MST | Minimum spanning tree |
| SHORTPATH | Shortest path |
| TRANSCL | Transitive closure |
| TSP | Traveling salesman |
| NONE | This value is used when you do not specify an algorithm to run. |

**OBJECTIVE**
indicates the objective value that is obtained by the solver at termination. For problem classes that do not have an explicit objective, such as cycle, the value of this keyword within the _OROPTMODEL_ macro variable is missing (.).

**RELATIVE_GAP**
indicates the relative gap between the best integer objective (BestInteger) and the best bound on the objective function value (BestBound) upon termination of the MILP solver. The relative gap is equal to

$$|\text{BestInteger} - \text{BestBound}| / (1\text{E}{-}10 + |\text{BestBound}|)$$

**ABSOLUTE_GAP**
indicates the absolute gap between the best integer objective (BestInteger) and the best bound on the objective function value (BestBound) upon termination of the MILP solver. The absolute gap is equal to $|\text{BestInteger} - \text{BestBound}|$.

**PRIMAL_INFEASIBILITY**
indicates the maximum (absolute) violation of the primal constraints by the solution.

**BOUND_INFEASIBILITY**
indicates the maximum (absolute) violation by the solution of the lower or upper bounds (or both).

**INTEGER_INFEASIBILITY**
indicates the maximum (absolute) violation of the integrality of integer variables returned by the MILP solver.

**BEST_BOUND**
indicates the best bound on the objective function value at termination. A missing value indicates that the MILP solver was not able to obtain such a bound.

**NODES**
indicates the number of nodes enumerated by the MILP solver by using the branch-and-bound algorithm.

**ITERATIONS**
indicates the number of simplex iterations taken to solve the problem.

**PRESOLVE_TIME**
indicates the time (in seconds) used in preprocessing.

**SOLUTION_TIME**
indicates the time (in seconds) taken to solve the problem, including preprocessing time.

**NOTE:** The time reported in PRESOLVE_TIME and SOLUTION_TIME is either CPU time or real time. The type is determined by the TIMETYPE= option.

When SOLUTION_STATUS has a value of OPTIMAL, CONDITIONAL_OPTIMAL, ITERATION_LIMIT_REACHED, or TIME_LIMIT_REACHED, all terms of the _OROPTMODEL_ macro variable are present; for other values of SOLUTION_STATUS, some terms do not appear.

The following keywords within the _OROPTMODEL_ macro variable appear only with certain algorithms. The keywords convey information about the number of solutions each algorithm found:

**NUM_ARTICULATION_POINTS**
indicates the number of articulation points found. This term appears only for biconnected components.

**NUM_CLIQUES**
indicates the number of cliques found. This term appears only for clique.

**NUM_COMPONENTS**
indicates the number of components that match the definitions of the corresponding problem class. This term appears only for connected components and biconnected components.

**NUM_CYCLES**
indicates the number of cycles found that satisfy the criteria you provide. This term appears only for cycles.

# Examples: Network Solver

## Example 9.1: Articulation Points in a Terrorist Network

This example considers the terrorist communications network from the attacks on the United States on September 11, 2001, described in Krebs 2002. Figure 9.78 shows this network, which was constructed after the attacks, based on collected intelligence information.

**Figure 9.78** Terrorist Communications Network from 9/11


The full network data include 153 links. The following statements show a small subset to illustrate the use of the BICONCOMP option in this context:

```
data LinkSetInTerror911;
   input from & $32. to & $32.;
   datalines;
Abu Zubeida                Djamal Beghal
Jean-Marc Grandvisir       Djamal Beghal
Nizar Trabelsi             Djamal Beghal
Abu Walid                  Djamal Beghal
Abu Qatada                 Djamal Beghal
Zacarias Moussaoui         Djamal Beghal
Jerome Courtaillier        Djamal Beghal
Kamel Daoudi               Djamal Beghal
Abu Walid                  Kamel Daoudi
Abu Walid                  Abu Qatada
Kamel Daoudi               Zacarias Moussaoui
Kamel Daoudi               Jerome Courtaillier
```

```
   ... more lines ...

Nawaf Alhazmi              Khalid Al-Mihdhar
Osama Awadallah            Khalid Al-Mihdhar
Abdussattar Shaikh         Khalid Al-Mihdhar
Abdussattar Shaikh         Osama Awadallah
;
```

Suppose that this communications network had been discovered before the attack on 9/11. If the investigators' goal was to disrupt the flow of communication between different groups within the organization, then they would want to focus on the people who are articulation points in the network.

To find the articulation points, use the following statements:

```
proc optmodel;
   set<str,str> LINKS;
   read data LinkSetInTerror911 into LINKS=[from to];
   set NODES = union{<i,j> in LINKS} {i,j};
   set<str> ARTPOINTS;

   solve with NETWORK /
      links     = (include=LINKS)
      biconcomp
      out       = (artpoints=ARTPOINTS)
   ;

   put ARTPOINTS;
   create data ArtPoints from [node]=ARTPOINTS artpoint=1;
quit;
```

The data set ArtPoints contains members of the network who are articulation points. Focusing investigations on cutting off these particular members could have caused a great deal of disruption in the terrorists' ability to communicate when formulating the attack.

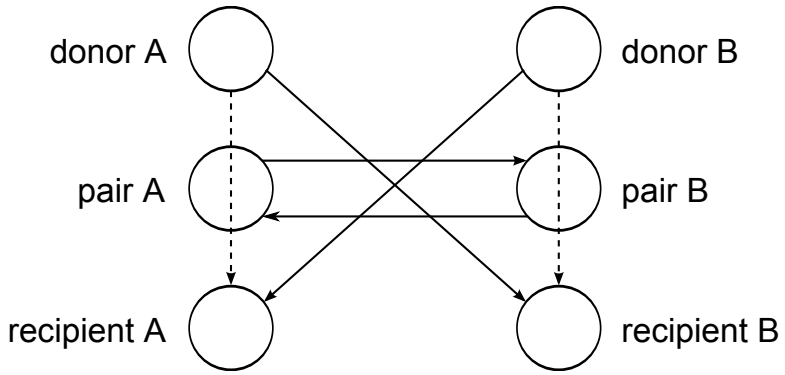
**Output 9.1.1** Articulation Points of Terrorist Communications Network from 9/11

| node | artpoint |
|---|---|
| Djamal Beghal | 1 |
| Zacarias Moussaoui | 1 |
| Essid Sami Ben Khemais | 1 |
| Mohamed Atta | 1 |
| Mamoun Darkazanli | 1 |
| Nawaf Alhazmi | 1 |

## Example 9.2: Cycle Detection for Kidney Donor Exchange

This example looks at an application of cycle detection to help create a kidney donor exchange. Suppose someone needs a kidney transplant and a family member is willing to donate one. If the donor and recipient are incompatible (because of blood types, tissue mismatch, and so on), the transplant cannot happen. Now suppose two donor-recipient pairs A and B are in this situation, but donor A is compatible with recipient B and donor B is compatible with recipient A. Then two transplants can take place in a two-way swap, shown graphically in Figure 9.79. More generally, an $n$-way swap can be performed involving $n$ donors and $n$ recipients (Willingham 2009).

