Chapter 67
The KDE Procedure

Contents

Overview: KDE Procedure ...................................................... 4958
Getting Started: KDE Procedure ............................................. 4958
Syntax: KDE Procedure .......................................................... 4961
  PROC KDE Statement ....................................................... 4961
  BIVAR Statement ........................................................... 4961
  BY Statement ............................................................... 4966
  FREQ Statement ............................................................ 4966
  SCORE Statement ........................................................... 4967
  UNIVAR Statement .......................................................... 4968
  WEIGHT Statement .......................................................... 4973
Details: KDE Procedure ......................................................... 4973
  Computational Overview ................................................... 4973
  Kernel Density Estimates .................................................. 4973
  Binning ............................................................................ 4975
  Kernel Distribution Estimates .......................................... 4976
  Convolutions .................................................................... 4978
  Fast Fourier Transform ...................................................... 4980
  Bandwidth Selection ......................................................... 4981
  ODS Table Names ............................................................ 4982
  ODS Graphics ................................................................... 4983
Examples: KDE Procedure ....................................................... 4985
  Example 67.1: Computing a Basic Kernel Density Estimate .......... 4985
  Example 67.2: Changing the Bandwidth .................................. 4987
  Example 67.3: Changing the Bandwidth (Bivariate) .................. 4988
  Example 67.4: Requesting Additional Output Tables ................ 4990
  Example 67.5: Univariate KDE Graphics ............................... 4993
  Example 67.6: Bivariate KDE Graphics ................................. 4996
References ............................................................................. 5000
Overview: KDE Procedure

The KDE procedure performs univariate and bivariate kernel density estimation. Statistical density estimation involves approximating a hypothesized probability density function from observed data. Kernel density estimation is a nonparametric technique for density estimation in which a known density function (the kernel) is averaged across the observed data points to create a smooth approximation. PROC KDE uses a Gaussian density as the kernel, and its assumed variance determines the smoothness of the resulting estimate. See Silverman (1986) for a thorough review and discussion.

You can use PROC KDE to compute a variety of common statistics, including estimates of the percentiles of the hypothesized probability density function. You can produce a variety of plots, including univariate and bivariate histograms, plots of the kernel density estimates, and contour plots. You can also save kernel density estimates into SAS data sets.

Getting Started: KDE Procedure

The following example illustrates the basic features of PROC KDE. Assume that 1000 observations are simulated from a bivariate normal density with means \((0, 0)\), variances \((10, 10)\), and covariance 9. The SAS DATA step to accomplish this is as follows:

```sas
data bivnormal;
  seed = 1283470;
  do i = 1 to 1000;
    z1 = rannor(seed);
    z2 = rannor(seed);
    z3 = rannor(seed);
    x = 3*z1+z2;
    y = 3*z1+z3;
    output;
  end;
  drop seed;
run;
```

The following statements request a bivariate kernel density estimate for the variables \(x\) and \(y\), with contour and surface plots:

```sas
ods graphics on;
proc kde data=bivnormal;
  bivar x y / plots=(contour surface);
run;
ods graphics off;
```
The contour plot and the surface plot of the estimate are displayed in Figure 67.1 and Figure 67.2, respectively. Note that the correlation of 0.9 in the original data results in oval-shaped contours. These graphs are produced by specifying the PLOTS= option in the BIVAR statement with ODS Graphics enabled. For general information about ODS Graphics, see Chapter 21, “Statistical Graphics Using ODS.” For specific information about the graphics available in the KDE procedure, see the section “ODS Graphics” on page 4983.

Figure 67.1 Contour Plot of Estimated Density
The default output tables for this analysis are shown in Figure 67.3.

**Figure 67.3** Default Bivariate Tables

### The KDE Procedure

<table>
<thead>
<tr>
<th>Inputs</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Data Set</td>
<td>WORK.BIVNORMAL</td>
</tr>
<tr>
<td>Number of Observations Used</td>
<td>1000</td>
</tr>
<tr>
<td>Variable 1</td>
<td>x</td>
</tr>
<tr>
<td>Variable 2</td>
<td>y</td>
</tr>
<tr>
<td>Bandwidth Method</td>
<td>Simple Normal Reference</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Controls</th>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>Grid Points</td>
<td>60</td>
<td>60</td>
</tr>
<tr>
<td>Lower Grid Limit</td>
<td>-14.27</td>
<td>-13.18</td>
</tr>
<tr>
<td>Upper Grid Limit</td>
<td>12.17</td>
<td>12.165</td>
</tr>
<tr>
<td>Bandwidth Multiplier</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
The “Inputs” table lists basic information about the density fit, including the input data set, the number of observations, and the variables. The bandwidth method is the technique used to select the amount of smoothing in the estimate. A simple normal reference rule is used for bivariate smoothing.

The “Controls” table lists the primary numbers controlling the kernel density fit. Here a $60 \times 60$ grid is fit to the entire range of the data, and no adjustment is made to the default bandwidth.

---

### Syntax: KDE Procedure

The following statements are available in the KDE procedure:

```
PROC KDE <options> ;
   BIVAR variable-list </options> ;
   UNIVAR variable-list </options> ;
   SCORE options ;
   BY variables ;
   FREQ variable ;
   WEIGHT variable ;
```

The PROC KDE statement invokes the procedure. The BIVAR statement computes one or more bivariate kernel density estimates on a uniformly spaced grid. The UNIVAR statement computes one or more univariate kernel density estimates on a uniformly spaced grid. The SCORE statement produces kernel density estimates at arbitrary locations for an associated BIVAR or UNIVAR statement. You can specify any number of BIVAR, UNIVAR, and SCORE statements.

The following sections describe the PROC KDE statement and then describe the other statements in alphabetical order.

---

### PROC KDE Statement

```
PROC KDE < DATA=SAS-data-set > ;
```

The PROC KDE statement invokes the procedure and optionally specifies the input data set.

You can specify the following option:

```
DATA=SAS-data-set
```

specifies the input SAS data set to be used by PROC KDE. The default is the most recently created data set.

---

### BIVAR Statement

```
BIVAR v1 <(v-options)> v2 <(v-options)> \ldots < vN <(v-options)> ] ] < /options> ;
BIVAR (v1 v2 ) <(v3 v4 ) \ldots (vN−1 vN ) > < /options> ;
```
The BIVAR statement computes bivariate kernel density estimates for the specified variables. The \textit{v-options} optionally specified in parentheses after a variable name apply only to that variable, and they override corresponding global \textit{options} that are specified following a slash (/).

You must specify at least two variables, \textit{v1} and \textit{v2}. If you specify more than two variables, PROC KDE computes a bivariate kernel density estimate for each distinct pair of variables in the list. For example, if you specify the following statement, then a bivariate kernel density estimate is computed for each of the variable pairs \((x, y), (x, z),\) and \((y, z)\):

\begin{verbatim}
bivar x y z;
\end{verbatim}

Alternatively, you can specify an explicit list of variable pairs, with each pair enclosed in parentheses. This requests a bivariate kernel density for each pair of variables. For example, if you specify the following statement, then bivariate kernel density estimates are computed for \((x, y)\) and \((y, z)\).

\begin{verbatim}
bivar (x y) (y z);
\end{verbatim}

\textbf{NOTE:} The \textit{VAR} statement supported by PROC KDE in SAS 8 and earlier releases is now obsolete. The \textit{VAR} statement has been replaced by the \textit{UNIVAR} and the \textit{BIVAR} statements, which enable you to produce multiple kernel density estimates with a single invocation of the procedure.

Table 67.1 summarizes the \textit{options} available in the \textit{BIVAR} statement.

\begin{table}[h]
\centering
\begin{tabular}{|l|l|}
\hline
\textbf{Option} & \textbf{Description} \\
\hline
BIVSTATS & Produces a table for each density estimate \\
BW= & Specifies the bandwidth \\
BWM= & Specifies the bandwidth multiplier \\
CDF & Produces the distribution function \\
GRIDL= & Specifies the lower grid limit \\
GRIDU= & Specifies the upper grid limit \\
LEVELS & Produces a table of levels for contours of the bivariate density \\
NGRID= & Specifies the number of grid points associated with each variable \\
NOPRINT & Suppresses output tables \\
OUT= & Specifies the name of the output data set \\
PERCENTILES & Produces a table of percentiles \\
PLOTS= & Requests one or more plots \\
TRUNCATE & Restricts the lower and upper grid limits to the minimum and maximum observed values, respectively, for each variable \\
UNISTATS & Produces, for each density estimate, a table that contains standard univariate statistics and the bandwidths \\
\hline
\end{tabular}
\caption{BIVAR Statement Options}
\end{table}

You can specify the following \textit{options} in the \textit{BIVAR} statement. Some \textit{options} can be used as \textit{v-options}, as indicated in the description of the \textit{option}.

\textbf{BIVSTATS}

produces, for each density estimate, a table that contains the covariance and correlation between the two variables.
BW=number

specifies the bandwidth to apply to each variable in each kernel density estimate. Larger bandwidths produce a smoother estimate, whereas smaller bandwidths produce a rougher estimate. To specify different bandwidths for different variables, specify BW=number as a v-option. By default, the bandwidth is set automatically by the simple normal reference method (see the section “Bandwidth Selection” on page 4981).

BWM=number

specifies the bandwidth multiplier to apply to the corresponding bandwidth for each variable. Values of number greater than 1 increase the effective bandwidth and produce a smoother estimate. Values less than 1 decrease the effective bandwidth and produce a rougher estimate. To specify different bandwidth multipliers for different variables, specify BWM=number as a v-option. By default, BWM=1.

CDF

computes the distribution function in addition to the density function for each pair of variables. The distribution function is obtained by a seminumerical technique as described in the section “Kernel Distribution Estimates” on page 4976.

GRIDL=number

specifies the lower grid limit to apply to each variable in each kernel density estimate. To specify different lower grid limits for different variables, specify GRIDL=number as a v-option. The default value for a particular variable is a function of both the kernel bandwidth and the minimum observed value for that variable.

