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STATISTICAL MODELING OF THE MECHANICAL BEHAVIOR OF THE SPRING, MASS AND DAMPER AEEMBLY USING SAS®

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ABSTRACT

Dampers in mechanical systems provide safety and comfort against dynamical and impact forces transmitted by external profiles. Since dampers are made of diverse elements, have a large range of performance variables. Therefore, analytical solution for predicting damper behavior is difficult. In the absence of analytical models, one of the popular and reliable methods is statistical modeling. In this study, a modeling approach for a continuously variable damper was successfully used with variable discrete time model through Euler's approximation. Effect of changing spring constants, forces, damper constants and initial displacement in a spring- mass-damper assembly for their effects upon damper characteristics were closely investigated via simulations using PROC MODEL. Initial values were specified in a VAR statement. The mass, time step, spring constants, and damper constants were declared and initialized by a CONTROL statement. Damping force equation was written in SAS® language and solved through a SOLVE statement. Multiple runs of simulation were compared by merging data sets and overlaying plots at discrete time intervals. Error monitoring and optimization were performed in PROC IML.

INTRODUCTION

PROBLEM FORMULATION

Damper is an important component of any mechanical system. In this particular analysis a simple model of four springs in series, mass (three cars connected in series) damper (suspension) system in series is used which is shown in Figure 1. A torque is input into the system at the motor which exerts a force on the first car which changes the position of the cars. The change in position of the car is measured by an encoder. There are shutdown safety switches to stop unstable operation. In this particular analysis, the force *F*(*t*) applied to the first car is the only input to the system and the displacement of the car (X(t)) is the output of concern. The goal of this project is to design a controller which can control the displacement of all the cars at each instant of time (t) for a specified force input. The value of displacement at the specified force input at each instant of time is estimated. In terms of a cars' suspension, when the car hits a bump a force will act on the tire. The tire will flex a certain amount to reduce the effect of the force on the passengers. The design goal is a suspension that will react to the force and reach equilibrium in the tire quickly. To get the system to respond quickly to the force, a series of springs are used with appropriate values of springs. This is equivalent to changing stiffness of the coil spring in the suspension system.

The force input F(t) can take various forms and can be modeled readily by standard mathematical functions: i) an unforced system which is modeled by using F(t) =0, ii) a constant applied force using F(t) = constant (F), iii) a constantly changing force (ramp input) with F(t) = Bt + C (B & C are constants), iv) a quadratically changing force, F(t) = $At^2 + Bt + C$ (A, B & C are constants), v) an oscillating force (sinusoidal input), F(t) = A sin ωt + Bcos ωt (here A,B & ω are constants where ω is the angular frequency of the applied oscillations), vi) an exponentially changing input, F(t) = Ae^{Bt} (A & B are constants). Extensions of the model to unforced, constant- force, ramp, sinusoidal and exponential force inputs are compared in this paper.

METHODS

MODEL FORMULATION

The first step taken in analyzing the system was to create a free body diagram and to sum the forces. The second step is taken to write the governing equations of motion of the system in terms of the system parameters and to create a simulink model. The system parameters are the mass of the three cars (m_1,m_2, m_3), spring constants of the four springs in series (k_1,k_2,k_3,k_4) and the damper coefficient (c). A model was developed for one car system and extended to the three car system. The free body diagram of the model for one car system and the forces acting on the one car model with mass $=m_1$ is shown in Figure 3A and 3B respectively.