**Figure 9.79** Kidney Donor Exchange Two-Way Swap




To model this problem, define a directed graph as follows. Each node is an incompatible donor-recipient pair. Link $(i, j)$ exists if the donor from node $i$ is compatible with the recipient from node $j$. The link weight is a measure of the quality of the match. By introducing dummy links whose weight is 0, you can also include altruistic donors who have no recipients or recipients who have no donors. The idea is to find a maximum-weight node-disjoint union of directed cycles. You want the union to be node-disjoint so that no kidney is donated more than once, and you want cycles so that the donor from node $i$ gives up a kidney if and only if the recipient from node $i$ receives a kidney.

Without any other constraints, the problem could be solved as a linear assignment problem, as described in the section "Linear Assignment (Matching)" on page 423. But doing so would allow arbitrarily long cycles in the solution. Because of practical considerations (such as travel) and to mitigate risk, each cycle must have no more than $L$ links. The kidney exchange problem is to find a maximum-weight node-disjoint union of short directed cycles.

One way to solve this problem is to explicitly generate all cycles whose length is at most $L$ and then solve a set packing problem. You can use PROC OPTMODEL to generate the cycles, formulate the set packing problem, call the mixed integer linear programming solver, and output the optimal solution.

The following DATA step sets up the problem, first creating a random graph on $n$ nodes with link probability $p$ and Uniform(0,1) weight:

```
/* create random graph on n nodes with arc probability p
   and uniform(0,1) weight */
%let n = 100;
%let p = 0.02;
data LinkSetIn;
   do from = 0 to &n - 1;
      do to = 0 to &n - 1;
         if from eq to then continue;
         else if ranuni(1) < &p then do;
            weight = ranuni(2);
            output;
         end;
      end;
   end;
run;
```

The following statements declare parameters and then read the input data:

```
%let max_length = 10;
proc optmodel;
   /* declare index sets and parameters, and read data */
   set <num,num> ARCS;
   num weight {ARCS};
   read data LinkSetIn into ARCS=[from to] weight;
   set<num,num,num> ID_ORDER_NODE;
```

The following statements use the network solver to generate all cycles whose length is greater than or equal to 2 and less than or equal to 10:

```
   /* generate all cycles with 2 <= length <= max_length */
   solve with NETWORK /
      loglevel        = moderate
      graph_direction = directed
      links           = (include=ARCS)
      cycle           = (mode=all_cycles minlength=2 maxlength=&max_length)
      out             = (cycles=ID_ORDER_NODE)
   ;
```

The network solver finds 224 cycles of the appropriate length, as shown in Output 9.2.1.

**Output 9.2.1** Cycles for Kidney Donor Exchange Network Solver Log

```
NOTE: There were 194 observations read from the data set WORK.LINKSETIN.
NOTE: The number of nodes in the input graph is 97.
NOTE: The number of links in the input graph is 194.
NOTE: Processing cycle detection.
NOTE: The graph has 224 cycles.
NOTE: Processing cycle detection used 0.27 (cpu: 0.27) seconds.
```

From the resulting set ID_ORDER_NODE, use the following statements to convert to one tuple per cycle-arc combination:

```
   /* extract <cid,from,to> triples from <cid,order,node> triples */
   set <num,num,num> ID_FROM_TO init {};
   num last init ., from, to;
   for {<cid,order,node> in ID_ORDER_NODE} do;
      from = last;
      to   = node;
      last = to;
      if order ne 1 then ID_FROM_TO = ID_FROM_TO union {<cid,from,to>};
   end;
```

Given the set of cycles, you can now formulate a mixed integer linear program (MILP) to maximize the total cycle weight. Let $C$ be the set of cycles of appropriate length, $N_c$ be the set of nodes in cycle $c$, $A_c$ be the set of links in cycle $c$, and $w_{ij}$ be the link weight for link $(i, j)$. Define a binary decision variable $x_c$. Set $x_c$ to 1 if cycle $c$ is used in the solution; otherwise, set it to 0. Then, the following MILP defines the problem that you want to solve (to maximize the quality of the kidney exchange):

$$\begin{aligned}
\text{maximize} \quad & \sum_{c \in C} \left( \sum_{(i,j) \in A_c} w_{ij} \right) x_c & & \\
\text{subject to} \quad & \sum_{c \in C : i \in N_c} x_c \leq 1 & i \in N & \qquad \text{(incomp_pair)} \\
& x_c \in \{0, 1\} & c \in C &
\end{aligned}$$

The constraint (incomp_pair) ensures that each node (incompatible pair) in the graph is intersected at most once. That is, a donor can donate a kidney only once. You can use PROC OPTMODEL to solve this mixed integer linear programming problem as follows:

```
   /* solve set packing problem to find maximum weight node-disjoint union
      of short directed cycles */
   set CYCLES = setof {<c,i,j> in ID_FROM_TO} c;
   set ARCS_c {c in CYCLES} = setof {<(c),i,j> in ID_FROM_TO} <i,j>;
   set NODES_c {c in CYCLES} = union {<i,j> in ARCS_c[c]} {i,j};
   set NODES = union {c in CYCLES} NODES_c[c];
   num cycle_weight {c in CYCLES} = sum {<i,j> in ARCS_c[c]} weight[i,j];

   /* UseCycle[c] = 1 if cycle c is used, 0 otherwise */
   var UseCycle {CYCLES} binary;

   /* declare objective */
   max TotalWeight
      = sum {c in CYCLES} cycle_weight[c] * UseCycle[c];

   /* each node appears in at most one cycle */
   con node_packing {i in NODES}:
      sum {c in CYCLES: i in NODES_c[c]} UseCycle[c] <= 1;

   /* call solver */
   solve with milp;

   /* output optimal solution */
   create data Solution from [c]={c in CYCLES: UseCycle[c].sol > 0.5}
      cycle_weight;
quit;
%put &_OROPTMODEL_;
```

PROC OPTMODEL solves the problem by using the mixed integer linear programming solver. As shown in Output 9.2.2, it was able to find a total weight (quality level) of 26.02.

