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Using the IML Procedure to Examine the Efficacy of a New Control Charting Technique

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ABSTRACT

For many years, control charts have been utilized to monitor processes, improve quality, and increase profitability. However, the body of literature tilts overwhelmingly toward charts monitoring normally distributed processes. In practice, the underlying distribution of a process may not follow a normal distribution, and many of those techniques may not be most effective. Mukherjee, McCracken, and Chakraborti (2015) suggested three control charts for simultaneous monitoring of the location and scale parameters for processes following the shifted exponential distribution. This study examines their proposed, Shifted Exponential Maximum Likelihood Estimator-Max Chart, (SEMLE-max) and suggests utilization of penalized maximum likelihood estimators (MLE) instead of traditional MLEs because of unbiasedness and minimum variance among unbiased estimators. The new chart, the Penalized SEMLE-max chart is constructed using similar methodology, and simulated data is used to compare average run lengths of the proposed chart to those obtained by the original chart.

INTRODUCTION

In the work published by Mukherjee, McCracken, and Chakraborti (2015), ample justification was given to the creation and practical implementation of non-normally distributed statistical process control charts. Specifically, the authors focus was upon monitoring time-to-event processes, as frequently modeled by the shifted exponential distribution:

$$f(x) = \frac{1}{\theta} e^{-\frac{1}{\theta}(x-\eta)}, \quad x > \eta, \quad -\infty < \eta < \infty$$

In the publication, the authors introduced three control charts explicitly designed for such processes. The control chart of interest in this study is their proposed, Shifted Exponential Maximum Likelihood Estimator Max Chart, or SEMLE-max. In this charting scheme, the maximum likelihood estimators (MLE) for the scale and location parameters are used to build two plotting statistics based upon the standard normal distribution. In order to avoid issues arising from joint monitoring of two processes, they authors suggest using a single plotting statistic for each respective sample taken, which is the absolute value of the maximum of the two respective statistics. The drawback to using the traditional MLEs in this circumstance is that they are biased estimators of the parameters. Sarhan (1954) derived two new estimators for the scale and location parameters of the shifted exponential distribution, using a least squares technique, which are unbiased and uniformly minimum variance for unbiased estimators. Zheng (2013) showed these estimators can be derived from a penalized likelihood function. Thus, in this study we recommend utilizing Sarhans estimators as opposed to the traditional MLEs for the desirable qualities aforementioned. We will use Monte Carlo simulation methods in “PROC IML” to compare the in-control and out-of-control average run lengths (IC-ARL & OOC-ARL) between the original chart, and the proposed modified chart.

ESTIMATORS

Similar to the derivation of the traditional MLEs for the scale and location paramters ($\hat{\theta}_{ML} = \bar{X} - X_{1:n}, \hat{\eta}_{ML} = X_{1:n}$), Zheng (2013) introduced a penalty to the likelihood function of the random sample. The purpose of the penalty is with specific respect to the overestimation of η, as $P[\hat{\eta}_{ML} = X_{1:n} > \eta] = 1$. Thus, the difference between the minimum order statistic and η is defined to be the likelihood function's penalty, and this difference scales the likelihood function, as follows:

$$(X_{1:n} - \eta)L(\theta, \eta) = \prod_{i=1}^{n} (X_{1:n} - \eta) \frac{1}{\theta} e^{-\frac{1}{\theta}(x_i - \eta)}$$

Using the traditional MLE derivation technique, the penalized MLEs for η and θ are:

$$\frac{\partial \ln\big(L(\theta, \eta)\big)}{\partial \theta} = -\frac{n}{\theta} + \frac{1}{\theta^2} \sum_{i=1}^{n} (x_i - \hat{\eta}) = 0$$

$$\hat{\theta}^* = \frac{1}{n} \sum_{i=1}^{n} (x_i - \hat{\eta})$$

And:

$$\frac{\partial \ln\big(L(\theta, \eta)\big)}{\partial \eta} = -\frac{1}{X_{1:n} - \eta} + \frac{n}{\theta} = 0$$

$$\hat{\eta}^* = X_{1:n} - \frac{\hat{\theta}^*}{n}$$

Plugging $\hat{\eta}^*$ into the equation for $\hat{\theta}^*$, the estimators then become:

$$\hat{\eta}^* = \frac{nX_{1:n} - \bar{X}}{n-1}$$

$$\hat{\theta}^* = \frac{n}{n-1} (\bar{X} - X_{1:n})$$

These estimators are unbiased:

$$E[\hat{\eta}^*] = E\left[\frac{nX_{1:n} - \bar{X}}{n-1}\right] = \frac{1}{n-1} (nE[X_{1:n}] - E[\bar{X}])$$

$$= \frac{1}{n-1} \left(n\left(\frac{\theta}{n} + \eta\right) - \eta - \theta\right) = \eta$$

$$E\big[\hat{\theta}^*\big] = E\left[\frac{n}{n-1} (\bar{X} - X_{1:n})\right] = \frac{n}{n-1} (E[\bar{X}] - E[X_{1:n}])$$

$$= \frac{n}{(n-1)} \left(\theta + \eta - \frac{\theta}{n} - n\right) = \theta$$

Sarhan (1954) also proved these estimators to be uniformly minimum variance among the unbiased estimators. It should also be noted that $\hat{\theta}^*$ is distributed as a gamma random variable, with parameters $\theta/(n-1)$ and $\kappa = n-1$, nearly identically to the original MLE, which was shown in Mukherjee et all (2015). A closed form for the distribution of $\hat{\eta}^*$ could not be derived, and thus, its density was estimated using simulation.

CONSTRUCTION OF THE CONTROL CHART

As noted in Mukherjee et al (2015), there are an array of issues associated with jointly monitoring two or more unknown parameters. One solution proposed in their work was calculating two identically distributed statistics, and designating the plotting statistic for a given sample as the maximum between the two. These two statistics are standardized (as they are not identically distributed in their original form) using the inverse cumulative density function (CDF) of the standard normal distribution, i.e.:

$$B_1 = \Phi^{-1}\big(F(x)\big)$$

$$B_2 = \Phi^{-1}\big(G(x)\big)$$

Where Φ^{-1} denotes the inverse CDF of the standard normal distribution, $F(x)$ denotes the CDF of $\hat{\eta}^*$, and $G(x)$ denotes the CDF of $\hat{\theta}^*$. For a given sample of the process being monitored, both B_1 and B_2 will be calculated and the maximum of their absolute value, say M_i, becomes the plotting statistic. Mathematically,

$$M_i = \max(|B_1|, |B_2|)$$

Since only positive values are yielded by the plotting statistic, only an upper control limit (UCL) is necessary to specify. For a desired in-control average run length (IC-ARL), called ARL_0, Mukherjee et al (2015) showed the UCL can be determined by:

$$UCL = \Phi^{-1}\left(0.5\left(1 + \sqrt{1 - (ARL_0)^{-1}}\right)\right)$$

Thus, for a desired IC-ARL of 500, the UCL to be used is 3.29. Note, this IC-ARL will be used in comparing out-of-control ARL (OOC-ARL) performance of the modified chart to the chart originally proposed in Mukherjee et al (2015). Therefore, for a given sample, the process is deemed OOC when $M_i > UCL$. Of note, an OOC signal can assist in indicating which of the two parameters has shifted away from its specified IC value. When $M_i = |B_1| > UCL$, this implies η may have shifted, and conversely, when $M_i = |B_2| > UCL$, this implies θ may have shifted. However, when both B_1 and B_2 are greater than the UCL, it becomes difficult to determine which of the two values (or possibly both) is signaling the OOC without physical investigation.

SIMULATION METHODS & RESULTS

Given the guidelines for the control charts construction as outlined in the previous section, the code for the simulation was written in similar steps. First, a random sample from the OOC shifted exponential distribution had to be simulated. This was performed using the inverse probability integral transform. This transformation utilizes the knowledge of a random variables cumulative density function (CDF) being distributed uniformly between 0 & 1 (i.e., $Y = F(x) \sim UNIF(0,1)$). Values from the uniform distribution are evaluated in a given random variables inverse CDF. In this case, the shifted exponential distributions CDF was set equal to Y and solved for X as given by:

$$Y = 1 - e^{-\frac{1}{\theta}(x-\eta)}$$

Solving for X:

$$X = \eta - \theta \ln(1 - Y)$$

Thus, values randomly sampled from $UNIF(0,1)$ are input for Y, and the resulting X values are distributed as the shifted exponential for a given value of η and θ. Using two “DO” loops in “the DATA step,” 10,000 samples, each of sample size $n = 5$, were simulated, and the estimators for the respective parameters from the OOC distribution were obtained using “PROC MEANS.” At this point, the plotting statistic for $\hat{\theta}^*$, B_2, was obtained using the “PROBCHI” and “QUANTILE” functions in “the DATA step.” Here, the probability associated with a given observed value of $\hat{\theta}^*$ (using PROBCHI) was inputted into the inverse CDF of the standard normal distribution (i.e., “QUANTILE(‘NORMAL’)”), and the absolute value of the resulting z-score is the value B_2 for a given sample.