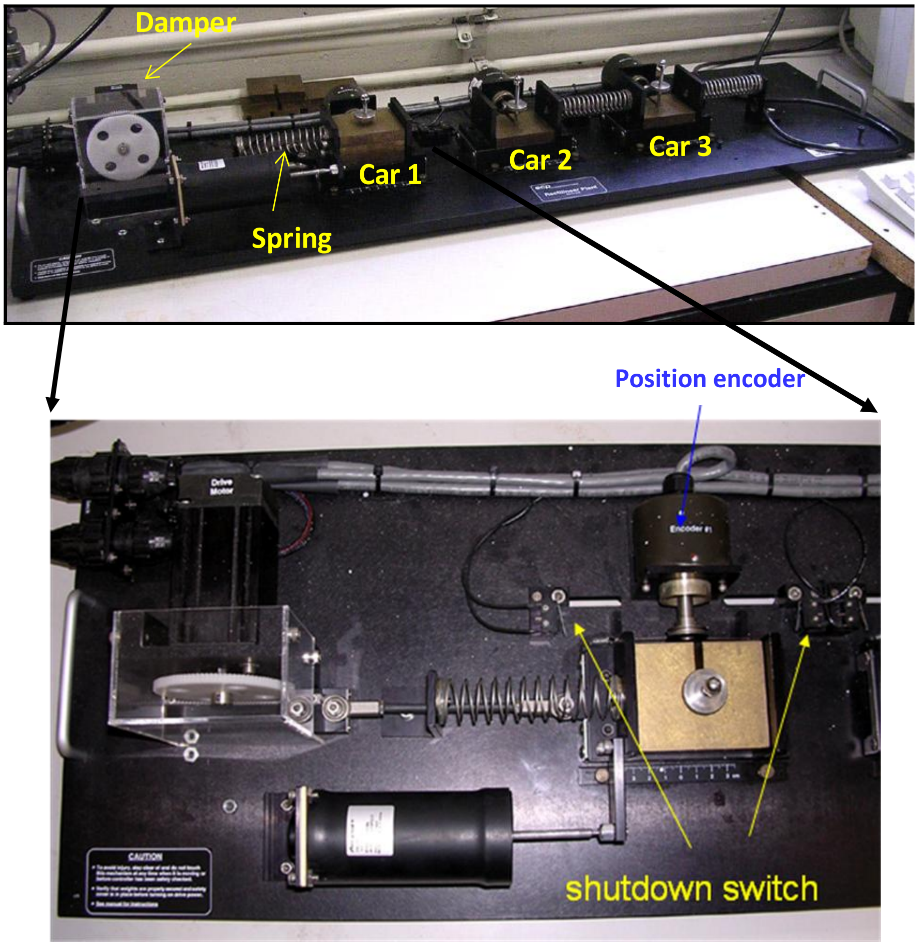

Figure 1: A model spring mass damper assembly with three cars connected in series.

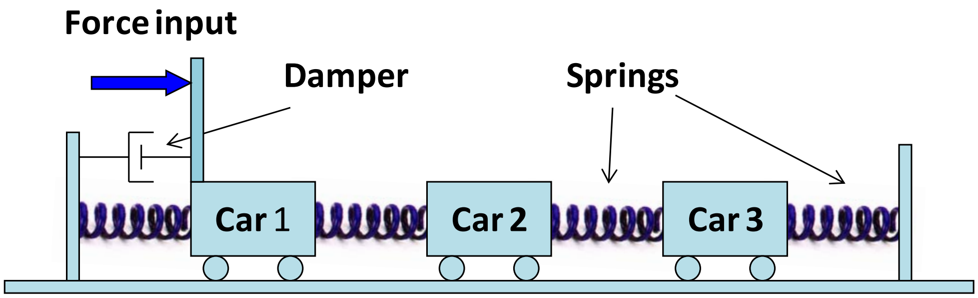

Figure 2: Free body diagram of the model spring, mass, damper assembly shown in Figure 1.

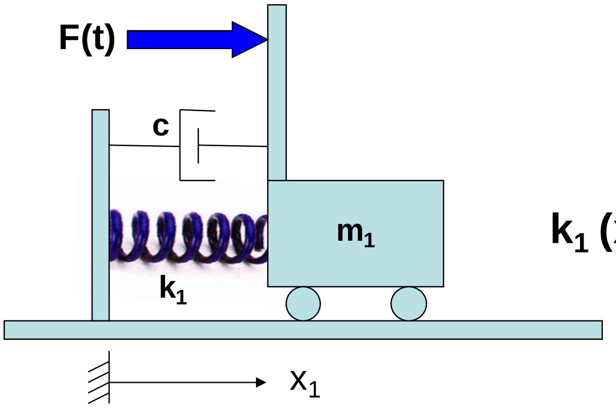

Figure 3A: Free body diagram of the model spring, mass and damper assembly for one car system

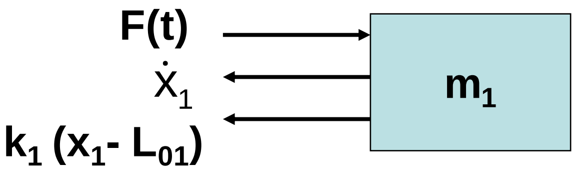

Figure 3B: Forces acting on car 1 with mass equal to m_1 kg

GOVERNING EQUATIONS

Balancing forces acting on car 1 (with mass = m_1 kg) gives the following governing equation (Eq. 1) for the system. The L_{01} is the free length of the spring which is ignored in calculations. The resulting governing equation (Eq. 2) is used to obtain the solutions. This is a second-order differential equation with parameters m_1, c and k_1. forcing term ($F(t)$) appears on the right hand side of the differential equation rather than the zero when the system is unforced.