GRIDU=number

specifies the upper grid limit to apply to each variable in each kernel density estimate. To specify different upper grid limits for different variables, specify GRIDU=number as a v-option. The default value for a particular variable is a function of both the kernel bandwidth and the maximum observed value for that variable.

LEVELS
LEVELS=(numlist)

computes a table of levels (called “Levels”) for contours of the bivariate density. The number of contours is equal to the number of values in numlist, where each value in numlist specifies a percentage to be used in calculating the density volume that is enclosed by the contour. The contours are defined such that the density has a constant level along each contour, and the volume enclosed by each contour corresponds to the total density volume minus the specified percentage of the total volume. In other words, the contours correspond to slices or levels of the density surface that are taken along the density axis. The “Levels” table also provides the minimum and maximum values for each contour along the directions of the two data variables. By default, LEVELS=(1, 5, 10, 50, 90, 95, 99, 100).

NGRID=number

NG=number

specifies the number of grid points to be associated with each variable in each kernel density estimate. To specify different numbers of grid points for different variables, specify NGRID=number as a v-option. By default, NGRID=60.
NOPRINT
suppresses output tables. You can use this option when you want to produce only graphical output.

OUT=SAS-data-set
names the output data set in which to save kernel density estimates. This output data set contains the following variables:

- var1, whose value is the name of the first variable in a bivariate kernel density estimate
- var2, whose value is the name of the second variable in a bivariate kernel density estimate
- value1, whose value corresponds to grid coordinates for the first variable
- value2, whose value corresponds to grid coordinates for the second variable
- density, whose values are equal to kernel density estimate at the associated grid point
- count, whose values represent the number of original observations contained in the bin that corresponds to a grid point
- distribution, whose values are equal to the distribution estimate at the associated grid point (this variable is included only when the CDF global option is specified)

PERCENTILES

PERCENTILES=numlist
produces a table of percentiles for each BIVAR variable. You can specify a list of percentiles to be computed in numlist. The default percentiles are 0.5, 1, 2.5, 5, 10, 25, 50, 75, 90, 95, 97.5, 99, and 99.5.

PLOTS=(plot-request<(options)> < ... plot-request <(options)> >)
specifies which plots of the bivariate data and kernel density estimate to produce. When you specify only one plot-request, you can omit the parentheses around the plot-request.

ODS Graphics must be enabled before plots can be requested. For example:

    ods graphics on;
    proc kde data=octane;
       bivar Rater Customer / plots=all;
    run;
    ods graphics off;

For more information about enabling and disabling ODS Graphics, see the section “Enabling and Disabling ODS Graphics” on page 607 in Chapter 21, “Statistical Graphics Using ODS.”

By default, if ODS Graphics is enabled and you do not specify the PLOTS= option, then the BIVAR statement creates a contour plot. If you specify the PLOTS= option, only the requested plots are created.

You can specify the following plot-requests:
ALL
produces all bivariate plots.

CONTOUR
produces a contour plot of the bivariate density estimate.

CONTOURSCATTER
produces a contour plot of the bivariate density estimate overlaid with a scatter plot of the data.

HISTOGRAM < (view-options) >
produces a bivariate histogram of the data. You can specify one or both of the following view-options within parentheses:

\textbf{ROTATE}=angle
rotates the histogram \textit{angle} degrees, where \(-180 < \textit{angle} < 180\). By default, \text{ROTATE}=54.

\textbf{TILT}=angle
tilts the histogram \textit{angle} degrees, where \(-180 < \textit{angle} < 180\). By default, \text{TILT}=20.

HISTSURFACE < (view-options) >
produces a bivariate histogram of the data overlaid with a surface plot of the bivariate kernel density estimate. You can specify one or both of the following view-options within parentheses:

\textbf{ROTATE}=angle
rotates the histogram and kernel density surface \textit{angle} degrees, where \(-180 < \textit{angle} < 180\). By default, \text{ROTATE}=54.

\textbf{TILT}=angle
tilts the histogram and kernel density surface \textit{angle} degrees, where \(-180 < \textit{angle} < 180\). By default, \text{TILT}=20.

NONE
suppresses all plots, including the contour plot that is produced by default when ODS Graphics is enabled and the \text{PLOTS=} option is not specified.

SCATTER
produces a scatter plot of the data.

SURFACE < (view-options) >
produces a surface plot of the bivariate kernel density estimate. You can specify one or both of the following view-options within parentheses:

\textbf{ROTATE}=angle
rotates the kernel density surface \textit{angle} degrees, where \(-180 < \textit{angle} < 180\). By default, \text{ROTATE}=54.

\textbf{TILT}=angle
tilts the kernel density surface \textit{angle} degrees, where \(-180 < \textit{angle} < 180\). By default, \text{TILT}=20.
TRUNCATE
sets the lower grid limit for each variable to the minimum observed for that variable, and sets the upper
grid limit for each variable to the maximum observed value for that variable.

NOTE: The GRIDL and GRIDU options take precedence over the TRUNCATE option. If one or both
are specified, the corresponding lower and upper grid limits are set accordingly.

UNISTATS
produces for each density estimate a table that contains standard univariate statistics for each of the
variable pairs and the bandwidths that are used to compute the kernel density estimate. The statistics
indexed in the table are the mean, variance, standard deviation, range, and interquartile range.

BY Statement

BY variables;
You can specify a BY statement with PROC KDE to obtain separate analyses of observations in groups that
are defined by the BY variables. When a BY statement appears, the procedure expects the input data set to be
sorted in order of the BY variables. If you specify more than one BY statement, only the last one specified is
used.
If your input data set is not sorted in ascending order, use one of the following alternatives:

• Sort the data by using the SORT procedure with a similar BY statement.

• Specify the NOTSORTED or DESCENDING option in the BY statement for the KDE procedure. The
NOTSORTED option does not mean that the data are unsorted but rather that the data are arranged
in groups (according to values of the BY variables) and that these groups are not necessarily in
alphabetical or increasing numeric order.

• Create an index on the BY variables by using the DATASETS procedure (in Base SAS software).

For more information about BY-group processing, see the discussion in SAS Language Reference: Concepts.
For more information about the DATASETS procedure, see the discussion in the Base SAS Procedures Guide.

FREQ Statement

FREQ variable;
The FREQ statement specifies a variable that provides frequencies for each observation in the DATA= data
set. Specifically, if \( n \) is the value of the FREQ variable for a given observation, then that observation is used
\( n \) times. If the value of the FREQ variable is missing or is less than 1, the observation is not used in the
analysis. If the value is not an integer, only the integer portion is used.
SCORE Statement

SCORE DATA=SAS-data-set OUT=SAS-data-set < / option> ;

The SCORE statement produces kernel density estimates at arbitrary locations for an associated UNIVAR or BIVAR statement.

You must specify the following arguments:

**DATA=SAS-data-set**

specifies the input SAS data set.

For a univariate density, the input data set contains the variable \( v_1 \), where \( v_1 \) is a variable that appears in a UNIVAR statement. The values of \( v_1 \) indicate arbitrary points for which a density estimate is requested.

For a bivariate density, the input data set contains variables \( v_1 \) and \( v_2 \), where \( v_1 \) and \( v_2 \) are variables that appear in a BIVAR statement. Pairs \((v_1, v_2)\) indicate arbitrary points for which a density estimate is requested.

When you specify a BY statement, the DATA= data set must not contain any of the BY variables. The entire data set is scored for each BY group.

**OUT=SAS-data-set**

names the output SAS data set to be produced by the SCORE statement.

For a univariate density, the output data set contains the following variables:

- \( \text{var} \), whose value is the name of the variable in the DATA= data set
- \( \text{value} \), whose values are taken from the DATA= data set
- \( \text{density} \), whose values are equal to the kernel density estimate
- \( \text{distribution} \), whose values are equal to the distribution estimate (this variable is included only when the CDF global option is specified in the corresponding UNIVAR statement)

For a bivariate density, the output data set has the following variables:

- \( \text{variable1} \), whose value is the name of the first variable in the DATA= data set
- \( \text{value1} \), whose values are taken from the DATA= data set
- \( \text{variable2} \), whose value is the name of the second variable in the DATA= data set
- \( \text{value2} \), whose values are taken from the DATA= data set
- \( \text{density} \), whose values are equal to the kernel density estimate
- \( \text{distribution} \), whose values are equal to the distribution estimate (this variable is included only when the CDF global option is specified in the corresponding BIVAR statement)

You can provide the following **option** in the SCORE statement.
**METHOD=INTERP | EXACT**

requests a particular scoring method. You can specify the following values:

- **EXACT** corresponds to the evaluation of the appropriate binned density equation from the section “Binning” on page 4975 and the analytical distribution equation from the section “Kernel Distribution Estimates” on page 4976.

- **INTERP** corresponds to linear interpolation of the density estimate and seminumerical distribution estimation. The seminumerical distribution technique is described in “Kernel Distribution Estimates” on page 4976. **NOTE:** Attempting to score outside the grid results in 0 for the density estimate.

By default, METHOD=INTERP.

You can include multiple SCORE statements. Each SCORE statement applies to the first UNIVAR or BIVAR statement that specifies the same variables as are included in the DATA= data set. The order of variables matters for the bivariate case. Therefore, a SCORE statement that specifies a DATA= data set consisting of variables x and y (in that order) will match only a BIVAR statement of one of the following forms:

- **BIVAR (x y)...<options>** ;
- **BIVAR x y ...<options>** ;

**Example**

Suppose the data set MyData contains the variables x and y and the data sets MyScoreInX, MyScoreInY, and MyScoreInXY contain the variables x, y, and (x, y) respectively. The following statements request both the individual (univariate) and joint (bivariate) kernel density estimates and distributions and selectively score them:

```plaintext
proc kde data=MyData;
  univar x y;
  bivar x y / CDF;
  score data=MyScoreInX out=MarginalX;
  score data=MyScoreInY out=MarginalY;
  score data=MyScoreInXY out=JointXY;
run;
```

The first SCORE statement is associated with the UNIVAR statement and produces the MarginalX output data set. This data set contains variables x, value, and density, where density is the density function at value.