Seeing as the density of $\hat{\eta}^*$ was estimated using simulation by necessity, a handful of extra procedures were implemented in order to obtain its plotting statistic, B_1. Using the same procedure described above, a random sample from the IC shifted exponential distribution was obtained. This random sample acts as the density for $\hat{\eta}^*$. Using “PROC IML,” the OOC value of $\hat{\eta}^*$ from each respective sample was iteratively compared to the simulated IC distribution in order to obtain a pseudo-probability. Z-scores for this vector of pseudo-probabilities was obtained in order to yield the plotting statistic B_1 in a nearly identical manner as described for B_2.

Now that values for B_1 and B_2 for each of the respective 10,000 samples are obtained, they can be plotted against the specified UCL of 3.29 in order to evaluate the effectiveness of these new estimators. This process was performed iteratively using “PROC IML.” When a sample plotted above the UCL, thus signaling an OOC process, the sample number was 4 recorded, and the mean difference between each observed OOC point was taken to be the OOC-ARL for a given pair of OOC values of θ and η. The entire aforementioned process was performed 10 times for each pair of OOC values of θ and η, and mean of those results were taken to be the OOC-ARL used for comparison against the Mukherjee et al (2015) chart's performance.

The procedure described above was performed for 20 paris of OOC values of η and θ, where the specified IC values were $\eta = 0$ and $\theta = 1$, respectively, and the sample size taken was $n = 5$. The results of this study are given in the below table, denoted by "ARL_S," and the results obtained in the original study are denoted "ARL_C."

Table 1: Comparison of Simulated OOC-ARL

η	θ	ARL_S	ARL_C
0	0.5	170.54	141.33
0.25	0.5	175.81	164.64
0.5	0.5	153.01	163.90
1	0.5	51.10	85.95
1.25	0.5	3.87	1.21
0	0.75	578.82	462.96
0.25	0.75	492.24	599.91
0.5	0.75	331.09	390.97
1	0.75	25.06	30.64
1.25	0.75	2.77	1.14
0	1	483.97	498.67
0.25	1	396.12	359.83
0.5	1	218.77	141.72
1	1	15.90	13.23
1.25	1	4.24	1.10
0	1.25	109.60	136.85
0.25	1.25	128.53	91.62
0.5	1.25	66.25	46.44
1	1.25	13.41	7.78
1.25	1.25	4.01	1.09

CONCLUSION

As shown in Table 1, the utilization of the penalized MLEs give improved performance over that of the standard estimators for the downward shifts in θ coupled with increases in η. Both charting schemes, as the location parameter shifts upward away from its nominal IC value, are highly effective in quickly detecting these large shifts, which is a similar conclusion drawn in Mukherjee et al (2015). However, considering the small scale, one may assume the analyzed OOC shifts are small, but the magnitude of the shifts is quite large. While the proposed modifications to the estimators yield improvements in some cases, such as the instance when OOC value of $\theta = 0.75$ and $\eta = 0.5$, an))C-ARL of 331.09, while an improvement over 390.97, is still far too slow for a 25% shift in θ and an increased η.

A big contributor to this relatively slow performance is the construction of the chart itself. While creatively thought out and mathematically elegant, this technique lends itself only to detection of very extreme values, as evidenced by the quick detection rate at large OOC η. However, consider the case of $\hat{\theta}$. Since it can be transformed into a χ_8^2 random variable in the case when $n = 5$, it can easily be determined what the z-score would be for a particular quantile of the χ_8^2 distribution. Say $\hat{\theta}$ is perfectly estimated to be 0.50, a 50% reduction in the IC value of θ. The probability associated with this estimate is about 0.24, which is associated with a z-score of about -0.70, which certainly would not signal a shift in θ. In fact, setting the UCL to 3.29 suggests that in order for an OOC signal to be detected, the lower-tailed probability associated with our estimators would have to be approximately 0.9995. Thus, as shown by the results in Table 1, unless η has shifted substantially from 0, it is highly improbable to detect an OOC shift. Consequently, it would not be recommended to implement this scheme with such a high UCL unless specific circumstances warrant it. While a smaller UCL (for example, 2.78) would sacrifice the IC performance of the charting scheme in both instances, it would make up the difference by being far more sensitive to shifts in either parameter.

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