$$m_1\ddot{X}_1 + c\dot{X}_1 + k_1(X_1 - L_{01}) = F(t) \qquad [1]$$

$$m_1\ddot{X}_1 + c\dot{X}_1 + k_1X_1 = F(t) \qquad [2]$$

Eq. 2 is a linear second-order differential equation. In mathematical terms, linearity means that, X_1, $\dot{X}_1$and $\ddot{X}_1$ occur only to the power of 1 (no X_1^2 ,$\ddot{X}_1^3$ terms, for example). In real-world terms, linearity means "What goes in comes out". If an oscillating force is applied to such a system, oscillations will result. If this mass-spring-damper system is held with a constant force, it will maintain a constant deflection from its datum position. Obtaining analytical solutions to such a system is extremely difficult. Reasonable solutions are possible through approximations of the continuous behavior of Eq. 2 by a discrete time model.

SIMULATION USING SAS®

Reasonable solutions to Eq. 2 can be obtained through statistical modeling. In this paper SA® procedures PROC MODEL and PROC IML were applied to simulate and optimize the performance of the damper assembly shown in Figure 1 varying system parameters. The continuous differential $\dot{X}_1$ was approximated with the following difference equation (Eq. 4).

$$\frac{\Delta X}{\Delta t} = V \qquad [4]$$

Where V is the velocity of the motion. Eq. 4 was written (as [(X_1 - $LAG(X_1)$)/dt] = V, (where dt is the time step used)), in SAS® to obtain the Euler's approximation for the following integral (Eq. 5).

$$X_1 = \int V\,dt \qquad [5]$$

The above discrete time model was simulated and solved using PROC MODEL (SAS® Code 1). Various time steps with respect to the changes in the system were chosen. At smaller time steps the approximation yielded results closer to the actual values. Unfortunately currently PROC MODEL does not allow varying step-sizes and error-monitoring of simulators for continuous systems. This was performed using the IML procedure. Codes were generated in PROC IML for error monitoring and for optimization of the system which is a new approach. No exogenous variables or endogenous data were required in the model. Observations were generated at specified discrete time intervals to perform the simulation (SAS® Code 1).

SAS® CODE 1

```
/*- Generate observations to drive the simulation time
periods-*/
   data damper;
      do n=0 to 1000 by 10;
         output;
      end;
   run;
```

Since the variables X and V were lagged, initial values were specified with a VAR statement to specify the starting state of the system. The mass(m_1), time step(dt), spring constant (k_1) and damper coefficient (c) were declared and initialized by a CONTROL statement (SAS® CODE 2) in PROC MODEL.

SAS® CODE 2

```
/*- Generate observations to drive the simulation time
periods-*/
   proc model data=damper;
      var force -200 disp  10  vel  0 accel -20  time 0;
      control  mass  10  c1  1.5  c2  1.5  dt 0.1  k1
            10  k2 10
                          a1 0 a2 0 a3 0;
            k=1/((1/k1)+(1/k2));
            c=1/((1/c1)+(1/c2));
              force = -k * disp -c * vel;
             disp  = lag(disp) + vel * dt;
             vel   = lag(vel) + accel * dt;
             accel = force / mass;
             time  = lag(time) + dt;
```

The damping force equation written in SAS® language was solved through a SOLVE statement with PROC MODEL. The simulation results from multiple runs were written to output data sets. Only solutions for three runs are shown in the results (SAS® CODE 3). The first run uses the original initial conditions specified in the VAR statement. The displacement scale was zeroed at the point where the force of gravity was offset. This way the gravity constant (g) was omitted from the force equation. Results are shown for two time durations (100 sec, and 10 sec).
The variations in displacement (X_1, specified and shown as 'disp' in SAS® codes) at discrete time intervals for three different constant force inputs (F=-200 kg, 400kg, and 600kg) were evaluated. The mass (m_1) was varied in the model and the effect of variation on displacement was evaluated. Similar results were also obtained for three values of m_1. The spring constant (k_1) and the damper coefficient (c) were varied and their effects on displacement were evaluated too. The solution results from each run were stored in separate SAS data sets and merged. The merged

solution results were plotted using the GPLOT procedure (SAS® CODE 4). The results from multiple runs were compared for initial evaluation of the system. The SAS® data sets with the simulation results were used as input data sets for optimization for maximum damper performance and for error analysis using PROC IML

.