The second SCORE statement is associated with the UNIVAR statement and produces the MarginalY output data set. This data set contains variables y, value and density, where density is the density function at value.

The third SCORE statement is associated with the BIVAR statement and produces the JointXY output data set. This data set contains variables x, value1, y, value2, density, and distribution, where density and distribution are the density and distribution functions, respectively, at (value1, value2).

**UNIVAR Statement**

```plaintext
UNIVAR variable <(v-options)> <...variable <(v-options)>> <options> ;
```
The UNIVAR statement computes univariate kernel density estimates. You can specify various *v-options* for each variable by enclosing them in parentheses after the variable name. You can also specify global *options* following a slash (/). Global *options* apply to all the variables specified in the UNIVAR statement. However, individual variable *v-options* override the global *options*.

**NOTE:** The VAR statement supported by PROC KDE in SAS 8 and earlier releases is now obsolete. The VAR statement has been replaced by the UNIVAR and BIVAR statements, which enable you to produce multiple kernel density estimates with a single invocation of the procedure.

Table 67.2 summarizes the *options* available in the UNIVAR statement.

<table>
<thead>
<tr>
<th>Option</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>BW=</td>
<td>Specifies a bandwidth</td>
</tr>
<tr>
<td>BWM=</td>
<td>Specifies a bandwidth multiplier</td>
</tr>
<tr>
<td>CDF</td>
<td>Produces the distribution function</td>
</tr>
<tr>
<td>GRIDL=</td>
<td>Specifies a lower grid limit</td>
</tr>
<tr>
<td>GRIDU=</td>
<td>Specifies an upper grid limit</td>
</tr>
<tr>
<td>LEVELS</td>
<td>Produces a table of levels for contours of the univariate density</td>
</tr>
<tr>
<td>METHOD=</td>
<td>Specifies which method to use to compute the bandwidth</td>
</tr>
<tr>
<td>NGRID=</td>
<td>Specifies a number of grid points</td>
</tr>
<tr>
<td>NOPRINT</td>
<td>Suppresses output tables</td>
</tr>
<tr>
<td>OUT=</td>
<td>Specifies the output SAS data set to contain the kernel density estimate</td>
</tr>
<tr>
<td>PERCENTILES</td>
<td>Produces a table of percentiles</td>
</tr>
<tr>
<td>PLOTS=</td>
<td>Produces plots of the univariate kernel density estimate</td>
</tr>
<tr>
<td>SJPIMAX=</td>
<td>Specifies the maximum grid value in determining the Sheather-Jones plug-in bandwidth</td>
</tr>
<tr>
<td>SJPIMIN=</td>
<td>Specifies the minimum grid value in determining the Sheather-Jones plug-in bandwidth</td>
</tr>
<tr>
<td>SJPINUM=</td>
<td>Specifies the number of grid values to be used in determining the Sheather-Jones plug-in bandwidth</td>
</tr>
<tr>
<td>SJPITOL=</td>
<td>Specifies the tolerance for termination of the bisection algorithm</td>
</tr>
<tr>
<td>TRUNCATE</td>
<td>Restricts the lower and upper grid limits to the minimum and maximum observed values, respectively, for each variable</td>
</tr>
<tr>
<td>UNISTATS</td>
<td>Produces, for each variable, a table that contains standard univariate statistics and the bandwidth</td>
</tr>
</tbody>
</table>

You can specify the following *options* in the UNIVAR statement. Some *options* can be used as *v-options*, as indicated in the description of the *option*.

**BW=** *number*

specifies the bandwidth to apply to each variable in each kernel density estimate. Larger bandwidths produce a smoother estimate, whereas smaller bandwidths produce a rougher estimate. To specify different bandwidths for different variables, specify **BW=** *number* as a *v-option*. By default, the bandwidth is set automatically by the Sheather-Jones plug-in method (see the section “Bandwidth Selection” on page 4981).
BWM=\textit{number}
specifies the bandwidth multiplier to apply to the corresponding bandwidth for each variable. Values of \textit{number} greater than 1 increase the effective bandwidth and produce a smoother estimate. Values less than 1 decrease the effective bandwidth and produce a rougher estimate. To specify different bandwidth multipliers for different variables, specify BWM=\textit{number} as a \textit{v-option}. By default, BWM=1.

CDF
computes the distribution function in addition to the density function for each variable. The distribution function is obtained by a seminumerical technique as described in the section “Kernel Distribution Estimates” on page 4976.

GRIDL=\textit{number}
specifies a lower grid limit for each kernel density estimate. To specify different lower grid limits for different variables, specify GRIDL=\textit{number} as a \textit{v-option}. The default value for a particular variable is a function of both the kernel bandwidth and the minimum observed value for that variable.

GRIDU=\textit{number}
specifies an upper grid limit for each kernel density estimate. To specify different upper grid limits for different variables, specify GRIDU=\textit{number} as a \textit{v-option}. The default value for a particular variable is a function of both the kernel bandwidth and the maximum observed value for that variable.

LEVELS
LEVELS=(\textit{numlist})
computes a table of levels (called “Levels”) for contours of the univariate density. The number of contours is equal to the number of values in \textit{numlist}, where each value in \textit{numlist} specifies a percentage to be used in calculating the density area that is enclosed by the contour. The contours are defined such that the density has a constant level along each contour, and the area enclosed by each contour corresponds to the total density area minus the specified percentage of the total area. In other words, the contours correspond to slices or levels of the density surface that are taken along the density axis. The “Levels” table also provides the minimum and maximum values for each contour along the direction of the data variable. By default, LEVELS=(1, 5, 10, 50, 90, 95, 99, 100).

METHOD=SJPI | SNR | SNRQ | SROT | OS
specifies the method for computing the bandwidth. You can specify the following values:

\begin{itemize}
\item \textbf{SJPI} \hspace{1cm} Sheather-Jones plug-in method
\item \textbf{SNR} \hspace{1cm} simple normal reference method
\item \textbf{SNRQ} \hspace{1cm} simple normal reference method using interquartile range
\item \textbf{SROT} \hspace{1cm} Silverman’s rule of thumb method
\item \textbf{OS} \hspace{1cm} oversmoothed method
\end{itemize}

For a description of these methods, see the section “Bandwidth Selection” on page 4981 and Jones, Marron, and Sheather (1996). By default, METHOD=SJPI.

\textbf{Note}: The BW= option takes precedence over the METHOD= option. If both are specified, the METHOD= option is ignored.
NGRID=number

specifies a number of grid points to use for each kernel density estimate. To specify different numbers of grid points for different variables, specify NGRID=number as a v-option. By default, NGRID=401.

NOPRINT

suppresses output tables. You can use this option when you want to produce only graphical output.

OUT=SAS-data-set

names the output SAS data set to contain the kernel density estimate. This output data set contains the following variables:

- var, whose value is the name of the variable in the kernel density estimate
- value, whose value corresponds to grid coordinates for the variable
- density, whose values are equal to the kernel density estimate at the associated grid point
- count, whose values indicate the number of original observations contained in the bin that corresponds to a grid point
- distribution, whose values are equal to the distribution estimate at the associated grid point (appears only when the CDF global option is specified)

PERCENTILES

PERCENTILES=numlist

produces a table of percentiles for each UNIVAR variable. You can specify a list of percentiles to be computed in numlist. The default percentiles are 0.5, 1, 2.5, 5, 10, 25, 50, 75, 90, 95, 97.5, 99, and 99.5.

PLOTS=(plot-request<(options)>< ... plot-request<(options)>>)

specifies which plots of the univariate kernel density estimate to produce. When you specify only one plot-request, you can omit the parentheses around the plot-request.

ODS Graphics must be enabled before plots can be requested. For example:

```
ods graphics on;

proc kde data=channel;
   univar length / plots=histdensity;
run;

ods graphics off;
```

For more information about enabling and disabling ODS Graphics, see the section “Enabling and Disabling ODS Graphics” on page 607 in Chapter 21, “Statistical Graphics Using ODS.”

You can specify the following plot-requests, each of which (except for DENSITYOVERLAY) produces a separate plot for every variable listed in the UNIVAR statement:

ALL produces all plots.

DENSITY produces the univariate kernel density estimate curve.
Chapter 67: The KDE Procedure

**DENSITYOVERLAY** produces the overlaid univariate kernel density estimate curves. If you specify more than one variable in the UNIVAR statement, PROC KDE overlays the density curves for all the variables on a single plot.

**HISTDENSITY** produces the univariate histogram of data overlaid with kernel density estimate curve.

**HISTOGRAM** produces the univariate histogram of data.

**NONE** suppresses all plots.

By default, if ODS Graphics is enabled and you do not specify the PLOTS= option, then the UNIVAR statement creates a histogram overlaid with a kernel density estimate. If you specify the PLOTS= option, only the requested plots are created.

**SJPIVMAX=number**

specifies the maximum grid value in determining the Sheather-Jones plug-in bandwidth. The default value is two times the oversmoothed estimate.

**SJPIVMIN=number**

specifies the minimum grid value in determining the Sheather-Jones plug-in bandwidth. The default value is the maximum value divided by 18.

**SJPIVNUM=number**

specifies the number of grid values to use in determining the Sheather-Jones plug-in bandwidth. By default, SJPIVNUM=21.

**SJPIVITOL=number**

specifies the tolerance for terminating the bisection algorithm that is used in computing the Sheather-Jones plug-in bandwidth. By default, SJPIVITOL=0.001.

**TRUNCATE**

sets the lower grid limit for each variable to the minimum observed for that variable, and sets the upper grid limit for each variable to the maximum observed value for that variable.

**NOTE:** The GRIDL and GRIDU options take precedence over the TRUNCATE option. If one or both are specified, the corresponding lower and upper grid limits are set accordingly.

**UNISTATS**

produces, for each variable a table that contains standard univariate statistics and the bandwidth that are used to compute its kernel density estimate. The statistics listed are the mean, variance, standard deviation, range, and interquartile range.

