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SAS® Magic Squares

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Abstract

Many SAS® users pursue recreational interests that can be investigated with SAS® software. A popular mathematical recreation that has intrigued enthusiasts for centuries is the magic square. A normal numeric magic square is an n by n matrix with cells filled with all positive integers from 1 to n^2 inclusive. The integers are arranged in such a way that every row, column and the main and secondary diagonals sum to the 'magic' number.

The primary goal of this project is to find an efficient method to generate all 4 by 4 magic squares using SAS® software. A secondary goal is to investigate the properties of these magic squares and to classify them into sub-types where possible.

Introduction

Mathematical recreationists typically attempt to create or solve numeric or logic puzzles. One type of puzzle that has become extremely popular in recent years is Sudoku. Creating and solving Sudoku puzzles can be very challenging, but these challenges have been met with the help of the powerful programming capabilities of SAS® software (Karwe, Seunarine & Razafindrakoto, 2006; First, 2007; First, 2008).

Long before Sudoku became popular the magic square was a source of entertainment and intrigue. Magic squares have appeared in jewelry, paintings, and carvings and have been the topic of many papers, books and more recently, web sites (see Wolfram). Magic squares have been studied in an attempt to create them and better understand their properties. A brief history of the magic square, which includes the earliest known example dating back to 650 BC in China, can be found at http://en.wikipedia.org/wiki/Magic_square.

There are many orders and types of normal magic squares, with each having its own properties. Odd order magic squares, for example, are formed by an n x n matrix where n is an odd number. Singly even order magic squares consist of a matrix where n is an even number which when divided by 2 results in an odd number (e.g., 6, 10, and 14). Doubly even order magic squares are defined by when n/2 results in another even number, such as when n equals 4, 8, or 12. Within an order there may be sub-types of squares for which an additional property is present such as when groupings of cells (other than rows, columns and diagonals) also sum to the magic number.

Regardless of a square's type or order, the magic number for a normal magic square of order n can be derived from the following formula: $(n^3 + n)/2$. For magic squares of order 4 the magic number is 34. The main goal of this paper is to generate all possible 4 by 4 magic squares using SAS® software. A secondary goal is to identify several sub-types of squares and reveal the relationships among them.

Algorithm to generate all 4 by 4 magic squares

There are 16! ways, which is nearly 2.1 x 10^{13}, to arrange the first 16 digits into the 16 cells of a 4 by 4 magic square. We know that a vast majority of these arrangements will not form magic squares so we must find a more efficient algorithm to generate the squares. To narrow the search strategy we first generate all blocks of 4 unique numbers, from the set of the first 16 positive integers, which sum to 34. This takes just a fraction of a second using the SAS® code below.

Code 1: Generate all blocks of 4 unique numbers that sum to 34 from the set of the first 16 positive integers

```
data temp;
  array ijkl (4) i j k l;
  do i = 1 to 16;
   do j = 1 to 16;
    do k = 1 to 16;
     do l = 1 to 16;
      if i ne j and i ne k and i ne l and
         j ne k and j ne l and k ne l and sum(i,j,k,l) = 34 then do;
         block = compress(i||','||j||','||k||','||l);
  output;
  end;
  end;end;end;end;
run;
```

In total there are 2064 blocks, the first and last 5 of which are shown in Table 1.

Table 1: A subset of 10 blocks

Obs	block		Obs	block
1	1,2,15,16		2060	9,8,3,14
2	1,2,16,15		2061	9,8,4,13
3	1,3,14,16		2062	9,8,5,12
4	1,3,16,14		2063	9,8,6,11
5	1,4,13,16	. . .	2064	9,8,7,10

The next step is to select four of these 2064 blocks to construct a magic square, and to continue this process until all possible magic squares have been generated. This is an iterative process that would involve evaluating 2064^3 variations (almost 8.8 billion) if all combinations of blocks were evaluated. This approach vastly reduces the number of iterations compared to 16!, but it still is not practical.

To further reduce the number of iterations the following approach was adopted. First, one block from the master set of 2064 blocks is selected. Then, from the master set of blocks, a subset of blocks is created by removing all blocks that contained any of the four numbers present in the selected block. From this subset a second block is selected and paired with the block selected in the first step. After selecting two blocks, a second subset of blocks is created from the first subset by removing all blocks that contained any of the numbers in the second block. This second subset includes all blocks that contain none of the numbers present in the two blocks selected thus far. Finally, a third block is selected from the second subset and combined with the two blocks already selected. At this point three blocks have been selected and there are no duplicate numbers among them. Four of the sixteen integers remain and if they can be arranged to generate a magic square we will have succeeded. Otherwise we know that this set of three blocks will not produce a magic square and we select the next block from the second subset. For each originally selected block we repeat this process for every block in the first subset, and for each of those blocks, repeat it for every block in the second subset.