SAS® CODE 3

```
/*- Simulate the model for the base case ---*/
      control Run '1';
      solve / out=s1;
     Label Run = 'Initial Displacement (mm)';
     run;
/*- Simulate the model for the 2nd run- twice the initial
displacement -*/
      control Run '2';
        var disp 20;
        solve / out=s2;
       run;
/*- Simulate the model for the third run- four times the initial
displacement -*/
      control Run '3 ';
      var disp  40;
        solve / out=s3;
       run;
```

SAS® CODE 4

```
/*-- Overlay Plot of Series simulations'--*/
 data s;
      set s1 s2 s3;
   run;
   /*- Plot the results --*/
   proc gplot data=s;
      plot disp*time=run /frame cframe=yellow;
       axis1  minor=none;
       axis2  minor=none label=(angle=90  rotate=0);
       Label time = 'Time (sec.)';
     Label disp = 'Displacement Series (mm)';
   run;
```

FORCING TERMS

The forcing term, which is the input to the system, given by $F(t)$ can be modeled readily by standard mathematical functions. Performance of five input force models (unforced, constant -force, ramp, sinusoidal, and exponential) were evaluated varying the system parameters. The different types of force input to the system are specified in PROC MODEL using control statements generating different output datasets for each run (SAS® CODE 5). The output can be merged and overlaid into one plot in GPLOT (SAS® Code 4).

SAS® CODE 5

```
/*- Changing the force input type in model statement-*/
   proc model data=damper;
      var      disp  10  vel  0 accel -20  time 0;
      control  mass   10  c1  1.5  c2  1.5  dt 0.3  k1
10  k2 10
                                a1 0 a2 0 a3 0;
        force1   = (a1*(dt*dt)) + (a2*dt) + a3;
       force2   = (a1*cos(a2*dt)) + (a3*sin(a2*dt));
       force3   = a1*(exp(a2*dt));

       force = -k * disp -c * vel;

     control Run 'F(t)= Zero   ';
     force=force1;
     solve / out=s1;
     run;

     control Run 'F(t)= Quadratic  ';
     control  a1 10 a2 10 a3 10;
     force=force1;
     solve / out=s2;
     run;

     control Run 'F(t)= Sinusoidal  ';
     control  a1 10 a2 10 a3 10;
     force=force2;
     solve / out=s3;
     run;

    control Run 'F(t)= Exponential  ';
     control  a1 10 a2 10;
     force=force3;
     solve / out=s4;
     run;
```

TIME CONSTANT AND THE TIME TO DECAY

The solution to the differential function comes as the sum of two parts: i) the complementary function which arises solely due to the spring-mass-damper system and ii) the particular integral which arises solely due to the force input term (*F*(*t*)). If the spring mass damper system is subjected to a constant force it will remain at constant motion from its datum position. If you apply oscillations to such a system oscillations will result. This is called steady state of the system. How the system gets to the steady state is governed by the system parameters (mass (M), spring constant (k) and the damper coefficient (c)). The way in which the mass reaches the steady state, called the transient is reflected in the complementary function and depends on the relative sizes of c^2 and 4Mk. The system with the force input taken out (*F*(*t*)=0) represents the transient response. The system is a second order differential equation (Eq. 2) and is related to a second order algebraic equation: $Ax^2+Bx+C=0$, where A,B, and C are constants. Like in second order algebraic equations the 'discriminant ', c^2-4Mk determines the type of solution to the differential equation that represents the spring-mass-damper system.