**Examples**

Suppose the data set *MyData* contains the variables *x1*, *x2*, *x3*, and *x4*. The following statements request a univariate kernel density estimate for each of these variables:

```plaintext
proc kde data=MyData;
   univar x1 x2 x3 x4;
run;
```

You can also specify different bandwidths and other options for each variable. For example, the following statements request kernel density estimates that use Silverman’s rule of thumb (SROT) method for all variables:

```plaintext
proc kde data=MyData;
   univar x1 x2 x3 x4;
      srot; 
run;
```
WEIGHT Statement

WEIGHT variable;

The WEIGHT statement specifies a variable that weights the observations in computing the kernel density estimate. Observations that have higher weights have more influence in the computations. If an observation has a nonpositive or missing weight, then the entire observation is omitted from the analysis. Be cautious in using data sets that have extreme weights, because they can produce unreliable results.

Details: KDE Procedure

Computational Overview

The two main computational tasks of PROC KDE are automatic bandwidth selection and the construction of a kernel density estimate after a bandwidth has been selected. An optional task is the construction of the kernel distribution estimate. The following sections discuss density and distribution estimation, kernel bandwidth selection, and efficient computation techniques.

Kernel Density Estimates

A weighted univariate kernel density estimate involves a variable \( X \) and a weight variable \( W \). Let \( (X_i, W_i), i = 1, 2, \ldots, n \), denote a sample of \( X \) and \( W \) of size \( n \). The weighted kernel density estimate of \( f(x) \), the density of \( X \), is as follows:

\[
\hat{f}(x) = \frac{1}{\sum_{i=1}^{n} W_i} \sum_{i=1}^{n} W_i \varphi_h(x - X_i)
\]

where \( h \) is the bandwidth and

\[
\varphi_h(x) = \frac{1}{\sqrt{2\pi}h} \exp\left( -\frac{x^2}{2h^2} \right)
\]

is the standard normal density rescaled by the bandwidth. If \( h \to 0 \) and \( nh \to \infty \), then the optimal bandwidth is

\[
h_{\text{AMISE}} = \left[ \frac{1}{2 \sqrt{\pi n } \int (f'(x))^2} \right]^{1/5}
\]
This optimal value is unknown, and so approximations methods are required. For a derivation and discussion of these results, see Silverman (1986, Chapter 3) and Jones, Marron, and Sheather (1996).

For the bivariate case, let \( X = (X, Y) \) be a bivariate random element taking values in \( \mathbb{R}^2 \) with joint density function

\[
f(x, y), \quad (x, y) \in \mathbb{R}^2
\]

and let \( X_i = (X_i, Y_i), \quad i = 1, 2, \ldots, n \), be a sample of size \( n \) drawn from this distribution. The kernel density estimate of \( f(x, y) \) based on this sample is

\[
\hat{f}(x, y) = \frac{1}{n} \sum_{i=1}^{n} \varphi_h(x - X_i, y - Y_i)
\]

where \( (x, y) \in \mathbb{R}^2, \ h_X > 0 \) and \( h_Y > 0 \) are the bandwidths, and \( \varphi_h(x, y) \) is the rescaled normal density

\[
\varphi_h(x, y) = \frac{1}{h_X h_Y} \varphi \left( \frac{x}{h_X}, \frac{y}{h_Y} \right)
\]

where \( \varphi(x, y) \) is the standard normal density function

\[
\varphi(x, y) = \frac{1}{2\pi} \exp \left( -\frac{x^2 + y^2}{2} \right)
\]

Under mild regularity assumptions about \( f(x, y) \), the mean integrated squared error (MISE) of \( \hat{f}(x, y) \) is

\[
\text{MISE}(h_X, h_Y) = \mathbb{E} \left( \int (\hat{f} - f)^2 \right)
\]

\[
= \frac{1}{4\pi n h_X h_Y} + \frac{h_X^4}{4} \int \left( \frac{\partial^2 f}{\partial X^2} \right)^2 \ dx \ dy
\]

\[
+ \frac{h_Y^4}{4} \int \left( \frac{\partial^2 f}{\partial Y^2} \right)^2 \ dx \ dy + O \left( h_X^4 + h_Y^4 + \frac{1}{n h_X h_Y} \right)
\]

as \( h_X \to 0, h_Y \to 0 \) and \( n h_X h_Y \to \infty \).

Now set

\[
\text{AMISE}(h_X, h_Y) = \frac{1}{4\pi n h_X h_Y} + \frac{h_X^4}{4} \int \left( \frac{\partial^2 f}{\partial X^2} \right)^2 \ dx \ dy
\]

\[
+ \frac{h_Y^4}{4} \int \left( \frac{\partial^2 f}{\partial Y^2} \right)^2 \ dx \ dy
\]

which is the asymptotic mean integrated squared error (AMISE). For fixed \( n \), this has a minimum at \( (h_{\text{AMISE,}X}, h_{\text{AMISE,}Y}) \) defined as

\[
h_{\text{AMISE,}X} = \left[ \frac{\int (\frac{\partial^2 f}{\partial X^2})^2}{4n\pi} \right]^{1/6} \left[ \frac{\int (\frac{\partial^2 f}{\partial Y^2})^2}{\int (\frac{\partial^2 f}{\partial X^2})^2} \right]^{2/3}
\]
These are the optimal asymptotic bandwidths in the sense that they minimize MISE. However, as in the univariate case, these expressions contain the second derivatives of the unknown density \( f \) being estimated, and so approximations are required. See Wand and Jones (1993) for further details.

Binning

Binning, or assigning data to discrete categories, is an effective and fast method for large data sets (Fan and Marron 1994). When the sample size \( n \) is large, direct evaluation of the kernel estimate \( \hat{f} \) at any point would involve \( n \) kernel evaluations, as shown in the preceding formulas. To evaluate the estimate at each point of a grid of size \( g \) would thus require \( ng \) kernel evaluations. When you use \( g = 401 \) in the univariate case or \( g = 60 \times 60 = 3600 \) in the bivariate case and \( n \geq 1000 \), the amount of computation can be prohibitively large. With binning, however, the computational order is reduced to \( g \), resulting in a much quicker algorithm that is nearly as accurate as direct evaluation.

To bin a set of weighted univariate data \( X_1, X_2, \ldots, X_n \) to a grid \( x_1, x_2, \ldots, x_g \), simply assign each sample \( X_i \), together with its weight \( W_i \), to the nearest grid point \( x_j \) (also called the bin center). When binning is completed, each grid point \( x_j \) has an associated number \( c_j \), which is the sum total of all the weights that correspond to sample points that have been assigned to \( x_j \). These \( c_j \)'s are known as the bin counts.

This procedure replaces the data \((X_i, W_i), i = 1, 2, \ldots, n\), with the smaller set \((x_j, c_j), i = 1, 2, \ldots, g\), and the estimation is carried out with these new data. This is so-called simple binning, versus the finer linear binning described in Wand (1994). PROC KDE uses simple binning for the sake of faster and easier implementation. Also, it is assumed that the bin centers \( x_1, x_2, \ldots, x_g \) are equally spaced and in increasing order. In addition, assume for notational convenience that \( \sum_{i=1}^{n} W_i = n \) and, therefore, \( \sum_{i=1}^{g} c_i = n \).

If you replace the data \((X_i, W_i), i = 1, 2, \ldots, n\), with \((x_i, c_i), i = 1, 2, \ldots, g\), the weighted estimator \( \hat{f} \) then becomes

\[
\hat{f}(x) = \frac{1}{n} \sum_{i=1}^{g} c_i \varphi_h(x - x_i)
\]

with the same notation as used previously. To evaluate this estimator at the \( g \) points of the same grid vector \( \text{grid} = (x_1, x_2, \ldots, x_g) \) is to calculate

\[
\hat{f}(x_i) = \frac{1}{n} \sum_{j=1}^{g} c_j \varphi_h(x_i - x_j)
\]

for \( i = 1, 2, \ldots, g \). This can be rewritten as

\[
\hat{f}(x_i) = \frac{1}{n} \sum_{j=1}^{g} c_j \varphi_h(|i - j| \delta)
\]

where \( \delta = x_2 - x_1 \) is the increment of the grid.
The same idea of binning works similarly with bivariate data, where you estimate $\hat{f}$ over the grid matrix $\text{grid} = \text{grid}_X \times \text{grid}_Y$ as follows:

$$\text{grid} = \begin{bmatrix}
(x_1, y_1) & (x_1, y_2) & \cdots & (x_1, y_{g_Y}) \\
(x_2, y_1) & (x_2, y_2) & \cdots & (x_2, y_{g_Y}) \\
\vdots & \vdots & \ddots & \vdots \\
(x_{g_X}, y_1) & (x_{g_X}, y_2) & \cdots & (x_{g_X}, y_{g_Y})
\end{bmatrix}$$

The density estimates are then

$$\hat{f}(x_i, y_j) = \frac{1}{n} \sum_{k=1}^{g_X} \sum_{l=1}^{g_Y} c_{k,l} \varphi_h |i - k|\delta_X, |j - l|\delta_Y$$

where $\delta_X = x_2 - x_1$ and $\delta_Y = y_2 - y_1$ are the increments of the grid.

### Kernel Distribution Estimates

The distribution function can be obtained by integrating the kernel density estimate (Azzalini 1981). PROC KDE provides both an analytical and a seminumerical integration approach, each of which involves the closed form solution to the integral of the binned density estimator from the previous section.

For the univariate case, the distribution function is

$$\hat{F}(x) = \int_{-\infty}^{x} \hat{f}(u) du = \frac{1}{N} \sum_{i=1}^{g} c_i \Phi_h (x - x_i)$$

where $h$ is the univariate kernel bandwidth and

$$\Phi_h(x) = \frac{1}{\sqrt{2\pi} h} \int_{-\infty}^{x} \exp \left( -\frac{u^2}{2h^2} \right) du$$

For the bivariate case, the distribution function is

$$\hat{F}(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} \hat{f}(u, v) du dv = \frac{1}{N} \sum_{j=1}^{g_Y} \Phi_{h_Y} (y - y_j) \sum_{i=1}^{g_X} c_{i,j} \Phi_{h_X} (x - x_i)$$

where $h_X$ and $h_Y$ are the bivariate kernel bandwidths for variables $x$ and $y$. 
The analytical integration approach is simply the direct evaluation of the appropriate distribution function equation. The seminumerical integration approach is a mixture of direct evaluation of the distribution function equation and numerical integration via the extended trapezoidal rule (Press et al. 1992). This mixture depends on whether the upper integration limits fall inside or outside the binning grid. In general, there are three cases:

- Integration limits precede leading grid edges.
- Integration limits fall within grid.
- Integration limits follow trailing grid edges.