**Output 9.2.2** Cycles for Kidney Donor Exchange MILP Solver Log

```
NOTE: Problem generation will use 4 threads.
NOTE: The problem has 224 variables (0 free, 0 fixed).
NOTE: The problem has 224 binary and 0 integer variables.
NOTE: The problem has 63 linear constraints (63 LE, 0 EQ, 0 GE, 0 range).
NOTE: The problem has 1900 linear constraint coefficients.
NOTE: The problem has 0 nonlinear constraints (0 LE, 0 EQ, 0 GE, 0 range).
NOTE: The MILP presolver value AUTOMATIC is applied.
NOTE: The MILP presolver removed 46 variables and 35 constraints.
NOTE: The MILP presolver removed 901 constraint coefficients.
NOTE: The MILP presolver modified 0 constraint coefficients.
NOTE: The presolved problem has 178 variables, 28 constraints, and 999
      constraint coefficients.
NOTE: The MILP solver is called.
NOTE: The parallel Branch and Cut algorithm is used.
NOTE: The Branch and Cut algorithm is using up to 4 threads.
          Node  Active    Sols    BestInteger      BestBound      Gap    Time
             0       1       3     22.7780692    868.9019355   97.38%       0
             0       1       3     22.7780692     26.7803921   14.94%       0
             0       1       4     23.2747070     26.0966379   10.81%       0
             0       1       5     26.0202871     26.0202871    0.00%       0
             0       0       5     26.0202871     26.0202871    0.00%       0
NOTE: The MILP solver added 12 cuts with 673 cut coefficients at the root.
NOTE: Optimal.
NOTE: Objective = 26.020287142.
NOTE: The data set WORK.SOLUTION has 6 observations and 2 variables.
STATUS=OK ALGORITHM=BAC SOLUTION_STATUS=OPTIMAL OBJECTIVE=26.020287142
RELATIVE_GAP=0 ABSOLUTE_GAP=0 PRIMAL_INFEASIBILITY=2.220446E-16
BOUND_INFEASIBILITY=2.220446E-16 INTEGER_INFEASIBILITY=1.44329E-15
BEST_BOUND=26.020287142 NODES=1 ITERATIONS=98 PRESOLVE_TIME=0.00
SOLUTION_TIME=0.02
```

The data set Solution, shown in Output 9.2.3, now contains the cycles that define the best exchange and their associated weight (quality).

**Output 9.2.3** Maximum Quality Solution for Kidney Donor Exchange

| c | cycle_weight |
|---|---|
| 12 | 5.84985 |
| 43 | 3.90015 |
| 71 | 5.44467 |
| 124 | 7.42574 |
| 222 | 2.28231 |
| 224 | 1.11757 |

## Example 9.3: Linear Assignment Problem for Minimizing Swim Times

A swimming coach needs to assign male and female swimmers to each stroke of a medley relay team. The swimmers' best times for each stroke are stored in a SAS data set. The LINEAR_ASSIGNMENT option evaluates the times and matches strokes and swimmers to minimize the total relay swim time.

The data are stored in matrix format, where the row identifier is the swimmer's name (variable name) and each swimming event is a column (variables: back, breast, fly, and free). In the following DATA step, the relay times are split into two categories, male (M) and female (F):

```
data RelayTimes;
   input name $ sex $ back breast fly free;
   datalines;
Sue     F 35.1 36.7 28.3 36.1
Karen   F 34.6 32.6 26.9 26.2
Jan     F 31.3 33.9 27.1 31.2
Andrea  F 28.6 34.1 29.1 30.3
Carol   F 32.9 32.2 26.6 24.0
Ellen   F 27.8 32.5 27.8 27.0
Jim     M 26.3 27.6 23.5 22.4
Mike    M 29.0 24.0 27.9 25.4
Sam     M 27.2 33.8 25.2 24.1
Clayton M 27.0 29.2 23.0 21.9
;
```

The following statements solve the linear assignment problem for both male and female relay teams:

```
proc contents data=RelayTimes
   out=stroke_data(rename=(name=stroke) where=(type=1));
run;

proc optmodel;
   set <str> STROKES;
   read data stroke_data into STROKES=[stroke];
   set <str> SWIMMERS;
   str sex {SWIMMERS};
   num time {SWIMMERS, STROKES};
   read data RelayTimes into SWIMMERS=[name] sex
      {stroke in STROKES} <time[name,stroke]=col(stroke)>;
   set FEMALES = {i in SWIMMERS: sex[i] = 'F'};
   set FNODES = FEMALES union STROKES;
   set MALES = {i in SWIMMERS: sex[i] = 'M'};
   set MNODES = MALES union STROKES;
   set <str,str> PAIRS;

   solve with NETWORK /
      graph_direction = directed
      links           = (weight=time)
      subgraph        = (nodes=FNODES)
      lap
      out             = (assignments=PAIRS)
   ;
   put PAIRS;
   create data LinearAssignF from [name assign]=PAIRS sex[name] cost=time;

   solve with NETWORK /
      graph_direction = directed
      links           = (weight=time)
      subgraph        = (nodes=MNODES)
      lap
      out             = (assignments=PAIRS)
   ;
```

```
   put PAIRS;
   create data LinearAssignM from [name assign]=PAIRS sex[name] cost=time;
quit;
```

The progress of the two SOLVE WITH NETWORK calls is shown in Output 9.3.1.