The discriminant, c^2-4Mk >0 produces a complementary function (transient) of the form $Y=Ae^{p1t}+Be^{p2t}$.with, A, B,p1 and p2 all constant and p1 and p2 both negative. How long it takes for the transient to die away depends upon the time constants of the two exponential decay terms. This corresponds to large 'c' values compared with M and k (c^2 > 4Mk) and so represents a heavily damped system. On the other hand c^2-4Mk <0 produces a complementary function (transient) of the form $Y = e^{pt}$ (A sinω t + B cosω t) with A, B, p and ω all constant and p negative. This produces a sinusoidal transient modulated by pure exponential decay. How long the sinusoids take to die away depend upon the time constant of the exponential hence corresponds to small values of 'c' compared with M and k (remember $c^2 < 4Mk$) and so is a lightly damped system. $c^2-4Mk = 0$ produces a complementary function (transient) of the form $Y = (A + Bt)e^{pt}$ with A, B and p constant and p negative. This is a linear function (A + Bt), modulated by exponential decay. How long this transient takes to die away depends on the time constant of the exponential. This is the fastest response possible without setting up oscillations in the system thus corresponds to a critically damped system. Thus the transient is the way in which the system responds during the time it takes to reach its steady state. The time to decay was determined by using a time constant (τ) in the exponential part of the transient (SAS® Code 6).

SAS® CODE 6

```
/*- Calculating the time constant -*/
   proc iml;
 mu = { 1 , 2, 3, 4, 5 }; m = {10 , 10 ,  10 ,  10,  10};
 c = {1.5, 1.5, 1.5, 1.5 ,1.5};   k = {5 , 5 , 5 , 5 , 5};

 nmu  = nrow(mu);
    a = j(nmu);
    t = j(nmu);
  csq = j(nmu);
    Y = j(nmu);
    do i = 1 to nmu;
       a[i] = 4*m[i] +k[i];
          t[i] = exp(-mu[i]);
     csq[i] = c[i]*c[i];
    end;

start damper( x ) global( m, csq, k, a, nmu, t, Y);
  A1 = 1.2; B1 = 2.3; p1 = 0.3; p2 = 0.5; w=5;
         do i = 1 to nmu;
     if csq[i]> a[i] then
        Y[i]=(A1*exp(p1*t[i]))+(B1*exp(p2*t[i]));
     if csq[i]< a[i] then
    Y[i]=((A1*sin(w*t[i]))+(B1*cos(w*t[i])))*(exp(p1*t[i]));
        else
    Y[i]=(A1+(B1*t[i]))*(exp(p2*t[i]));
         return(Y);
   end;
 finish;
      print t[format=6.2];  print Y[format=6.2];
```

RESULTS

The damper characteristics changed significantly upon changing the initial displacement, initial mass, spring constant and damper coefficient. The results are shown below. The force input did not influence the damper performance.

EFFECT OF INITIAL DISPLACEMENT

Smaller initial displacements resulted in smaller oscillations in the system and better damper characteristics (Figure 4). There were no differences in the frequencies of oscillations observed. However, the amplitudes of oscillations increased with increasing initial displacement.

EFFECT OF INITIAL MASS

The damper characteristics also changed upon changing the mass of the car. Smaller mass resulted in smaller frequencies and smaller amplitudes of oscillations in the system thus better damper characteristics (Figure 5).

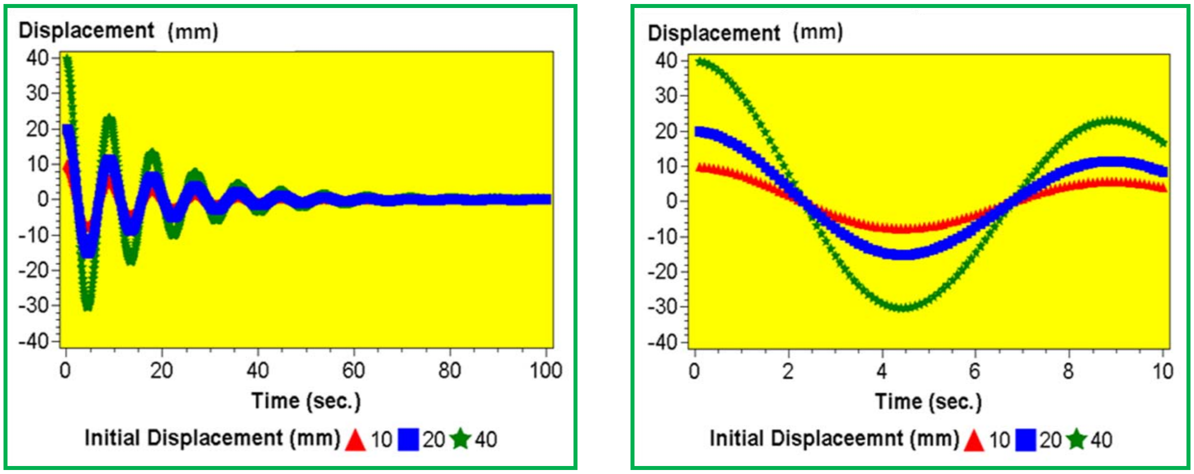

Figure 4: Effect of three values of initial displacement on displacement of cars, shown for two time periods (10 sec. and 100 sec.).