When the integration limits lead or trail the grid edges, the seminumerical approach is identical to the analytical approach. Otherwise, the seminumerical approach splits the overall integral into integrals from minus infinity to the lower grid edges, and an integral from the leading grid edges to the upper integration limits. For the univariate case, this split becomes

$$
\hat{F}(x) = \int_{-\infty}^{x_1} \hat{f}(u)du + \int_{x_1}^{x} \hat{f}(u)du = \hat{F}(x_1) + \int_{x_1}^{x} \hat{f}(u)du
$$

The term $\hat{F}(x_1)$ is simply the analytical distribution function up to the leading grid edge. The second term is evaluated numerically:

- If $x$ coincides with a grid element $x_k$, the overall integral is

$$
\hat{F}(x) \approx \hat{F}(x_1) + \hat{F}(x_k)
$$

where

$$
\hat{F}(x_k) = \left[ \sum_{m=1}^{k} \hat{f}(x_m) - \frac{1}{2}(\hat{f}(x_1) + \hat{f}(x_k)) \right] \delta
$$

- If $x$ does not coincide with a grid element, then the numerical integral is approximated by linear interpolation that uses the nearest grid elements $p$ and $p+1$:

$$
\hat{F}(x) = \hat{F}(x_p) + \frac{(\hat{F}(x_{p+1}) - \hat{F}(x_p))}{x_{p+1} - x_p}(x - x_p).
$$
The bivariate case is similar to the univariate case, although there are multiple analytical terms to evaluate due to the nature of the integral:

\[
\hat{F}(x, y) = \hat{F}(x, y_1) + \hat{F}(x_1, y) - \hat{F}(x_1, y_1) + \int_{x_1}^{x} \int_{y_1}^{y} \hat{f}(u, v) \, du \, dv
\]

As with the univariate case, the fourth term is evaluated numerically:

- If \((x, y)\) coincides with a grid element, then the overall integral is

\[
\hat{F}(x_k, y_\ell) \approx \hat{F}(x, y_1) + \hat{F}(x_1, y) - \hat{F}(x_1, y_1) + \hat{F}(x_k, y_\ell)
\]

where \(\hat{F}(x_k, y_\ell)\) is recursively computed via

\[
\begin{align*}
\hat{F}(x_k) &= \sum_{n=1}^{\ell} \hat{f}(x_k, y_n) - \frac{1}{2} (\hat{f}(x_k, y_1) + \hat{f}(x_k, y_\ell)) \quad \delta_Y \\
\hat{F}(x_k, y_\ell) &\approx \sum_{m=1}^{k} \hat{F}(x_m) - \frac{1}{2} (\hat{F}(x_1) + \hat{F}(x_k)) \quad \delta_X
\end{align*}
\]

- If \((x, y)\) does not coincide with a grid element, then the numerical integral is approximated by bilinear interpolation that uses the nearest grid elements \(p, p+1, q,\) and \(q+1:\)

\[
\begin{align*}
\hat{F}(y_q) &= \hat{F}(x_p, y_q) + \frac{(\hat{F}(x_{p+1}, y_q) - \hat{F}(x_p, y_q))}{x_{p+1} - x_p} (x - x_p) \\
\hat{F}(y_{q+1}) &= \hat{F}(x_p, y_{q+1}) + \frac{(\hat{F}(x_{p+1}, y_{q+1}) - \hat{F}(x_p, y_{q+1}))}{x_{p+1} - x_p} (x - x_p) \\
\hat{F}(x, y) &= \hat{F}(y_q) + \frac{(\hat{F}(y_{q+1}) - \hat{F}(y_q))}{y_{q+1} - y_q} (y - y_q)
\end{align*}
\]

**Convolutions**

The formulas for the binned estimator \(\hat{f}\) in the previous subsection are in the form of a convolution product between two matrices, one of which contains the bin counts, the other of which contains the rescaled kernels evaluated at multiples of grid increments. This section defines these two matrices explicitly, and shows that \(\hat{f}\) is their convolution.
Beginning with the weighted univariate case, define the following matrices:

\[
K = \frac{1}{n} (\varphi_h(0), \varphi_h(\delta), \ldots, \varphi_h((g-1)\delta))^t \\
C = (c_1, c_2, \ldots, c_g)^t
\]

The first thing to note is that many terms in \( K \) are negligible. The term \( \varphi_h(i\delta) \) is taken to be 0 when \( |i\delta/h| \geq 5 \), so you can define

\[
l = \min(g-1, \text{floor}(5h/\delta))
\]

as the maximum integer multiple of the grid increment to get nonzero evaluations of the rescaled kernel. Here \( \text{floor}(x) \) denotes the largest integer less than or equal to \( x \).

Next, let \( p \) be the smallest power of 2 that is greater than \( g + l + 1 \),

\[
p = 2^{\text{ceil}(\log_2(g+l+1))}
\]

where \( \text{ceil}(x) \) denotes the smallest integer greater than or equal to \( x \).

Modify \( K \) as follows:

\[
K = \frac{1}{n} (\varphi_h(0), \varphi_h(\delta), \ldots, \varphi_h((l\delta), 0, \ldots, 0, \varphi_h(l\delta), \ldots, \varphi_h(\delta))^t
\]

Essentially, the negligible terms of \( K \) are omitted, and the rest are symmetrized (except for one term). The whole matrix is then padded to size \( p \times 1 \) with zeros in the middle. The dimension \( p \) is a highly composite number—that is, one that decomposes into many factors—leading to the most efficient fast Fourier transform operation (see Wand 1994).

The third operation is to pad the bin count matrix \( C \) with zeros to the same size as \( K \):

\[
C = (c_1, c_2, \ldots, c_g, 0, \ldots, 0)^t
\]

The convolution \( K \ast C \) is then a \( p \times 1 \) matrix, and the preceding formulas show that its first \( g \) entries are exactly the estimates \( \hat{f}(x_i) \), \( i = 1, 2, \ldots, g \).

For bivariate smoothing, the matrix \( K \) is defined similarly as

\[
K = \begin{bmatrix}
\kappa_{0,0} & \kappa_{0,1} & \ldots & \kappa_{0,l_Y} & 0 & \kappa_{0,l_Y} & \ldots & \kappa_{0,1} \\
\kappa_{1,0} & \kappa_{1,1} & \ldots & \kappa_{1,l_Y} & 0 & \kappa_{1,l_Y} & \ldots & \kappa_{1,1} \\
\vdots & & & & & & & \\
\kappa_{l_X,0} & \kappa_{l_X,1} & \ldots & \kappa_{l_X,l_Y} & 0 & \kappa_{l_X,l_Y} & \ldots & \kappa_{l_X,1} \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
\kappa_{l_X,0} & \kappa_{l_X,1} & \ldots & \kappa_{l_X,l_Y} & 0 & \kappa_{l_X,l_Y} & \ldots & \kappa_{l_X,1} \\
\vdots & & & & & & & \\
\kappa_{1,0} & \kappa_{1,1} & \ldots & \kappa_{1,l_Y} & 0 & \kappa_{1,l_Y} & \ldots & \kappa_{1,1}
\end{bmatrix}_{p_X \times p_Y}
\]

where \( l_X = \min(g_X - 1, \text{floor}(5h_X/\delta_X)) \), \( p_X = 2^{\text{ceil}(\log_2(g_X+l_X+1))} \), and so forth, and \( \kappa_{i,j} = \frac{1}{n} \varphi_h(i\delta_X, j\delta_Y) i = 0, 1, \ldots, l_X, j = 0, 1, \ldots, l_Y \).
The bin count matrix $C$ is defined as
\[
C = \begin{bmatrix}
c_{1,1} & c_{1,2} & \cdots & c_{1,gY} & 0 & \cdots & 0 \\
c_{2,1} & c_{2,2} & \cdots & c_{2,gY} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
c_{gX,1} & c_{gX,2} & \cdots & c_{gX,gY} & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\end{bmatrix}_{pX \times pY}
\]

As with the univariate case, the $gX \times gY$ upper-left corner of the convolution $K \ast C$ is the matrix of the estimates $\hat{f}(\text{grid})$.

Most of the results in this subsection are found in Wand (1994).

### Fast Fourier Transform

As shown in the last subsection, kernel density estimates can be expressed as a submatrix of a certain convolution. The fast Fourier transform (FFT) is a computationally effective method for computing such convolutions. For a reference on this material, see Press et al. (1988).

The discrete Fourier transform of a complex vector $z = (z_0, \ldots, z_{N-1})$ is the vector $Z = (Z_0, \ldots, Z_{N-1})$, where
\[
Z_j = \sum_{l=0}^{N-1} z_l e^{2\pi i lj/N}, \quad j = 0, \ldots, N - 1
\]
and $i$ is the square root of $-1$. The vector $z$ can be recovered from $Z$ by applying the inverse discrete Fourier transform formula
\[
z_l = N^{-1} \sum_{j=0}^{N-1} Z_j e^{-2\pi i lj/N}, \quad l = 0, \ldots, N - 1
\]

Discrete Fourier transforms and their inverses can be computed quickly using the FFT algorithm, especially when $N$ is highly composite; that is, it can be decomposed into many factors, such as a power of 2. By the discrete convolution theorem, the convolution of two vectors is the inverse Fourier transform of the element-by-element product of their Fourier transforms. This, however, requires certain periodicity assumptions, which explains why the vectors $K$ and $C$ require zero-padding. This is to avoid wrap-around effects (see Press et al. 1988, pp. 410–411). The vector $K$ is actually mirror-imaged so that the convolution of $C$ and $K$ will be the vector of binned estimates. Thus, if $S$ denotes the inverse Fourier transform of the element-by-element product of the Fourier transforms of $K$ and $C$, then the first $g$ elements of $S$ are the estimates.