After all the iterations for the originally selected block are complete, a second block is selected from the master set and the entire process is repeated. This process is repeated for every block in the master set of 2064 blocks. At the completion of these steps all 7040 magic squares will have been generated.

Fundamental magic squares

Every magic square can be rotated or reflected to generate 8 squares that look unique but are simply different views of the same fundamental square. Thus, within the 7040 squares there are only 880 (7040/8) fundamental magic squares. To illustrate this consider the hypothetical magic square shown in diagram 1 of Table 2, where each letter from A to P represents an integer from 1 through 16. This magic square can be represented by the seven other diagrams by simply rotating or reflecting the square.

Table 2: Rotations and reflections of a fundamental magic square

1. Magic square

A	B	C	D
E	F	G	H
I	J	K	L
M	N	O	P

2. Vertical reflection

D	C	B	A
H	G	F	E
L	K	J	I
P	O	N	M

3. Horizontal reflection

M	N	O	P
I	J	K	L
E	F	G	H
A	B	C	D

4. Rotate left

D	H	L	P
C	G	K	O
B	F	J	N
A	E	I	M

5. Rotate right

M	I	E	A
N	J	F	B
O	K	G	C
P	L	H	D

6. Rotate 180^{0}

P	O	N	M
L	K	J	I
H	G	F	E
D	C	B	A

7. Secondary diagonal reflection

P	L	H	D
O	K	G	C
N	J	F	B
M	I	E	A

8. Main diagonal reflection

A	E	I	M
B	F	J	N
C	G	K	O
D	H	L	P

Blocks that are used and not used

Not all of the 2064 blocks are used in the full set of 7040 magic squares. It turns out that 1096 (about 53%) of them are used and that 968 (about 47%) of them are not. With so many unused blocks it would have been more efficient if we could have identified them prior to the search for all magic squares and eliminate them from the process. In that way we could have reduced the master set of blocks from 2064 to 1096 and sped up the search dramatically. The number of blocks in the first and second subset of blocks, for each number in the master set, would also have been reduced significantly, making the process that much more efficient. Unfortunately, these unused blocks could not be identified in advance.

Since almost half of the blocks were unused, we investigated further to find out more about the relationship between integer value and the number of blocks used. Table 3 shows the total number of blocks that each integer was found in and the number of those blocks that are used in the construction of magic squares. The first column contains a list of integers and the second shows the total number of blocks in which each integer appears. The third column contains the number of used blocks and the fourth the number of used blocks as a proportion of the total number of blocks. By way of an example, the integers 1 and 16 each appear in 632 blocks, but only 456 of them can be found in the complete set of magic squares.