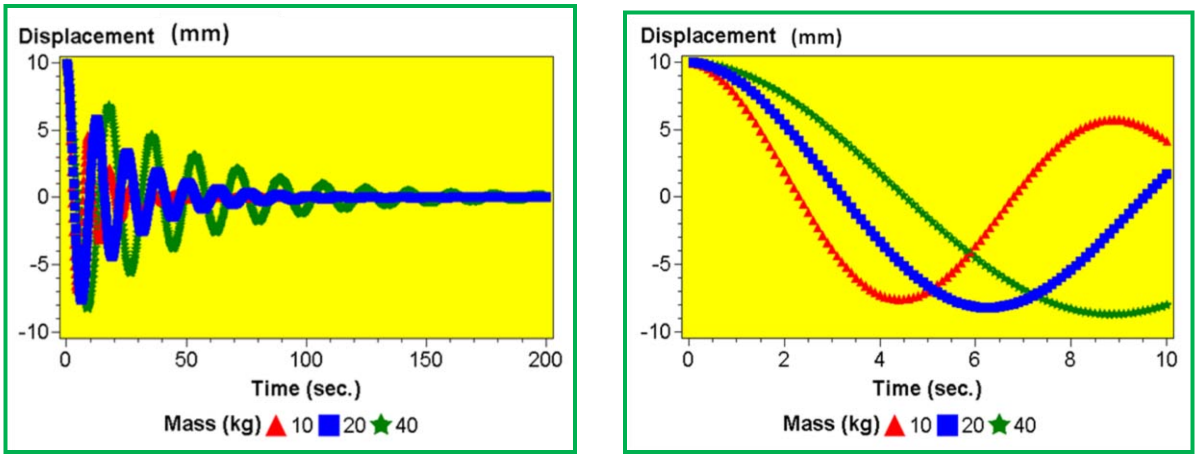

Figure 5: Effect of three values of mass of car on displacement, shown for two time periods (10 sec. and 100 sec.)

EFFECT OF SPRING CONSTANT

The damper characteristics also changed dramatically upon changing the spring constant. Smaller spring constant resulted in larger oscillations in the system and poorer damper characteristics (Figure 6). The optimum spring constant was larger than or equal to 8. Effect of spring constant was also independent of other parameters.

EFFECT OF DRIVING FORCE AND DAMPER COEFFICIENT

Change in the constant driving force did not influence the damper performance (Figure 7). An optimum damper coefficient was required for optimum damper characteristics. Although no change in frequencies of oscillations observed, amplitudes reduced slightly (Figure 8) with increasing damper coefficient. .

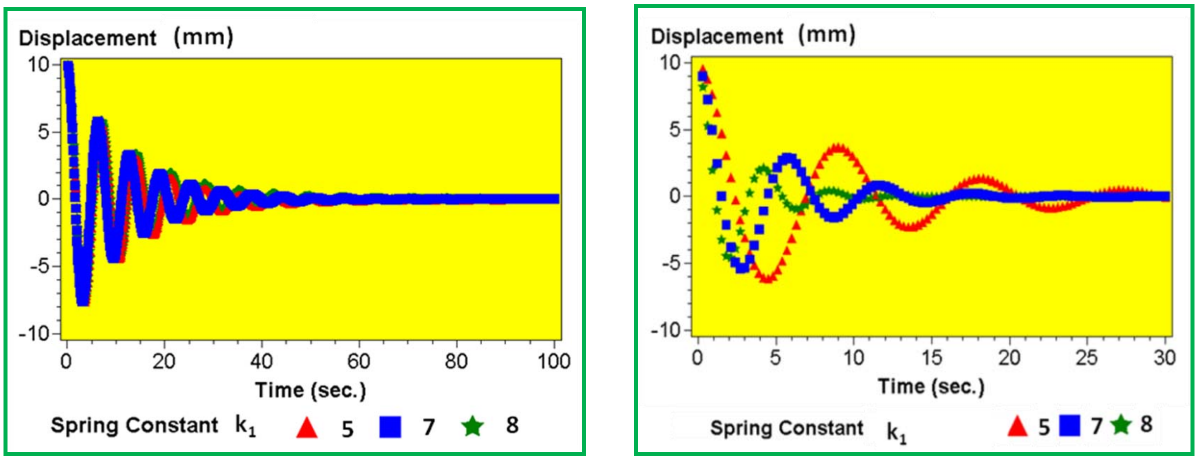

Figure 6: Effect of three values of mass of car on displacement, shown for two time periods (30 sec. and 100 sec.)