The bivariate Fourier transform of an $N_1 \times N_2$ complex matrix having $(l_1 + 1, l_2 + 1)$ entry equal to $z_{l_1,l_2}$ is the $N_1 \times N_2$ matrix with $(j_1 + 1, j_2 + 1)$ entry given by
\[
Z_{j_1,j_2} = \sum_{l_1=0}^{N_1-1} \sum_{l_2=0}^{N_2-1} z_{l_1,l_2} e^{2\pi i (l_1 j_1/N_1 + l_2 j_2/N_2)}
\]
and the formula of the inverse is

\[ z_{l_1 l_2} = (N_1 N_2)^{-1} \sum_{j_1=0}^{N_1-1} \sum_{j_2=0}^{N_2-1} Z_{j_1 j_2} e^{-2\pi i (l_1 j_1 / N_1 + l_2 j_2 / N_2)} \]

The same discrete convolution theorem applies, and zero-padding is needed for matrices C and K. In the case of K, the matrix is mirror-imaged twice. Thus, if S denotes the inverse Fourier transform of the element-by-element product of the Fourier transforms of K and C, then the upper-left \( g_X \times g_Y \) corner of S contains the estimates.

### Bandwidth Selection

Several different bandwidth selection methods are available in PROC KDE in the univariate case. Following the recommendations of Jones, Marron, and Sheather (1996), the default method follows a plug-in formula of Sheather and Jones.

This method solves the fixed-point equation

\[
h = \left[ \frac{R(\varphi)}{nR(\hat{f}'_{g(h)})(\int x^2 \varphi(x) \, dx)^2} \right]^{1/5}
\]

where \( R(\varphi) = \int \varphi^2(x) \, dx \).

PROC KDE solves this equation by first evaluating it on a grid of values spaced equally on a log scale. The largest two values from this grid that bound a solution are then used as starting values for a bisection algorithm.

The simple normal reference rule works by assuming \( \hat{f} \) is Gaussian in the preceding fixed-point equation. This results in

\[
h = \hat{\sigma} \left[ \frac{4}{(3n)} \right]^{1/5} = 1.06 \hat{\sigma} n^{-1/5}
\]

where \( \hat{\sigma} \) is the sample standard deviation.

Alternatively, the bandwidth can be computed using the interquartile range, \( Q \):

\[
h = 1.06 \hat{\sigma} n^{-1/5} \\
\approx 1.06 (Q/1.34)n^{-1/5} \\
\approx 0.785 Q n^{-1/5}
\]

Silverman’s rule of thumb (Silverman 1986, Section 3.4.2) is computed as

\[
h = 0.9 \min[\hat{\sigma}, Q/1.34] n^{-1/5}
\]

The oversmoothed bandwidth is computed as

\[
h = 3\hat{\sigma} \left[ \frac{1}{(70\sqrt{n})} \right]^{1/5}
\]
When you specify a WEIGHT variable, PROC KDE uses weighted versions of $Q_3$, $Q_1$, and $\hat{\sigma}$ in the preceding expressions. The weighted quartiles are computed as weighted order statistics, and the weighted variance takes the form

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{n} W_i (X_i - \bar{X})^2}{\sum_{i=1}^{n} W_i}$$

where $\bar{X} = (\sum_{i=1}^{n} W_i X_i) / (\sum_{i=1}^{n} W_i)$ is the weighted sample mean.

For the bivariate case, Wand and Jones (1993) note that automatic bandwidth selection is both difficult and computationally expensive. Their study of various ways of specifying a bandwidth matrix also shows that using two bandwidths, one in each coordinate’s direction, is often adequate. PROC KDE enables you to adjust the two bandwidths by specifying a multiplier for the default bandwidths recommended by Bowman and Foster (1993):

$$h_X = \hat{\sigma}_X n^{-1/6}$$
$$h_Y = \hat{\sigma}_Y n^{-1/6}$$

Here $\hat{\sigma}_X$ and $\hat{\sigma}_Y$ are the sample standard deviations of $X$ and $Y$, respectively. These are the optimal bandwidths for two independent normal variables that have the same variances as $X$ and $Y$. They are, therefore, conservative in the sense that they tend to oversmooth the surface.

You can specify the BWM= option to adjust the aforementioned bandwidths to provide the appropriate amount of smoothing for your application.

**ODS Table Names**

PROC KDE assigns a name to each table it creates. You can use these names to reference the table when using the Output Delivery System (ODS) to select tables and create output data sets. These names are listed in Table 67.3. For more information about ODS, see Chapter 20, “Using the Output Delivery System.”

<table>
<thead>
<tr>
<th>ODS Table Name</th>
<th>Description</th>
<th>Statement</th>
<th>Option</th>
</tr>
</thead>
<tbody>
<tr>
<td>BivariateStatistics</td>
<td>Bivariate statistics</td>
<td>BIVAR</td>
<td>BIVSTATS</td>
</tr>
<tr>
<td>Controls</td>
<td>Control variables</td>
<td>default</td>
<td></td>
</tr>
<tr>
<td>Inputs</td>
<td>Input information</td>
<td>default</td>
<td></td>
</tr>
<tr>
<td>Levels</td>
<td>Levels of density estimate</td>
<td>BIVAR</td>
<td>LEVELS</td>
</tr>
<tr>
<td>Percentiles</td>
<td>Percentiles of data</td>
<td>BIVAR / UNIVAR</td>
<td>PERCENTILES</td>
</tr>
<tr>
<td>UnivariateStatistics</td>
<td>Basic statistics</td>
<td>BIVAR / UNIVAR</td>
<td>UNISTATS</td>
</tr>
</tbody>
</table>
ODS Graphics

Statistical procedures use ODS Graphics to create graphs as part of their output. ODS Graphics is described in detail in Chapter 21, “Statistical Graphics Using ODS.”

Before you create graphs, ODS Graphics must be enabled (for example, by specifying the ODS GRAPHICS ON statement). For more information about enabling and disabling ODS Graphics, see the section “Enabling and Disabling ODS Graphics” on page 607 in Chapter 21, “Statistical Graphics Using ODS.”

The overall appearance of graphs is controlled by ODS styles. Styles and other aspects of using ODS Graphics are discussed in the section “A Primer on ODS Statistical Graphics” on page 606 in Chapter 21, “Statistical Graphics Using ODS.”

ODS Graph Names

PROC KDE assigns a name to each graph it creates using the Output Delivery System (ODS). You can use these names to reference the graphs when using ODS. The names are listed in Table 67.4.

<table>
<thead>
<tr>
<th>ODS Graph Name</th>
<th>Plot Description</th>
<th>Statement</th>
<th>PLOTS= Option</th>
</tr>
</thead>
<tbody>
<tr>
<td>BivariateHistogram</td>
<td>Bivariate histogram of data</td>
<td>BIVAR</td>
<td>HISTOGRAM</td>
</tr>
<tr>
<td>ContourPlot</td>
<td>Contour plot of bivariate kernel density estimate</td>
<td>BIVAR</td>
<td>CONTOUR</td>
</tr>
<tr>
<td>ContourScatterPlot</td>
<td>Contour plot of bivariate kernel density estimate overlaid with scatter plot</td>
<td>BIVAR</td>
<td>CONTOURSCATTER</td>
</tr>
<tr>
<td>DensityPlot</td>
<td>Univariate kernel density estimate curve</td>
<td>UNIVAR</td>
<td>DENSITY</td>
</tr>
<tr>
<td>DensityOverlayPlot</td>
<td>Overlaid univariate kernel density estimate curves</td>
<td>UNIVAR</td>
<td>DENSITYOVERLAY</td>
</tr>
<tr>
<td>HistogramDensity</td>
<td>Univariate histogram overlaid with kernel density estimate curve</td>
<td>UNIVAR</td>
<td>HISTDENSITY</td>
</tr>
<tr>
<td>Histogram</td>
<td>Univariate histogram of data</td>
<td>UNIVAR</td>
<td>HISTOGRAM</td>
</tr>
<tr>
<td>HistogramSurface</td>
<td>Bivariate histogram overlaid with surface plot of bivariate kernel density estimate</td>
<td>BIVAR</td>
<td>HISTSURFACE</td>
</tr>
<tr>
<td>ScatterPlot</td>
<td>Scatter plot of data</td>
<td>BIVAR</td>
<td>SCATTER</td>
</tr>
<tr>
<td>SurfacePlot</td>
<td>Surface plot of bivariate kernel density estimate</td>
<td>BIVAR</td>
<td>SURFACE</td>
</tr>
</tbody>
</table>
Bivariate Plots

You can specify the PLOTS= option in the BIVAR statement to request graphical displays of bivariate kernel density estimates.

By default, if ODS Graphics is enabled and you do not specify the PLOTS= option, then the BIVAR statement creates a contour plot. If you specify the PLOTS= option, only the requested plots are created.

Univariate Plots

You can specify the PLOTS= option in the UNIVAR statement to request graphical displays of univariate kernel density estimates.

By default, if ODS Graphics is enabled and you do not specify the PLOTS= option, then the UNIVAR statement creates a histogram overlaid with a kernel density estimate. If you specify the PLOTS= option, only the requested plots are created.

Binning of Bivariate Histogram

Let \((X_1, Y_1), \ldots, (X_n, Y_n)\) be a sample of size \(n\) drawn from a bivariate distribution. For the marginal distribution of \(X_i, i = 1, 2, \ldots, n\), the number of bins \((\text{Nbins}_X)\) in the bivariate histogram is calculated according to the formula

\[
\text{Nbins}_X = \text{ceil} \left( \frac{\text{range}_X}{\text{width}_X} \right)
\]

where \(\text{ceil}(x)\) denotes the smallest integer greater than or equal to \(x\),

\[
\text{range}_X = \max_{1 \leq i \leq n} (X_i) - \min_{1 \leq i \leq n} (X_i)
\]

and the optimal bin width is obtained, following Scott (1992, p. 84), as

\[
\text{width}_X = 3.504 \hat{\sigma}_X \left(1 - \hat{\rho}^2\right)^{3/8} n^{-1/4}
\]

Here, \(\hat{\sigma}_X\) and \(\hat{\rho}\) are the sample variance and the sample correlation coefficient, respectively. When you specify a WEIGHT variable, PROC KDE uses weighted versions of \(\hat{\sigma}_X\) and \(\hat{\rho}\) in the preceding expressions.