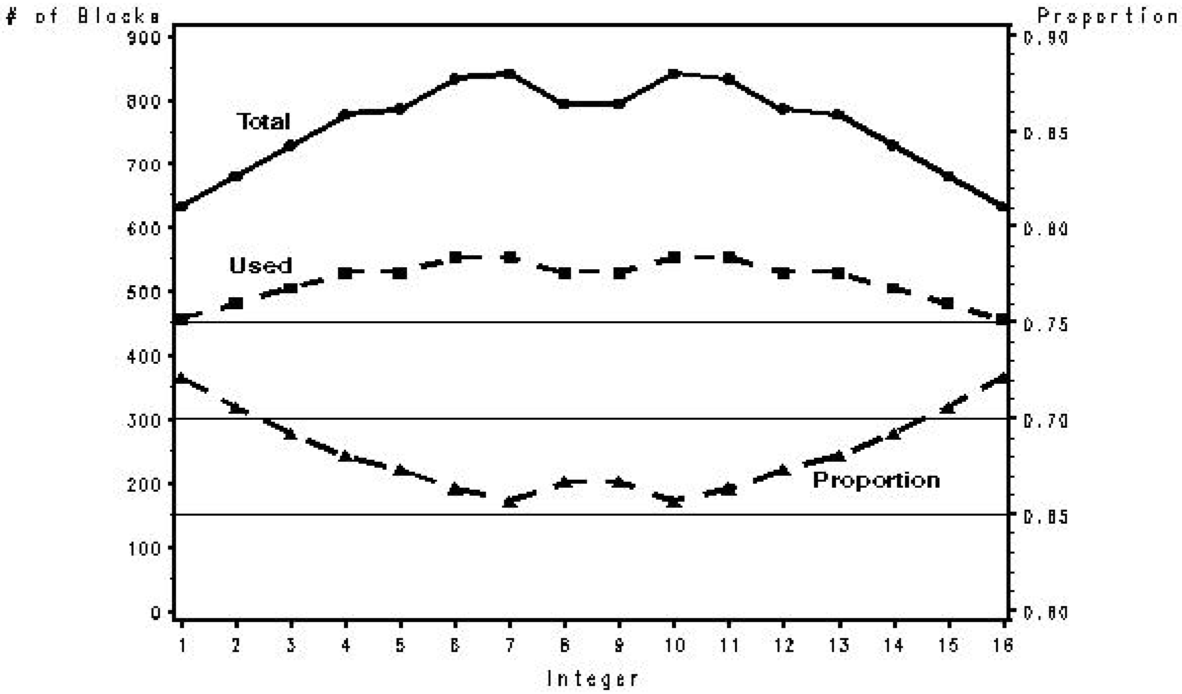
Table 3: Number of blocks by integer

Integer	Total number of blocks	Number of used blocks	Used blocks as a proportion of total number
1	632	456	0.722
2	680	480	0.706
3	728	504	0.692
4	776	528	0.680
5	784	528	0.674
6	832	552	0.664
7	840	552	0.657
8	792	528	0.667
9	792	528	0.667
10	840	552	0.657
11	832	552	0.664
12	784	528	0.674
13	776	528	0.680
14	728	504	0.692
15	680	480	0.706
16	632	456	0.722

Figure 1 shows the three columns of numbers in Table 3 as separate functions of integer. The line labeled 'Total' shows the relationship between the total number of blocks and integer, and the line labeled 'Used' shows the relationship between the number of used blocks and integer. The appropriate y-axis for these two lines is found at the left side of the graph and is labeled '# of Blocks'. The bottom line labeled 'Proportion' plots the relationship between the proportion of used blocks and integer. The appropriate y-axis for this line is found at the right side of the graph.

The 'Total' function shows that integers are not equally represented within the complete set of 2064 blocks. Generally speaking the total number of blocks increases as integers increase from 1 to 7 or decrease from 16 to 10. The integers 8 and 9 are anomalies in that they deviate slightly from this pattern. This anomaly is probably due to the magic square being an even ordered square in which no single integer forms the unique mid point between 1 and n^2. The 'Used' function follows a similar pattern as 'Total' but it is somewhat attenuated. Interestingly, when the proportion of used blocks is plotted as a function of integer the function is u-shaped – that is, it decreases as integers increase from 1 to 7 or decrease from 16 to 10. Again, the middle two numbers, 8 and 9, depart slightly from this trend.

Figure 1: Number and proportion of blocks as a function of integer



Combinations that are used and not used

Many blocks contain the same set of four integers but the integers appear in different orders. These blocks can be considered to be combinations of a set of 24 permutations. Of all 86 combinations, the integers in 51 (59.3%) of them do and 35 (40.7%) do not appear in the full set of magic squares. All completely unused combinations are shown in Table 4. The number in the first column identifies the combination and the remaining four numbers constitute the unused combination.

Table 4: Completely unused combinations

number	combination
1	1,5,13,15
2	1,6,13,14
3	1,7,11,15
4	1,9,10,14
5	1,9,11,13
6	1,10,11,12
7	2,4,12,16
8	2,5,13,14
9	2,6,10,16
10	2,6,12,14
11	2,8,10,14
12	2,9,10,13
13	2,9,11,12
14	3,4,11,16
15	3,4,12,15
16	3,5,11,15
17	3,7,8,16
18	3,7,9,15
19	3,7,11,13
20	3,8,11,12
21	3,9,10,12
22	4,6,8,16
23	4,6,10,14
24	4,7,8,15
25	4,7,11,12
26	4,8,10,12
27	4,9,10,11
28	5,6,7,16
29	5,6,8,15
30	5,6,9,14
31	5,6,10,13
32	5,7,8,14
33	5,7,9,13
34	6,7,8,13
35	7,8,9,10

Because each of these combinations represents 4! or 24 blocks, together they account for 35 x 24 = 840 of the 968 unused blocks. This leaves 128 unused blocks (968-840) unaccounted for. These 128 blocks can be found in the 15 combinations shown below, where some but not all 24 blocks are used. The last column shows the number of blocks (from a maximum of 24) that are used to form magic squares.