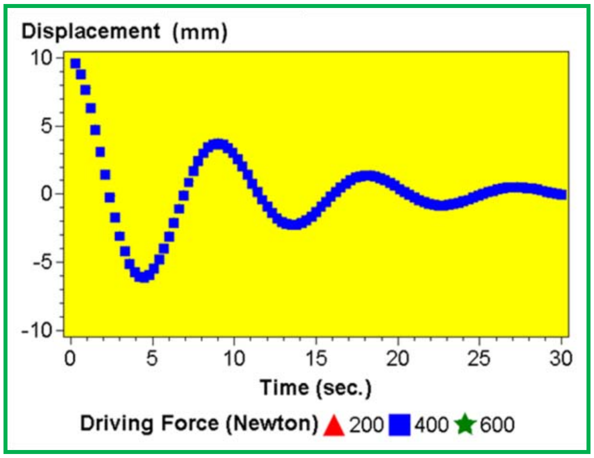

Figure 7: Effect of three values of constant input force on displacement

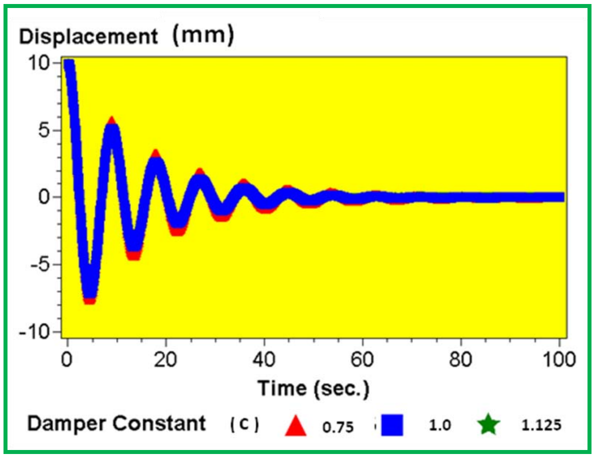

Figure 8: Effect of three values of damper coefficient on displacement

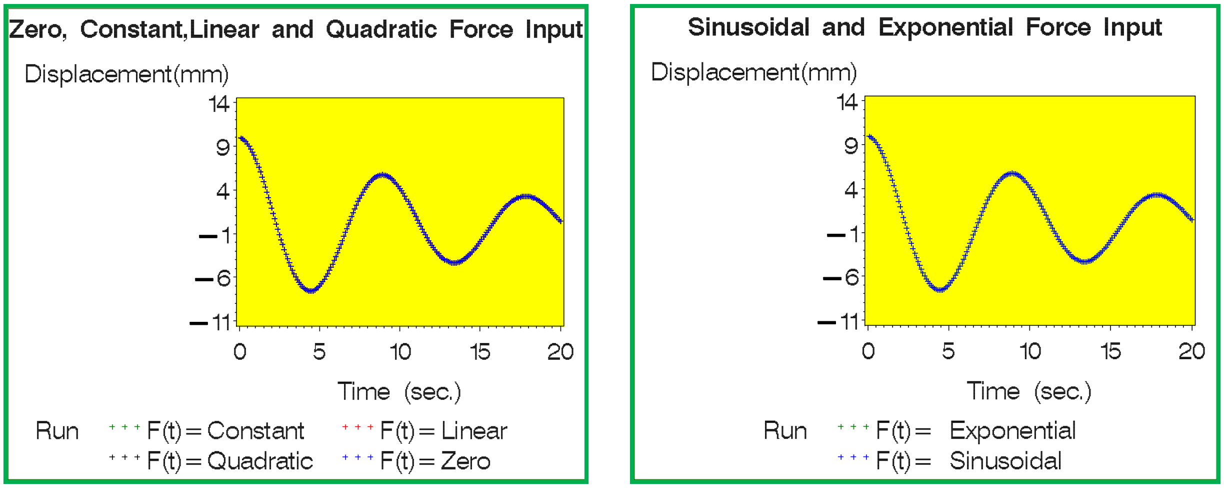

Figure 9: Effect of six types (zero, constant, linearly varying, quadratically varying, exponential, and sinusoidal) of driving force input on the displacement of the cars (shown for 20 sec.)