Similar formulas are used to compute the number of bins for the marginal distribution of \(Y_i, i = 1, 2, \ldots, n\). Further details can be found in Scott (1992).

Notice that if \(|\hat{\rho}| > 0.99\), then \(\text{Nbins}_X\) is calculated as in the univariate case (see Terrell and Scott 1985). In this case \(\text{Nbins}_Y = \text{Nbins}_X\).
**Examples: KDE Procedure**

### Example 67.1: Computing a Basic Kernel Density Estimate

This example illustrates the basic functionality of the UNIVAR statement. The effective channel length (in microns) is measured for 1225 field effect transistors. The channel lengths are saved as values of the variable length in a SAS data set named channel; see the file kdex1.sas in the SAS Sample Library. These statements create the channel data set:

```sas
data channel;
  input length @@;
datalines;
0.91 1.01 0.95 1.13 1.12 0.86 0.96 1.17 1.36 1.10
0.98 1.27 1.13 0.92 1.15 1.26 1.14 0.88 1.03 1.00
0.98 0.94 1.09 0.92 1.05 1.05 1.11 1.15
1.11 0.98 0.78 1.09 0.94 1.05 0.89 1.16 0.88 1.19

... more lines ...

2.13 2.05 1.90 2.07 2.15 1.96 2.15 1.89 2.15 2.04
1.95 1.93 2.22 1.74 1.91
;
```

The following statements request a kernel density estimate of the variable length:

```sas
ods graphics on;
proc kde data=channel;
  univar length;
run;
```

Because ODS Graphics is enabled, PROC KDE produces a histogram with an overlaid kernel density estimate by default, although the PLOTS= option is not specified. The resulting graph is shown in Output 67.1.1. For general information about ODS Graphics, see Chapter 21, “Statistical Graphics Using ODS.” For specific information about the graphics available in the KDE procedure, see the section “ODS Graphics” on page 4983.
The default output tables for this analysis are the “Inputs” and “Controls” tables, shown in Output 67.1.2.

### Output 67.1.2 Univariate Inputs Table

<table>
<thead>
<tr>
<th>The KDE Procedure</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Inputs</strong></td>
</tr>
<tr>
<td>Data Set</td>
</tr>
<tr>
<td>Number of Observations Used</td>
</tr>
<tr>
<td>Variable</td>
</tr>
<tr>
<td>Bandwidth Method</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>Controls</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>length</td>
</tr>
<tr>
<td>Grid Points</td>
</tr>
<tr>
<td>Lower Grid Limit</td>
</tr>
<tr>
<td>Upper Grid Limit</td>
</tr>
<tr>
<td>Bandwidth Multiplier</td>
</tr>
</tbody>
</table>
Example 67.2: Changing the Bandwidth

Continuing with Example 67.1, you can specify different bandwidth multipliers that determine the smoothness of the kernel density estimate. The following statements show kernel density estimates for the variable length by specifying two different bandwidth multipliers with the BWM= option:

```sas
proc kde data=channel;
   univar length(bwm=2) length(bwm=0.25);
run;
ods graphics off;
```

Output 67.2.1 shows an oversmoothed estimate because the bandwidth multiplier is 2. Output 67.2.2 is created by specifying BWM=0.25, so it is an undersmoothed estimate.

Output 67.2.1  Histogram with Oversmoothed Kernel Density Estimate
Example 67.3: Changing the Bandwidth (Bivariate)

Recall the analysis from the section “Getting Started: KDE Procedure” on page 4958. Suppose you would like a slightly smoother estimate. You could then rerun the analysis with a larger bandwidth:

```plaintext
ods graphics on;
proc kde data=bivnormal;
  bivar x y / bwm=2;
run;
```

The BWM= option requests bandwidth multipliers of 2 for both x and y. With ODS Graphics enabled, the BIVAR statement produces a contour plot, as shown in Output 67.3.1.
Multiple Bandwidths

You can also specify multiple bandwidths with only one run of the KDE procedure. Notice that by specifying pairs of variables inside parentheses, a kernel density estimate is computed for each pair. In the following statements the first kernel density is computed with the default bandwidth, but the second kernel density specifies a bandwidth multiplier of 0.5 for the variable $x$ and a multiplier of 2 for the variable $y$:

```latex
proc kde data=bivnormal;
   bivar (x y) (x (bwm=0.5) y (bwm=2));
run;
ods graphics off;
```

The contour plot of the second kernel density estimate is shown in Output 67.3.2.
Example 67.4: Requesting Additional Output Tables

This example illustrates how to request output tables with summary statistics in addition to the default output tables. Using the same data as in the section “Getting Started: KDE Procedure” on page 4958, the following statements request univariate and bivariate summary statistics, percentiles, and levels of the kernel density estimate:

```plaintext
proc kde data=bivnormal;
   bivar x y / bivstats levels percentiles unistats;
run;
```

The resulting output is shown in Output 67.4.1.
Output 67.4.1 Bivariate Kernel Density Estimate Tables

**The KDE Procedure**

<table>
<thead>
<tr>
<th>Inputs</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Data Set</td>
<td>WORK.BIVNORMAL</td>
</tr>
<tr>
<td>Number of Observations Used</td>
<td>1000</td>
</tr>
<tr>
<td>Variable 1</td>
<td>x</td>
</tr>
<tr>
<td>Variable 2</td>
<td>y</td>
</tr>
<tr>
<td>Bandwidth Method</td>
<td>Simple Normal Reference</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Controls</th>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>Grid Points</td>
<td>60</td>
<td>60</td>
</tr>
<tr>
<td>Lower Grid Limit</td>
<td>-14.27</td>
<td>-13.18</td>
</tr>
<tr>
<td>Upper Grid Limit</td>
<td>12.17</td>
<td>12.165</td>
</tr>
<tr>
<td>Bandwidth Multiplier</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Univariate Statistics</th>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>-0.075</td>
<td>-0.070</td>
</tr>
<tr>
<td>Variance</td>
<td>9.73</td>
<td>9.93</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>3.12</td>
<td>3.15</td>
</tr>
<tr>
<td>Range</td>
<td>20.39</td>
<td>19.09</td>
</tr>
<tr>
<td>Interquartile Range</td>
<td>4.46</td>
<td>4.51</td>
</tr>
<tr>
<td>Bandwidth</td>
<td>0.99</td>
<td>1.00</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Bivariate Statistics</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Covariance</td>
<td>8.88</td>
</tr>
<tr>
<td>Correlation</td>
<td>0.90</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Percentiles</th>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>-7.71</td>
<td>-8.44</td>
</tr>
<tr>
<td>1.0</td>
<td>-7.08</td>
<td>-7.46</td>
</tr>
<tr>
<td>2.5</td>
<td>-6.17</td>
<td>-6.31</td>
</tr>
<tr>
<td>5.0</td>
<td>-5.28</td>
<td>-5.23</td>
</tr>
<tr>
<td>10.0</td>
<td>-4.18</td>
<td>-4.11</td>
</tr>
<tr>
<td>25.0</td>
<td>-2.24</td>
<td>-2.30</td>
</tr>
<tr>
<td>50.0</td>
<td>-0.11</td>
<td>-0.058</td>
</tr>
<tr>
<td>75.0</td>
<td>2.22</td>
<td>2.21</td>
</tr>
<tr>
<td>90.0</td>
<td>3.81</td>
<td>3.94</td>
</tr>
<tr>
<td>95.0</td>
<td>4.88</td>
<td>5.22</td>
</tr>
<tr>
<td>97.5</td>
<td>6.03</td>
<td>5.94</td>
</tr>
<tr>
<td>99.0</td>
<td>6.90</td>
<td>6.77</td>
</tr>
<tr>
<td>99.5</td>
<td>7.71</td>
<td>7.07</td>
</tr>
</tbody>
</table>
Chapter 67: The KDE Procedure

Output 67.4.1 continued

<table>
<thead>
<tr>
<th>Levels</th>
<th>Density</th>
<th>Lower for x</th>
<th>Upper for x</th>
<th>Lower for y</th>
<th>Upper for y</th>
</tr>
</thead>
<tbody>
<tr>
<td>Percent</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.001034</td>
<td>-8.45</td>
<td>8.58</td>
<td>-8.89</td>
<td>8.30</td>
</tr>
<tr>
<td>5</td>
<td>0.003076</td>
<td>-7.10</td>
<td>7.24</td>
<td>-7.17</td>
<td>7.01</td>
</tr>
<tr>
<td>10</td>
<td>0.004886</td>
<td>-6.21</td>
<td>5.90</td>
<td>-6.31</td>
<td>6.15</td>
</tr>
<tr>
<td>50</td>
<td>0.01586</td>
<td>-3.52</td>
<td>3.65</td>
<td>-3.73</td>
<td>4.00</td>
</tr>
<tr>
<td>90</td>
<td>0.02338</td>
<td>-1.28</td>
<td>1.41</td>
<td>-1.58</td>
<td>0.99</td>
</tr>
<tr>
<td>95</td>
<td>0.02464</td>
<td>-1.28</td>
<td>0.97</td>
<td>-1.15</td>
<td>0.57</td>
</tr>
<tr>
<td>99</td>
<td>0.02547</td>
<td>-0.83</td>
<td>0.069</td>
<td>-0.72</td>
<td>0.14</td>
</tr>
<tr>
<td>100</td>
<td>0.02564</td>
<td>-0.38</td>
<td>-0.38</td>
<td>-0.29</td>
<td>-0.29</td>
</tr>
</tbody>
</table>

The “Univariate Statistics” table contains standard univariate statistics for each variable, as well as statistics associated with the density estimate. Note that the estimated variances for both x and y are fairly close to the true values of 10.