Table 5: Partially unused combinations

combination number	combination	number of used blocks
1	1,2,15,16	16
2	1,3,14,16	16
3	1,5,12,16	16
4	1,8,9,16	16
5	2,4,13,15	16
6	2,6,11,15	16
7	2,7,10,15	16
8	3,4,13,14	16
9	3,6,11,14	16
10	3,7,10,14	16
11	4,5,12,13	16
12	4,8,9,13	16
13	5,6,11,12	8
14	5,7,10,12	16
15	6,8,9,11	16

For one combination [5, 6, 11, 12], 24-8=16 blocks are unused in the complete set of magic squares. For each of the remaining 14 combinations, 24-16=8 blocks are not used. In total then, these partially unused combinations account for an additional 128 unused blocks. We have now accounted for all 968 unused blocks.

Sub-types of 4 by 4 magic squares

Normal

Normal magic squares are defined as having the properties that every row, column, and the main and secondary diagonals sum to the 'magic' number. In addition to these properties it turns out that the groups of cells identified in Table 6 also sum to 34 for every 4 by 4 magic square.

Table 6: Additional properties of all normal 4 by 4 magic squares

```
1. * B C *   2. * * * *   3. * * * *   4. A * * D
   * * * *      E * * H      * F G *      * * * *
   * * * *      I * * L      * J K *      * * * *
   * N O *      * * * *      * * * *      M * * P
```

Pan-magic

Pan-magic squares have the additional property that the so-called broken diagonals also sum to the magic number. The full set of 6 broken diagonals is shown in Table 7. In each of these hypothetical squares, the numbers in the cells represented by the letters must sum to 34 for the square to qualify as pan-magic.

Table 7: Broken diagonals

```
1.  * B * *   2. * * * D   3. * * C *   4. A * * *    5. * * C *   6. * B * *
    * * G *      E * * *      * F * *      * * * H       * * * H      E * * *
    * * * L      * J * *      I * * *      * * K *       I * * *      * * * L
    M * * *      * * O *      * * * P      * N * *       * N * *      * * O *
```

There are 384 pan-magic squares, or 48 (384/8) fundamental pan-magic squares. One surprising feature of these squares is that the following 4-cell units also sum to 34. An example of a pan-magic square is shown in Table 9:

Table 8: Additional properties of pan-magic squares

```
1.  A B * *    2. * * C D    3. A * * D    4. * * * *
    * * * *       * * * *       E * * H       * * * *
    * * * *       * * * *       * * * *       I * * L
    M N * *       * * O P       * * * *       M * * P

5.  A B * *    6. * B C *    7. * * C D    8. * * * *    9. * * * *
    E F * *       * F G *       * * G H       E F * *       * * G H
    * * * *       * * * *       * * * *       I J * *       * * K L
    * * * *       * * * *       * * * *       * * * *       * * * *

10. * * * *   11. * * * *   12. * * * *
    * * * *       * * * *       * * * *
    I J * *       * J K *       * * K L
    M N * *       * N O *       * * O P
```

Table 9: A pan-magic square

Row	Column 1	2	3	4	Total
1	8	11	5	10	34
2	13	2	16	3	34
3	12	7	9	6	34
4	1	14	4	15	34
Total	34	34	34	34	

Associative

An associative magic square is a normal magic square with the additional property that the eight pairs of numbers that are symmetrically opposite the center of the square sum to $n^2 + 1$, or 17 in the case of a 4 by 4 magic square. There are 384 associative magic squares, or 48 (384/8) fundamental associative magic squares. Using the notation shown in Table 2.1, the eight symmetrically opposite pairs of numbers are AP, BO, CN, DM, EL, FK, GJ, and HI. An example of an associative magic square is shown in Table 10.

Table 10: An associative magic square

Row	Column 1	2	3	4	Total
1	9	16	7	2	34
2	4	5	14	11	34
3	6	3	12	13	34
4	15	10	1	8	34
Total	34	34	34	34	

Quadrant associative

A second type of associative magic square, which we call quadrant associative, has the property that all diagonally opposite pairs of numbers within the four main quadrants sum to $n^2 + 1$, or 17. The four main quadrants of a 4 by 4 matrix are shown in Table 11 with the numerals 1, 2, 3, and 4 identifying the quadrants.

Table 11: Four main quadrants of a magic square

1	1	2	2
1	1	2	2
3	3	4	4
3	3	4	4

Using the notation shown in Table 2.1, the diagonally opposite pairs of numbers within the four quadrants are: AF, BE, CH, DG, IN, JM, KP, and LO. In total, there are 384 quadrant associative magic squares, or 48 (384/8) fundamental quadrant associative magic squares. Table 12 shows an example of a quadrant associative magic square.