EFFECT OF DRIVING FORCE TYPE

The type of driving force did not show significant influence on the frequency or the amplitude of the displacement of the cars. The damper performance is not affected by the type on force input to the system (Figure 9). Angular frequency did not influence the displacements measured. Altering the initial displacement of the first car with the sinusoidal and the exponential driving force input were analyzed. Smaller initial displacements resulted in smaller amplitudes of oscillations indicating enhanced damper performance. The sinusoidal and the exponential input did not vary in their amplitudes with the change in initial displacements (Figure 10).

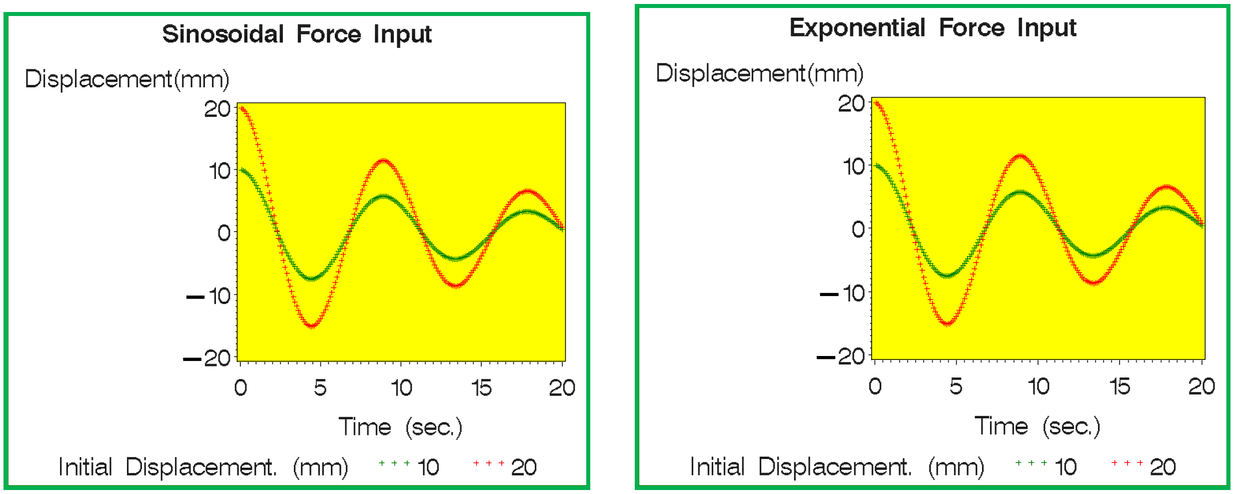

Figure 10: Effect of varying initial displacement with sinusoidal and exponential force inputs on displacement of the cars (shown for 20 sec.)

TIME CONSTANT

The time constant was calculated to be 5τ. The contribution of $e^{-t/\tau}$ died away to 'practically nothing (0.67%)' at a time constant equal to 5τ (Table 1). The steady state solution was calculated to be equal to 1.0

Table 1: Time constant and the time to decay.

t	$e^{-t/\tau} * 100\%$
0	100.00
1τ	36.79
2τ	13.53
3τ	4.98
4τ	1.83
5τ	0.67

CONCLUSIONS

In designing damper system, the only parameter that can be changed is the value of spring constant. All of the other variables do not impose significant changes in damping behavior. If the system oscillates, the spring constant needs to be lowered to improve damper performance. The over shoots in the system do not always mean that the system is always under-damped. Instead, the overshoots can be ignored and the system can be considered critically damped when it reaches equilibrium quickly after the overshoot. Time constant of 5 gives important results, at which the contribution of the exponential term dies away. Statistical modeling is a useful approach to evaluate the performance of the spring-mass-damper system. Useful results obtained in this study for a wide range of input parameters indicate that modeling similar systems statistically using SAS® is powerful and can replace the analytical solutions.

REFERENCES

SAS Institute, Inc. (1989). SAS/IML® Software: Usage and Reference, Version 6, First Edition. Cary, NC: SAS Institute, Inc.

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