**Output 9.3.1** Network Solver Log: Linear Assignment for Swim Times

```
NOTE: The data set WORK.STROKE_DATA has 4 observations and 41 variables.
NOTE: There were 4 observations read from the data set WORK.STROKE_DATA.
NOTE: There were 10 observations read from the data set WORK.RELAYTIMES.
NOTE: The SUBGRAPH= option filtered 16 elements from 'time.'
NOTE: The number of nodes in the input graph is 10.
NOTE: The number of links in the input graph is 24.
NOTE: Processing the linear assignment problem.
NOTE: Objective = 111.5.
NOTE: Processing the linear assignment problem used 0.00 (cpu: 0.00) seconds.
{<'Karen','breast'>,<'Jan','fly'>,<'Carol','free'>,<'Ellen','back'>}
NOTE: The data set WORK.LINEARASSIGNF has 4 observations and 4 variables.
NOTE: The SUBGRAPH= option filtered 24 elements from 'time.'
NOTE: The number of nodes in the input graph is 8.
NOTE: The number of links in the input graph is 16.
NOTE: Processing the linear assignment problem.
NOTE: Objective = 96.6.
NOTE: Processing the linear assignment problem used 0.00 (cpu: 0.00) seconds.
{<'Jim','free'>,<'Mike','breast'>,<'Sam','back'>,<'Clayton','fly'>}
NOTE: The data set WORK.LINEARASSIGNM has 4 observations and 4 variables.
```

The data sets LinearAssignF and LinearAssignM contain the optimal assignments. Note that in the case of the female data, there are more people (set $S$) than there are strokes (set $T$). Therefore, the solver allows for some members of $S$ to remain unassigned.

**Output 9.3.2** Optimal Assignments for Best Female Swim Times

| name | assign | sex | cost |
|---|---|---|---|
| Karen | breast | F | 32.6 |
| Jan | fly | F | 27.1 |
| Carol | free | F | 24.0 |
| Ellen | back | F | 27.8 |
| | | | **111.5** |

**Output 9.3.3** Optimal Assignments for Best Male Swim Times

| name | assign | sex | cost |
|---|---|---|---|
| Jim | free | M | 22.4 |
| Mike | breast | M | 24.0 |
| Sam | back | M | 27.2 |
| Clayton | fly | M | 23.0 |
| | | | **96.6** |

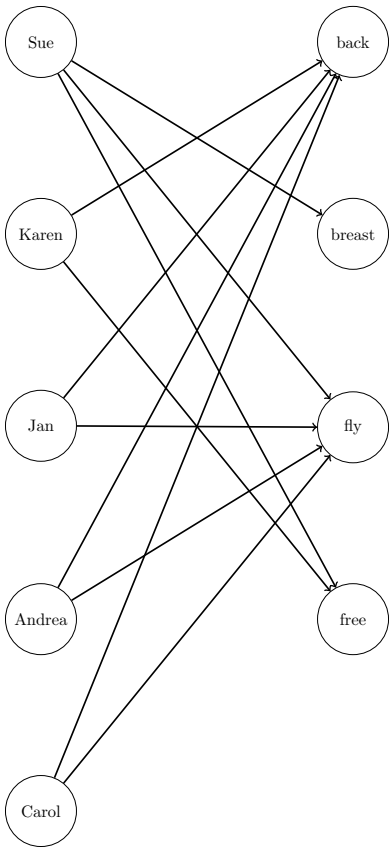
## Example 9.4: Linear Assignment Problem, Sparse Format versus Dense Format

This example looks at the problem of assigning swimmers to strokes based on their best times. However, in this case certain swimmers are not eligible to perform certain strokes. A missing (.) value in the data matrix identifies an ineligible assignment. For example:

```
data RelayTimesMatrix;
   input name $ sex $ back breast fly free;
   datalines;
Sue      F    .  36.7 28.3 36.1
Karen    F 34.6    .    . 26.2
Jan      F 31.3    . 27.1    .
Andrea   F 28.6    . 29.1    .
Carol    F 32.9    . 26.6    .
;
```

Recall that the linear assignment problem can also be interpreted as the minimum-weight matching in a bipartite directed graph. The eligible assignments define links between the rows (swimmers) and the columns (strokes), as in Figure 9.80.

**Figure 9.80** Bipartite Graph for Linear Assignment Problem



You can represent the same data in RelayTimesMatrix by using a links data set as follows:

```
data RelayTimesLinks;
   input name $ attr $ cost;
   datalines;
Sue     breast 36.7
Sue     fly    28.3
Sue     free   36.1
Karen   back   34.6
Karen   free   26.2
Jan     back   31.3
Jan     fly    27.1
Andrea  back   28.6
Andrea  fly    29.1
Carol   back   32.9
Carol   fly    26.6
;
```

This graph must be bipartite (such that $S$ and $T$ are disjoint). If it is not, the network solver returns an error.

Now, you can use either input format to solve the same problem, as follows:

```
proc contents data=RelayTimesMatrix
   out=stroke_data(rename=(name=stroke) where=(type=1));
run;

proc optmodel;
   set <str> STROKES;
   read data stroke_data into STROKES=[stroke];
   set <str> SWIMMERS;
   str sex {SWIMMERS};
   num time {SWIMMERS, STROKES};
   read data RelayTimesMatrix into SWIMMERS=[name]
      sex
      {stroke in STROKES} <time[name,stroke]=col(stroke)>;
   set SWIMMERS_STROKES =
      {name in SWIMMERS, stroke in STROKES: time[name,stroke] ne .};
   set <str,str> PAIRS;

   solve with NETWORK /
      graph_direction = directed
      links           = (weight=time)
      subgraph        = (links=SWIMMERS_STROKES)
      lap
      out             = (assignments=PAIRS)
   ;

   put PAIRS;
   create data LinearAssignMatrix from [name assign]=PAIRS
      sex[name] cost=time;
quit;

proc sql;
   create table stroke_data as
   select distinct attr as stroke
```

```
   from RelayTimesLinks;
quit;

proc optmodel;
   set <str> STROKES;
   read data stroke_data into STROKES=[stroke];
   set <str> SWIMMERS;
   str sex {SWIMMERS};
   set <str,str> SWIMMERS_STROKES;
   num time {SWIMMERS_STROKES};
   read data RelayTimesLinks into SWIMMERS_STROKES=[name attr] time=cost;
   set <str,str> PAIRS;

   solve with NETWORK /
      graph_direction = directed
      links           = (weight=time)
      lap
      out             = (assignments=PAIRS)
   ;

   put PAIRS;
   create data LinearAssignLinks from [name attr]=PAIRS cost=time;
quit;
```

The data sets LinearAssignMatrix and LinearAssignLinks now contain the optimal assignments, as shown in Output 9.4.1 and Output 9.4.2.