Table 12: A quadrant associative magic square

Row	Column 1	2	3	4	Total
1	9	5	16	4	34
2	12	8	13	1	34
3	7	11	2	14	34
4	6	10	3	15	34
Total	34	34	34	34	

Distributive

Distributive magic squares have the property that each of the four integers in the following sets of numbers (1,2,3,4) and (5,6,7,8) and (9,10,11,12) and (13,14,15,16) are located in a row and column where none of the other three numbers in the set are located. There are 2432 distributive magic squares or 304 (2432/8) fundamental distributive magic squares. An example of a distributive magic square (which is also pan-magic) is shown in Table 13.

Table 13: An example of a distributive magic square

Row	Column 1	2	3	4	Total
1	8	10	3	13	34
2	1	15	6	12	34
3	14	4	9	7	34
4	11	5	16	2	34
Total	34	34	34	34	

Summary of sub-types

Table 14 summarizes the total number of squares of each sub-type for all 4 by 4 magic squares. In summary, there are 4224 normal magic squares that do not have the property of any of the other sub-types. There are 2432 distributive magic squares and 384 of each of the other sub-types.

Table 14: Summary of magic square sub-types

Sub-type of magic square	Total number of squares	Number of fundamental squares
Pan-magic	384	48
Associative	384	48
Quadrant	384	48
Distributive	2432	304
None of the above types	4224	528
All types	7040	880

Some magic squares have properties of one sub-type only while others have properties of two sub-types. However, no magic square has properties of three or more sub-types. Table 15 shows the relationships among sub-types of 4 by 4 magic squares. This table has been organized into five columns, where the first four represent the four sub-types of magic squares and the rightmost column the total number of squares. Each of the first four columns contains either a 1 or a 0 in every row. If n1 squares were common to all four types of magic squares then there would be a 1 under each type of square (first four columns) and n1 in the rightmost column. Similarly, if n2 squares had properties of both associative and distributive magic squares then n2 would be in the last column on the row where there is a 1 under associative and distributive and a 0 under pan-magic and quadrant associative.

The total number of magic squares of each sub-type can be determined from this table by summing the number of squares from the relevant rows. For example, the total number of pan-magic squares is determined by summing the numbers in the rightmost column wherever there is a 1 in the pan-magic column.

Table 15: Relationships among magic square types

pan-magic	associative	quadrant associative	distributive	#squares	
0	0	0	0	4224	(normal, without any other properties)
0	0	0	1	1664	
0	0	1	0	128	
0	0	1	1	256	
0	1	0	0	128	
0	1	0	1	256	
0	1	1	1	0	
1	0	0	0	128	
1	0	0	1	256	
1	0	1	0	0	
1	0	1	1	0	
1	1	0	0	0	
1	1	0	1	0	
1	1	1	0	0	
1	1	1	1	0	
Total				7040	

This table informs us that there are 1664 distributive magic squares that do not have the defining properties of pan, associative, or quadrant associative magic squares. One example of these squares is shown in Table 16.

Table 16: A distributive magic square

Row	Column 1	2	3	4	Total
1	7	4	14	9	34
2	10	13	3	8	34
3	1	6	12	15	34
4	16	11	5	2	34
Total	34	34	34	34	

Similarly, there are 384 associative squares, 256 of which are also distributive. An example of an associative distributive magic square is shown in Table 17. The other 128 associative squares do not share the properties of pan, distributive or quadrant associative magic squares.

Table 17: An associative distributive magic square

Row	Column 1	2	3	4	Total
1	3	16	10	5	34
2	6	9	15	4	34
3	13	2	8	11	34
4	12	7	1	14	34
Total	34	34	34	34	

Discussion

Using basic SAS® programming statements it was easy to generate all 7040 four by four magic squares. Four sub-types of squares (i.e., distributive, pan, associative, quadrant associative) were identified and it was determined that 4224 of the 7040 squares cannot be classified into any of these four sub-types. It was further determined that a square can be of one sub-type only with the exception that a small number of distributive squares also have the property of one other sub-type (i.e., pan, associative, or quadrant associative).

There are 880 fundamental magic squares of order four. According to Wolfram there are 275,305,224 fundamental magic squares of order five, and a very much larger but as yet undetermined number of order six. Given these large numbers it is doubtful if the current algorithm would be of much practical use in enumerating magic squares of order 5 or greater.

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