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Computing Expected
Means Squares

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Computing Expected Mean Squares

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ABSTRACT

This evaluation of the expected mean squares arising from the analysis of unbalanced mixed ANOVA models has long been an analytically intractable problem. This paper presents a theoretically and computationally simple technique.

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1. INTRODUCTION

Hartley's [1967] method of "synthesis" was one of the first general methods developed for computing the coefficients in the expected values of mean squares for the mixed model. Gaylor et al. [1970] described the computation of expected values arising from the set of mean squares of the Forward Doolittle.

Speed and Hocking [1974] developed the most computationally simple technique to date. The method presented here may be viewed as minor modification of the Speed and Hocking technique.

2. THE MODEL

Following Searle [1971] the mixed model is represented as:

$$Y = X_0 \beta_0 + \sum_{i=1}^k X_i \beta_i + e \quad (1)$$

where,

- (1) Y is an n-vector of observations
- (2) X_0, X_1, \dots, X_k are $n \times m_i$ known matrices
- (3) β_0 is a vector of fixed effects
- (4) The vectors $\beta_i (i=1, \dots, k)$ are assumed independent of each other and e and are distributed $N(0, \sigma_i^2 I_{m_i})$
- (5) e is an n-vector assumed $N(0, \sigma_e^2 I)$

On making the above normality assumptions,

$$E(Y) = X_0 \beta_0$$
$$\text{Var}(Y) = \sum_{i=1}^k X_i X_i' \sigma_i^2 + I \sigma_e^2$$

and any symmetric quadratic form $Y'QY$ has

$$E(Y'QY) = \beta'_0 X'_0 Q X_0 \beta_0 + \sum_{i=1}^k \text{tr}(X'_i Q X_i) \sigma_i^2 + \text{tr}(Q) \sigma_e^2 \quad (2)$$

Letting $X = [X_0 | X_1 | \dots | X_k]$, note that all of the matrices except Q , needed in (2) are submatrices of $X'QX$.

3. COMPUTING SUMS OF SQUARES

A common practice for computing the sums of squares in mixed models is to formulate testable hypotheses as if all effects were fixed and then compute the corresponding SS.

Letting $\beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_k \end{bmatrix}$ and treating β as if it were fixed

and letting $(X'X)^{-}$ be any generalized inverse of $X'X$, then

$$SS_L = SS(H_0: L\beta=0) = (Lb)'(L(X'X)^{-}L')^{-1}(Lb) \quad (3)$$

where $b = (X'X)^{-}X'Y$ and L is any matrix of full row rank in the row space of X .

Equation (3) is a quadratic form in Y , namely $Y'Q_L Y$, where

$$Q_L = X(X'X)^{-}L'(L(X'X)^{-}L')^{-1}L(X'X)^{-}X'$$

Only the elements of $X'Q_L X$ are needed to evaluate the expected value of (3) when (1) is assumed to be the true model. Since L is in the row space of X , $L = L(X'X)^{-}X'X$.

Therefore

$$X'Q_LX = L'(L(X'X)^{-1}L')^{-1}L \quad (4)$$

Although (4) reduces the computational burden of computing expected values of SS's, a further reduction is developed in the next section.

4. EXPECTED VALUES OF SS

Theorem: If L is of full row rank and in the row space of X, and SS_L is computed using (3) then there exists a matrix $C = [C_0 | C_1 | \dots | C_k]$ of the same dimensions of L such that:

(i) $C = ML$ and

$$(ii) E(SS_L) = \beta'_0 C'_0 C_0 \beta_0 + \sum_{i=1}^k SSQ(C_i) \sigma_i^2 + n_L \sigma_e^2$$

where $SSQ(C_i)$ = sum of squares of the elements of the C_i submatrices, and

n_L = the number of rows in L

Proof: If the matrix

$$[L(X'X)^{-1}L' | L] \quad (5)$$

is formed and Cholesky adjustments are performed on the diagonals of the left hand matrix then the following matrix results:

$$[U | C] \quad (6)$$

Here U is the Cholesky decomposition (upper) of $L(X'X)^{-1}L'$ and $C = (U')^{-1}L$.

Thus $X'Q_LX = L'(L(X'X)^{-1}L')^{-1}L = L'(U'U)^{-1}L = C'C$

and the expected value of SS_L may be computed from the elements of

$$C'C = \begin{bmatrix} C'_0 C_0 & C'_0 C_1 & \dots & C'_0 C_k \\ C'_1 C_0 & C'_1 C_1 & \dots & C'_1 C_k \\ \vdots & \vdots & \ddots & \vdots \\ C'_k C_0 & C'_k C_1 & \dots & C'_k C_k \end{bmatrix}$$

Therefore

$$E(SS_L) = \beta'_0 C'_0 C_0 \beta_0 + \sum_{i=1}^k \text{tr}(C'_i C_i) \sigma_i^2 + n_L \sigma_e^2$$

but $\text{tr}(C'_i C_i) = \text{SSQ}(C_i)$, thus $C'C$ need not be computed and point (ii) of the theorem is proven. Since

$$\begin{aligned} C &= (U')^{-1} L \\ &= ML \end{aligned}$$

point (i) is also proven.

Several minor lemmas are a consequence of the above theorem.

Lemma 1: If any submatrix (L_i) of

$$L = [L_0 | L_1 | \dots | L_k]$$

is zero, then the expected value of SS_L does not involve the i^{th} effect.

Proof: $C = [ML_0 | ML_1 | \dots | ML_k]$

Lemma 2: If L consists of any number of the non-zero rows of the Cholesky matrix of $X'X$, then $C = L$.

Proof: This is a direct consequence of the method of "synthesis."

5. CONCLUSION

When working with an un-reparameterized model, computing the expected values of mean squares is no more difficult than computing the mean squares themselves.

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