**Output 9.4.1** Optimal Assignments for Swim Times (Dense Input)

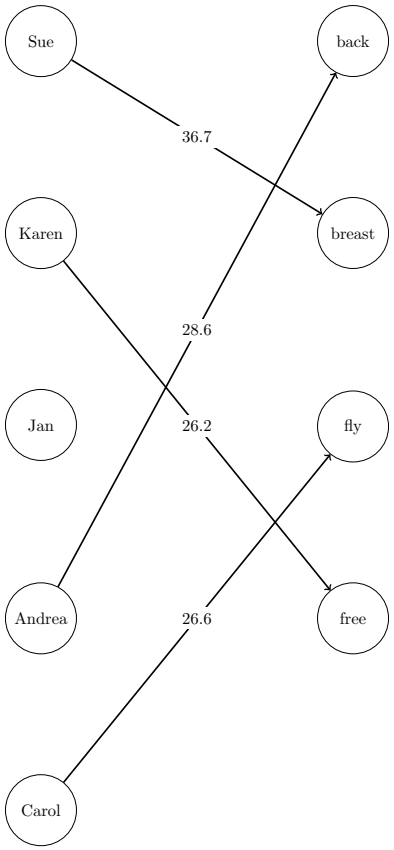
| name | assign | sex | cost |
|---|---|---|---|
| Sue | breast | F | 36.7 |
| Karen | free | F | 26.2 |
| Andrea | back | F | 28.6 |
| Carol | fly | F | 26.6 |
| | | | **118.1** |

**Output 9.4.2** Optimal Assignments for Swim Times (Sparse Input)

| name | attr | cost |
|---|---|---|
| Sue | breast | 36.7 |
| Karen | free | 26.2 |
| Andrea | back | 28.6 |
| Carol | fly | 26.6 |
| | | **118.1** |

The optimal assignments are shown graphically in Figure 9.81.

**Figure 9.81** Optimal Assignments for Swim Times

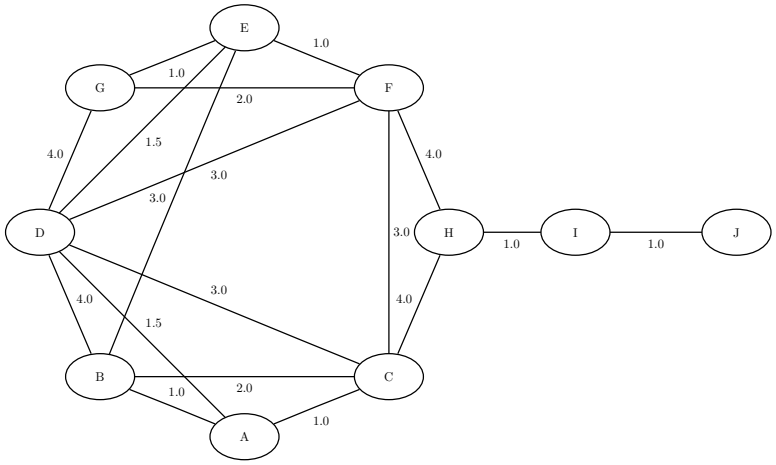



For large problems where a number of links are forbidden, the sparse format can be faster and can save a great deal of memory. Consider an example that uses the dense format with 15,000 columns ($|S| = 15,000$) and 4,000 rows ($|T| = 4,000$). To store the dense matrix in memory, the network solver needs to allocate approximately $|S| \cdot |T| \cdot 8/1024/1024 = 457$ MB. If the data have mostly ineligible links, then the sparse (graph) format is much more efficient with respect to memory. For example, if the data have only 5% of the eligible links ($15,000 \cdot 4,000 \cdot 0.05 = 3,000,000$), then the dense storage would still need 457 MB. The sparse storage for the same example needs approximately $|S| \cdot |T| \cdot 0.05 \cdot 12/1024/1024 = 34$ MB. If the problem is fully dense (all links are eligible), then the dense format is more efficient.

## Example 9.5: Minimum Spanning Tree for Computer Network Topology

Consider the problem of designing a small network of computers in an office. In designing the network, the goal is to make sure that each machine in the office can reach every other machine. To accomplish this goal, Ethernet lines must be constructed and run between the machines. The construction costs for each possible link are based approximately on distance and are shown in Figure 9.82. Besides distance, the costs also reflect some restrictions due to physical boundaries. To connect all the machines in the office at minimal cost, you need to find a minimum spanning tree on the network of possible links.

**Figure 9.82** Potential Office Computer Network




Define the link data set as follows:

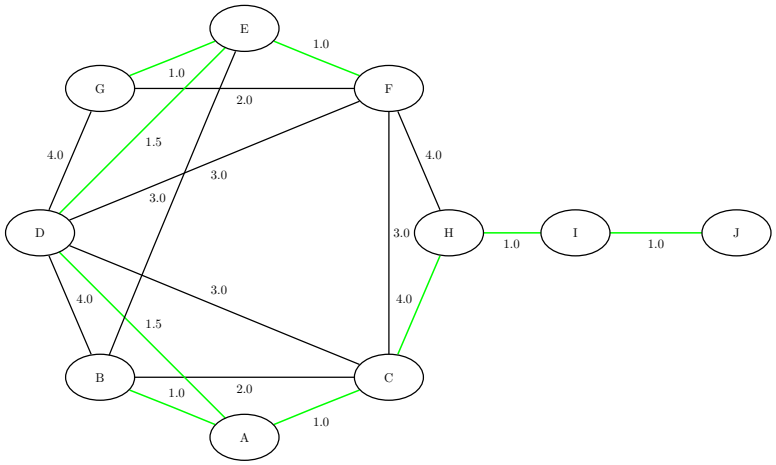
```
data LinkSetInCompNet;
   input from $ to $ weight @@;
   datalines;
A B 1.0  A C 1.0  A D 1.5  B C 2.0  B D 4.0
B E 3.0  C D 3.0  C F 3.0  C H 4.0  D E 1.5
D F 3.0  D G 4.0  E F 1.0  E G 1.0  F G 2.0
F H 4.0  H I 1.0  I J 1.0
;
```

The following statements find a minimum spanning tree:

```
proc optmodel;
   set<str,str> LINKS;
   num weight{LINKS};
   read data LinkSetInCompNet into LINKS=[from to] weight;
   set<str,str> FOREST;

   solve with NETWORK /
      links       = (weight=weight)
      minspantree
      out         = (forest=FOREST)
   ;

   put FOREST;
   put (sum {<i,j> in FOREST} weight[i,j]);
   create data MinSpanTree from [from to]=FOREST weight;
quit;
```

Output 9.5.1 shows the resulting data set MinSpanTree, which is displayed graphically in Figure 9.83 with the minimal cost links shown in green.