The “Bivariate Statistics” table lists the covariance and correlation between the two variables. Note that the estimated correlation is equal to its true value to two decimal places.

The “Percentiles” table lists percentiles for each variable.

The “Levels” table lists contours of the density corresponding to percentiles of the bivariate data, and the minimum and maximum values of each variable on those contours. For example, 5% of the observed data have a density value less than 0.0030. The minimum x and y values on this contour are −7.10 and −7.14, respectively (the Lower for x and Lower for y columns), and the maximum values are 7.07 and 6.77, respectively (the Upper for x and Upper for y columns).

You can also request “Percentiles” or “Levels” tables with specific percentiles:

```plaintext
proc kde data=bivnormal;
   bivar x y / levels=2.5, 50, 97.5
      percentiles=2.5, 25, 50, 75, 97.5;
run;
```

The resulting “Percentiles” and “Levels” tables are shown in Output 67.4.2.

Output 67.4.2 Customized Percentiles and Levels Tables

The KDE Procedure

<table>
<thead>
<tr>
<th>Percentiles</th>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5</td>
<td>-6.17</td>
<td>-6.31</td>
</tr>
<tr>
<td>25.0</td>
<td>-2.24</td>
<td>-2.30</td>
</tr>
<tr>
<td>50.0</td>
<td>-0.11</td>
<td>-0.058</td>
</tr>
<tr>
<td>75.0</td>
<td>2.22</td>
<td>2.21</td>
</tr>
<tr>
<td>97.5</td>
<td>6.03</td>
<td>5.94</td>
</tr>
</tbody>
</table>
Example 67.5: Univariate KDE Graphics

This example uses data from the section “Getting Started: KDE Procedure” to illustrate the use of ODS Graphics. The following statements request the available univariate plots in PROC KDE:

```plaintext
ods graphics on;
proc kde data=bivnormal;
  univar x / plots=(density histogram histdensity);
  univar x y / plots=densityoverlay;
run;
ods graphics off;
```

Graphs are requested by specifying the PLOTS= option in the UNIVAR statement with ODS Graphics enabled. Output 67.5.1, Output 67.5.2, and Output 67.5.3 show the kernel density estimate, histogram, and histogram with kernel density estimate overlaid, respectively, produced by the first UNIVAR statement.

Output 67.5.1 Kernel Density Estimate

<table>
<thead>
<tr>
<th>Percent</th>
<th>Density</th>
<th>Lower for x</th>
<th>Upper for x</th>
<th>Lower for y</th>
<th>Upper for y</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5</td>
<td>0.001990</td>
<td>-7.55</td>
<td>8.14</td>
<td>-8.03</td>
<td>7.44</td>
</tr>
<tr>
<td>50.0</td>
<td>0.01586</td>
<td>-3.52</td>
<td>3.65</td>
<td>-3.73</td>
<td>4.00</td>
</tr>
<tr>
<td>97.5</td>
<td>0.02504</td>
<td>-0.83</td>
<td>0.52</td>
<td>-0.72</td>
<td>0.57</td>
</tr>
</tbody>
</table>
Output 67.5.2 Histogram

Output 67.5.3 Histogram with Overlaid Kernel Density Estimate
Output 67.5.4 shows the plot produced by the second UNIVAR statement, in which the kernel density estimates for x and y are overlaid.

Output 67.5.4  Overlaid Kernel Density Estimates

For general information about ODS Graphics, see Chapter 21, “Statistical Graphics Using ODS.” For specific information about the graphics available in the KDE procedure, see the section “ODS Graphics” on page 4983.
Example 67.6: Bivariate KDE Graphics

This example illustrates the available bivariate graphics in PROC KDE. The octane data set comes from Rodriguez and Taniguchi (1980), where it is used for predicting customer octane satisfaction by using trained-rater observations. The variables in this data set are Rater and Customer. Either variable might have missing values. See the file kdex3.sas in the SAS Sample Library. The following statements create the octane data set:

```sas
data octane;
  input Rater Customer;
  label Rater = 'Rater'
            Customer = 'Customer';
  datalines;
  94.5 92.0
  94.0 88.0
  94.0 90.0
  93.0 93.0
  ... more lines ...
  88.0 84.0
  .H 90.0
;  
```

The following statements request all the available bivariate plots in PROC KDE:

```sas
ods graphics on;
proc kde data=octane;
  bivar Rater Customer / plots=all;
run;
ods graphics off;
```

Output 67.6.1 shows a scatter plot of the data, Output 67.6.2 shows a bivariate histogram of the data, Output 67.6.3 shows a contour plot of bivariate density estimate, Output 67.6.4 shows a contour plot of bivariate density estimate overlaid with a scatter plot of data, Output 67.6.5 shows a surface plot of bivariate kernel density estimate, and Output 67.6.6 shows a bivariate histogram overlaid with a bivariate kernel density estimate. These graphical displays are requested by specifying the PLOTS= option in the BIVAR statement with ODS Graphics enabled. For general information about ODS Graphics, see Chapter 21, “Statistical Graphics Using ODS.” For specific information about the graphics available in the KDE procedure, see the section “ODS Graphics” on page 4983.
Output 67.6.1 Scatter Plot

Distribution of Rater by Customer

Output 67.6.2 Bivariate Histogram

Distribution of Rater by Customer
Output 67.6.3 Contour Plot

Kernel Density for Rater and Customer

Output 67.6.4 Contour Plot with Overlaid Scatter Plot

Distribution and Kernel Density for Rater and Customer
**Output 67.6.5** Surface Plot

Kernel Density for Rater and Customer

**Output 67.6.6** Bivariate Histogram with Overlaid Surface Plot

Distribution and Kernel Density for Rater and Customer
References


Subject Index

bandwidth
  selection (KDE), 4981

binning
  KDE procedure, 4975

bivariate histogram
  KDE procedure, 4984

computational details
  KDE procedure, 4973

convolution
  KDE procedure, 4978

distribution
  KDE procedure, 4976

fast Fourier transform
  KDE procedure, 4980

KDE procedure
  bandwidth selection, 4981
  binning, 4975
  bivariate histogram, 4984
  computational details, 4973
  convolution, 4978
  distribution, 4976
  examples, 4985
  fast Fourier transform, 4980
  ODS graph names, 4983
  options, 4961
  output table names, 4982

kernel density estimates
  KDE procedure, 4957

ODS graph names
  KDE procedure, 4983

output table names
  KDE procedure, 4982
## Syntax Index

**BIVAR statement**
- KDE procedure, 4961

**BIVSTATS option**
- BIVAR statement, 4962

**BW= option**
- BIVAR statement, 4963
- UNIVAR statement, 4969

**BWM= option**
- BIVAR statement, 4963
- UNIVAR statement, 4970

**BY statement**
- KDE procedure, 4966

**CDF option**
- BIVAR statement, 4963
- UNIVAR statement, 4970

**DATA= option**
- PROC KDE statement, 4961

**FREQ statement**
- KDE procedure, 4966

**GRIDL= option**
- BIVAR statement, 4963
- UNIVAR statement, 4970

**GRIDU= option**
- BIVAR statement, 4963
- UNIVAR statement, 4970

**KDE, 4957**

**KDE procedure, 4957**
- syntax, 4961
- BIVAR statement, 4961

**KDE procedure, BIVAR statement, 4961**
- BIVSTATS option, 4962
- BW= option, 4963
- BWM= option, 4963
- CDF option, 4963
- GRIDL= option, 4963
- GRIDU= option, 4963
- LEVELS= option, 4963
- NGRID= option, 4963
- NOPRINT option, 4964
- OUT= option, 4964
- PERCENTILES option, 4964
- PLOTS= option, 4964
- TRUNCATE option, 4966
- UNISTATS option, 4966
- KDE procedure, BY statement, 4966

**KDE procedure, FREQ statement, 4966**

**KDE procedure, PROC KDE statement, 4961**
- DATA= option, 4961

**KDE procedure, SCORE statement, 4967**
- METHOD= option, 4968

**KDE procedure, UNIVAR statement, 4968**
- BW= option, 4969
- BWM= option, 4970
- CDF option, 4970
- GRIDL= option, 4970
- GRIDU= option, 4970
- LEVELS= option, 4970
- NGRID= option, 4971
- NOPRINT option, 4971
- OUT= option, 4971
- PERCENTILES= option, 4971
- PLOTS= option, 4971
- SJPI MAX= option, 4972
- SJPI MIN= option, 4972
- SJPI NUM= option, 4972
- SJPI TOL= option, 4972
- TRUNCATE option, 4972
- UNISTATS option, 4972

**KDE procedure, WEIGHT statement, 4973**
- LEVELS= option
  - BIVAR statement, 4963
  - UNIVAR statement, 4970

**METHOD= option**
- SCORE statement, 4968

**NGRID= option**
- BIVAR statement, 4963
- UNIVAR statement, 4971

**NOPRINT**
- BIVAR statement, 4964
- UNIVAR statement, 4971

**OUT= option**
- BIVAR statement, 4964
- UNIVAR statement, 4971

**PERCENTILES option**
- BIVAR statement, 4964

**PERCENTILES= option**
- UNIVAR statement, 4971

**PLOTS= option**
- BIVAR statement, 4964
UNIVAR statement, 4971
PROC KDE statement, see KDE procedure

SCORE statement
   KDE procedure, 4967
  SJPI MAX= option
   UNIVAR statement, 4972
  SJPI MIN= option
   UNIVAR statement, 4972
  SJPI NUM= option
   UNIVAR statement, 4972
  SJPI TOL= option
   UNIVAR statement, 4972

TRUNCATE option
   BIVAR statement, 4966
   UNIVAR statement, 4972

UNISTATS option
   BIVAR statement, 4966
   UNIVAR statement, 4972

UNIVAR statement
   KDE procedure, 4968

WEIGHT statement
   KDE procedure, 4973