**Figure 9.83** Minimum Spanning Tree for Office Computer Network




**Output 9.5.1** Minimum Spanning Tree of a Computer Network Topology

| from | to | weight |
|---|---|---|
| I | J | 1.0 |
| A | C | 1.0 |
| E | F | 1.0 |
| E | G | 1.0 |
| H | I | 1.0 |
| A | B | 1.0 |
| D | E | 1.5 |
| A | D | 1.5 |
| C | H | 4.0 |
| | | **13.0** |

## Example 9.6: Transitive Closure for Identification of Circular Dependencies in a Bug Tracking System

Most software bug tracking systems have some notion of *duplicate bugs* in which one bug is declared to be the same as another bug. If bug A is considered a duplicate (DUP) of bug B, then a fix for B would also fix A. You can represent the DUPs in a bug tracking system as a directed graph where you add a link $A \rightarrow B$ if A is a DUP of B.

The bug tracking system needs to check for two situations when users declare a bug to be a DUP. The first situation is called a *circular dependence*. Consider bugs A, B, C, and D in the tracking system. The first user declares that A is a DUP of B and that C is a DUP of D. Then, a second user declares that B is a DUP of C, and a third user declares that D is a DUP of A. You now have a circular dependence, and no primary bug is defined on which the development team should focus. You can easily see this circular dependence in the graph representation, because $A \rightarrow B \rightarrow C \rightarrow D \rightarrow A$. Finding such circular dependencies can be done using cycle detection, which is described in the section "Cycle" on page 417. However, the second situation

that needs to be checked is more general. If a user declares that A is a DUP of B and another user declares that B is a DUP of C, this chain of duplicates is already an issue. The bug tracking system needs to provide one primary bug to which the rest of the bugs are duplicated. The existence of these chains can be identified by calculating the transitive closure of the directed graph that is defined by the DUP links.

Given the original directed graph $G$ (defined by the DUP links) and its transitive closure $G^T$, any link in $G^T$ that is not in $G$ exists because of some chain that is present in $G$.

Consider the following data that define some duplicated bugs (called *defects*) in a small sample of the bug tracking system:

```
data DefectLinks;
   input defectId $ linkedDefect $ linkType $ when datetime16.;
   format when datetime16.;
   datalines;
D0096978 S0711218 DUPTO 20OCT10:00:00:00
S0152674 S0153280 DUPTO 30MAY02:00:00:00
S0153280 S0153307 DUPTO 30MAY02:00:00:00
S0153307 S0152674 DUPTO 30MAY02:00:00:00
S0162973 S0162978 DUPTO 29NOV10:16:13:16
S0162978 S0165405 DUPTO 29NOV10:16:13:16
S0325026 S0575748 DUPTO 01JUN10:00:00:00
S0347945 S0346582 DUPTO 03MAR06:00:00:00
S0350596 S0346582 DUPTO 21MAR06:00:00:00
S0539744 S0643230 DUPTO 10MAY10:00:00:00
S0575748 S0643230 DUPTO 15JUN10:00:00:00
S0629984 S0643230 DUPTO 01JUN10:00:00:00
;
```

The following statements calculate cycles in addition to the transitive closure of the graph $G$ that is defined by the duplicated defects in DefectLinks. The output data set Cycles contains any circular dependencies, and the data set TransClosure contains the transitive closure $G^T$. To identify the chains, you can use PROC SQL to identify the links in $G^T$ that are not in $G$.

```
proc optmodel;
   set<str,str> LINKS;
   read data DefectLinks into LINKS=[defectId linkedDefect];
   set<num,num,str> CYCLES;
   set<str,str> CLOSURE;

   solve with NETWORK /
      loglevel        = moderate
      graph_direction = directed
      links           = (include=LINKS)
      cycle           = (mode=first_cycle)
      out             = (cycles=CYCLES)
   ;

   put CYCLES;
   create data Cycles from [cycle order node]=CYCLES;
```

```
   solve with NETWORK /
      loglevel           = moderate
      graph_direction    = directed
      links              = (include=LINKS)
      transitive_closure
      out                = (closure=CLOSURE)
   ;

   put CLOSURE;
   create data TransClosure from [defectId linkedDefect]=CLOSURE;
quit;

proc sql;
   create table Chains as
   select defectId, linkedDefect from TransClosure
      except
   select defectId, linkedDefect from DefectLinks;
quit;
```

The progress of the procedure is shown in Output 9.6.1.

**Output 9.6.1** Network Solver Log: Transitive Closure for Identification of Circular Dependencies in a Bug Tracking System

```
NOTE: There were 12 observations read from the data set WORK.DEFECTLINKS.
NOTE: The number of nodes in the input graph is 16.
NOTE: The number of links in the input graph is 12.
NOTE: Processing cycle detection.
NOTE: The graph does have a cycle.
NOTE: Processing cycle detection used 0.00 (cpu: 0.00) seconds.
{<1,1,'S0152674'>,<1,2,'S0153280'>,<1,3,'S0153307'>,<1,4,'S0152674'>}
NOTE: The data set WORK.CYCLES has 4 observations and 3 variables.
NOTE: The number of nodes in the input graph is 16.
NOTE: The number of links in the input graph is 12.
NOTE: Processing the transitive closure.
NOTE: Processing the transitive closure used 0.00 (cpu: 0.00) seconds.
{<'D0096978','S0711218'>,<'S0152674','S0153280'>,<'S0153280','S0153307'>,<
'S0153307','S0152674'>,<'S0162973','S0162978'>,<'S0162978','S0165405'>,<
'S0325026','S0575748'>,<'S0347945','S0346582'>,<'S0350596','S0346582'>,<
'S0539744','S0643230'>,<'S0575748','S0643230'>,<'S0629984','S0643230'>,<
'S0153280','S0152674'>,<'S0162973','S0165405'>,<'S0325026','S0643230'>,<
'S0153307','S0153280'>,<'S0153280','S0153280'>,<'S0152674','S0152674'>,<
'S0152674','S0153307'>,<'S0153307','S0153307'>}
NOTE: The data set WORK.TRANSCLOSURE has 20 observations and 2 variables.
NOTE: Table WORK.CHAINS created, with 8 rows and 2 columns.
```

The data set Cycles contains one case of a circular dependence in which the DUPs start and end at S0152674.

**Output 9.6.2** Cycle in Bug Tracking System

| cycle | order | node |
|---|---|---|
| 1 | 1 | S0152674 |
| 1 | 2 | S0153280 |
| 1 | 3 | S0153307 |
| 1 | 4 | S0152674 |

The data set Chains contains the chains in the bug tracking system that come from the links in $G^T$ that are not in $G$.

**Output 9.6.3** Chains in Bug Tracking System

| defectId | linkedDefect |
|---|---|
| S0152674 | S0152674 |
| S0152674 | S0153307 |
| S0153280 | S0152674 |
| S0153280 | S0153280 |
| S0153307 | S0153280 |
| S0153307 | S0153307 |
| S0162973 | S0165405 |
| S0325026 | S0643230 |

## Example 9.7: Traveling Salesman Tour through US Capital Cities

Consider a cross-country trip where you want to travel the fewest miles to visit all of the capital cities in all US states except Alaska and Hawaii. Finding the optimal route is an instance of the traveling salesman problem, which is described in section "Traveling Salesman Problem" on page 453.

The following PROC SQL statements use the built-in data set maps.uscity to generate a list of the capital cities and their latitude and longitude:

```
/* Get a list of the state capital cities (with lat and long) */
proc sql;
   create table Cities as
   select unique statecode as state, city, lat, long
      from maps.uscity
      where capital='Y' and statecode not in ('AK' 'PR' 'HI');
quit;
```

From this list, you can generate a links data set CitiesDist that contains the distances, in miles, between each pair of cities. The distances are calculated by using the SAS function GEODIST.

```
/* Create a list of all the possible pairs of cities */
proc sql;
   create table CitiesDist as
   select
      a.city as city1, a.lat as lat1, a.long as long1,
      b.city as city2, b.lat as lat2, b.long as long2,
      geodist(lat1, long1, lat2, long2, 'DM') as distance
      from Cities as a, Cities as b
      where a.city < b.city;
quit;
```

The following PROC OPTMODEL statements find the optimal tour through each of the capital cities:

```
/* Find optimal tour by using the network solver */
proc optmodel;
   set<str,str> CAPPAIRS;
   set<str> CAPITALS = union {<i,j> in CAPPAIRS} {i,j};
   num distance{i in CAPITALS, j in CAPITALS: i < j};
   read data CitiesDist into CAPPAIRS=[city1 city2] distance;
   set<str,str> TOUR;
   num order{CAPITALS};

   solve with NETWORK /
      loglevel  = moderate
      links     = (weight=distance)
      tsp
      out       = (order=order tour=TOUR)
   ;

   put (sum{<i,j> in TOUR} distance[i,j]);
   /* Create tour-ordered pairs (rather than input-ordered pairs) */
   str CAPbyOrder{1..card(CAPITALS)};
   for {i in CAPITALS} CAPbyOrder[order[i]] = i;
   set TSPEDGES init
      setof{i in 2..card(CAPITALS)} <CAPbyOrder[i-1],CAPbyOrder[i]>
      union {<CAPbyOrder[card(CAPITALS)],CAPbyOrder[1]>};
   num distance2{<i,j> in TSPEDGES} =
      if i < j then distance[i,j] else distance[j,i];
   create data TSPTourNodes from [node] tsp_order=order;
   create data TSPTourLinks from [city1 city2]=TSPEDGES distance=distance2;
quit;
```

The progress of the procedure is shown in Output 9.7.1. The total mileage needed to optimally traverse the capital cities is 10, 627.75 miles.

**Output 9.7.1** Network Solver Log: Traveling Salesman Tour through US Capital Cities

```
NOTE: There were 1176 observations read from the data set WORK.CITIESDIST.
NOTE: The number of nodes in the input graph is 49.
NOTE: The number of links in the input graph is 1176.
NOTE: Processing the traveling salesman problem.
NOTE: The initial TSP heuristics found a tour with cost 10645.918753 using 0.05
      (cpu: 0.02) seconds.
NOTE: The MILP presolver value NONE is applied.
NOTE: The MILP solver is called.
NOTE: The Branch and Cut algorithm is used.
          Node  Active    Sols    BestInteger      BestBound      Gap    Time
             0       1       1  10645.9187534  10040.5139714    6.03%       0
             0       1       1  10645.9187534  10241.6970024    3.95%       0
             0       1       1  10645.9187534  10262.9074205    3.73%       0
             0       1       1  10645.9187534  10267.6251909    3.68%       0
             0       1       1  10645.9187534  10275.1505973    3.61%       0
             0       1       1  10645.9187534  10283.2134412    3.53%       0
             0       1       1  10645.9187534  10345.3151313    2.91%       0
             0       1       1  10645.9187534  10350.0790852    2.86%       0
             0       1       1  10645.9187534  10355.3956630    2.81%       0
             0       1       1  10645.9187534  10381.4538838    2.55%       0
             0       1       1  10645.9187534  10481.2454442    1.57%       0
             0       1       1  10645.9187534  10560.1837745    0.81%       0
             0       1       1  10645.9187534  10576.0823291    0.66%       0
             0       1       1  10645.9187534  10612.1627809    0.32%       0
             0       1       1  10645.9187534  10619.6942572    0.25%       0
             0       1       2  10627.7543183  10627.7543183    0.00%       0
             0       0       2  10627.7543183  10627.7543183    0.00%       0
NOTE: The MILP solver added 17 cuts with 5262 cut coefficients at the root.
NOTE: Optimal.
NOTE: Objective = 10627.754318.
NOTE: Processing the traveling salesman problem used 0.07 (cpu: 0.03) seconds.
10627.754318
NOTE: The data set WORK.TSPTOURNODES has 49 observations and 2 variables.
NOTE: The data set WORK.TSPTOURLINKS has 49 observations and 3 variables.
```

The following PROC GPROJECT and PROC GMAP statements produce a graphical display of the solution:

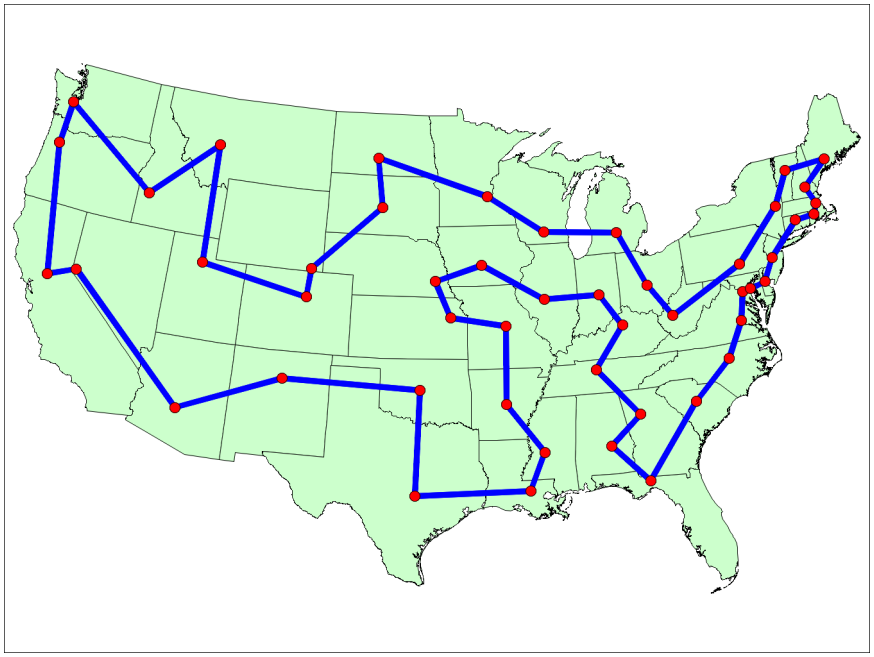
```
/* Merge latitude and longitude */
proc sql;
   /* merge in the lat & long for city1 */
   create table TSPTourLinksAnno1 as
   select unique TSPTourLinks.*, cities.lat as lat1, cities.long as long1
      from TSPTourLinks left join cities
      on TSPTourLinks.city1=cities.city;
   /* merge in the lat & long for city2 */
   create table TSPTourLinksAnno2 as
```

```
   select unique TSPTourLinksAnno1.*, cities.lat as lat2, cities.long as long2
      from TSPTourLinksAnno1 left join cities
      on TSPTourLinksAnno1.city2=cities.city;
quit;

/* Create the annotated data set to draw the path on the map
    (convert lat & long degrees to radians, since the map is in radians) */
data anno_path;
   set TSPTourLinksAnno2;
   length function color $8;
   xsys='2'; ysys='2'; hsys='3'; when='a'; anno_flag=1;
   function='move';
   x=atan(1)/45 * long1;
   y=atan(1)/45 * lat1;
   output;
   function='draw';
   color="blue"; size=0.8;
   x=atan(1)/45 * long2;
   y=atan(1)/45 * lat2;
   output;
run;

/* Get a map with only the contiguous 48 states */
data states;
   set maps.states (where=(fipstate(state) not in ('HI' 'AK' 'PR')));
run;

data combined;
   set states anno_path;
run;

/* Project the map and annotate the data */
proc gproject data=combined out=combined dupok;
   id state;
run;

data states anno_path;
   set combined;
   if anno_flag=1 then output anno_path;
   else                output states;
run;

/* Get a list of the endpoints locations */
proc sql;
   create table anno_dots as
   select unique x, y from anno_path;
quit;

/* Create the final annotate data set */
data anno_dots;
   set anno_dots;
   length function color $8;
   xsys='2'; ysys='2'; when='a'; hsys='3';
   function='pie';
```

```
   rotate=360; size=0.8; style='psolid'; color="red";
   output;
   style='pempty'; color="black";
   output;
run;

/* Generate the map with GMAP */
pattern1 v=s c=cxccffcc repeat=100;
proc gmap data=states map=states anno=anno_path all;
   id state;
   choro state / levels=1 nolegend coutline=black
                 anno=anno_dots des='' name="tsp";
run;
```

The minimal cost tour through the capital cities is shown on the US map in Output 9.7.2.

**Output 9.7.2** Optimal Traveling Salesman Tour through US Capital Cities



The data set TSPTourLinks contains the links in the optimal tour. To display the links in the order they are to be visited, you can use the following DATA step:

```
/* Create the directed optimal tour */
data TSPTourLinksDirected(drop=next);
   set TSPTourLinks;
   retain next;
   if _N_ ne 1 and city1 ne next then do;
      city2 = city1;
      city1 = next;
   end;
   next = city2;
run;
```

The data set TSPTourLinksDirected is shown in Figure 9.84.

**Figure 9.84** Links in the Optimal Traveling Salesman Tour

| city1 | city2 | distance |
|---|---|---|
| Montgomery | Tallahassee | 177.14 |
| Tallahassee | Columbia | 311.23 |
| Columbia | Raleigh | 182.99 |
| Raleigh | Richmond | 135.58 |
| Richmond | Washington | 97.96 |
| Washington | Annapolis | 27.89 |
| Annapolis | Dover | 54.01 |
| Dover | Trenton | 83.88 |
| Trenton | Hartford | 151.65 |
| Hartford | Providence | 65.56 |
| Providence | Boston | 38.41 |
| Boston | Concord | 66.30 |
| Concord | Augusta | 117.36 |
| Augusta | Montpelier | 139.32 |
| Montpelier | Albany | 126.19 |
| Albany | Harrisburg | 230.24 |
| Harrisburg | Charleston | 287.34 |
| Charleston | Columbus | 134.64 |
| Columbus | Lansing | 205.08 |
| Lansing | Madison | 246.88 |
| Madison | Saint Paul | 226.25 |
| Saint Paul | Bismarck | 391.25 |
| Bismarck | Pierre | 170.27 |
| Pierre | Cheyenne | 317.90 |
| Cheyenne | Denver | 98.33 |
| Denver | Salt Lake City | 373.05 |
| Salt Lake City | Helena | 403.40 |
| Helena | Boise City | 291.20 |
| Boise City | Olympia | 401.31 |
| Olympia | Salem | 146.00 |
| Salem | Sacramento | 447.40 |
| Sacramento | Carson City | 101.51 |
| Carson City | Phoenix | 577.84 |
| Phoenix | Santa Fe | 378.27 |
| Santa Fe | Oklahoma City | 474.92 |
| Oklahoma City | Austin | 357.38 |
| Austin | Baton Rouge | 394.78 |
| Baton Rouge | Jackson | 139.75 |
| Jackson | Little Rock | 206.87 |
| Little Rock | Jefferson City | 264.75 |
| Jefferson City | Topeka | 191.67 |
| Topeka | Lincoln | 132.94 |
| Lincoln | Des Moines | 168.10 |
| Des Moines | Springfield | 243.02 |
| Springfield | Indianapolis | 186.46 |
| Indianapolis | Frankfort | 129.90 |
| Frankfort | Nashville-Davidson | 175.58 |
| Nashville-Davidson | Atlanta | 212.61 |
| Atlanta | Montgomery | 145.39 |
| | | **10,627.75** |

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