

SAS/OR® 14.2 User's Guide
Mathematical Programming Examples

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SAS/OR® 14.2 User's Guide: Mathematical Programming Examples

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Introduction

This book contains all 29 examples from the classic book *Model Building in Mathematical Programming* by H. Paul Williams. For each example, the problem statement is first repeated verbatim from Williams (1999) for the first 24 chapters and from Williams (2013) for the remaining chapters.[1] Then the problem is solved using the OPTMODEL procedure in SAS/OR software.

The examples cover linear programming, mixed integer linear programming, and quadratic programming. In most cases, the problem is solved with a single call to one of the mathematical programming solvers available in PROC OPTMODEL. The purpose of this book is to supplement the *SAS/OR User's Guide: Mathematical Programming* with additional examples that demonstrate best practices.

Each chapter contains five sections, described as follows.

- Problem Statement:
  Repeats verbatim the problem description, including any tables and figures, from Williams (1999) or Williams (2013).
- Mathematical Programming Formulation:
  Describes the index sets, parameters, decision variables, objectives, and constraints for one formulation of the problem.
- Input Data:
  Creates the input data sets and macro variables to be used by PROC OPTMODEL.
- PROC OPTMODEL Statements and Output:
  Shows and discusses the PROC OPTMODEL statements that declare sets and parameters, read the input data, formulate the mathematical programming problem, solve the problem, and output the solution. Also shows the output that is created by PROC OPTMODEL and occasionally other SAS procedures.
- Features Demonstrated:
  Lists the important PROC OPTMODEL features demonstrated in this example.

[1] Figures and tables are numbered differently so that they match the chapter organization of this book. To be consistent with the verbatim problem statement, all other sections use British spelling. However, for clarity, large numbers and decimals are punctuated in American style (for example, 10,000 instead of 10 000 and 0.5 instead of 0·5), words are occasionally added or changed (with the changes shown inside square brackets), and punctuation is occasionally changed.

Although PROC OPTMODEL is case-insensitive, in the interest of clarity a few typographical conventions are observed regarding capitalization of names:

OPTMODEL Expression	**Capitalization**
Index set names	All uppercase
Index set member names	All lowercase
Parameter names	All lowercase
Variable names (including implicit variables)	First letter of each word
Objective names	First letter of each word
Constraint names	First letter

The examples shown here are small and not computationally challenging. Throughout, a separation between data and model is maintained so that you can solve larger or more difficult instances without modifying the PROC OPTMODEL statements. A user who learns the techniques demonstrated in these examples will be well-prepared to use PROC OPTMODEL to tackle similar modeling challenges that arise in real-world problems.

Chapter 1
Food Manufacture 1: When to Buy and How to Blend

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Problem Statement

A food is manufactured by refining raw oils and blending them together.[1] The raw oils come in two categories:

vegetable oils	VEG 1
	VEG 2
non-vegetable oils	OIL 1
	OIL 2
	OIL 3

Each oil may be purchased for immediate delivery (January) or bought on the futures market for delivery in a subsequent month. Prices now and in the futures market are given below (in £/ton):

	VEG 1	VEG 2	OIL 1	OIL 2	OIL 3
January	110	120	130	110	115
February	130	130	110	90	115
March	110	140	130	100	95
April	120	110	120	120	125
May	100	120	150	110	105
June	90	100	140	80	135

The final product sells at £150 per ton.

Vegetable oils and non-vegetable oils require different production lines for refining. In any month it is not possible to refine more than 200 tons of vegetable oils and more than 250 tons of non-vegetable oils. There is no loss of weight in the refining process and the cost of refining may be ignored.

[1] Reproduced with permission of John Wiley & Sons Ltd. (Williams 1999, pp. 231–232).

It is possible to store up to 1000 tons of each raw oil for use later. The cost of storage for vegetable and non-vegetable oil is £5 per ton per month. The final product cannot be stored, nor can refined oils be stored.

There is a technological restriction of hardness on the final product. In the units in which hardness is measured this must lie between 3 and 6. It is assumed that hardness blends linearly and that the hardnesses of the raw oils are

VEG 1	8.8
VEG 2	6.1

OIL 1	2.0
OIL 2	4.2
OIL 3	5.0

What buying and manufacturing policy should the company pursue in order to maximize profit?

At present there are 500 tons of each type of raw oil in storage. It is required that these stocks will also exist at the end of June.

Mathematical Programming Formulation

Index Sets and Their Members

The following index sets and their members are used in this example:

- oil $\in$ OILS
- period $\in$ PERIODS
- VEG $\subseteq$ OILS: vegetable oils
- NONVEG $=$ OILS $\setminus$ VEG: non-vegetable oils

Parameters

Table 1.1 shows the parameters that are used in this example.

Table 1.1 Parameters

Parameter Name	Interpretation
cost[oil,period]	Cost of raw oil per period
hardness[oil]	Hardness of raw oil
revenue_per_ton	Revenue per ton of final product
veg_ub	Tons of vegetable oils that can be refined per period
nonveg_ub	Tons of non-vegetable oils that can be refined per period
store_ub	Tons of raw oil that can be stored per period
storage_cost_per_ton	Storage cost per ton of raw oil per period
hardness_lb	Lower bound on hardness of final product
hardness_ub	Upper bound on hardness of final product
init_storage	Initial tons of each type of raw oil in storage
hardness_sol[period]	Hardness of final product per period

Variables

Table 1.2 shows the variables that are used in this example.

Table 1.2 Variables

Variable Name	Interpretation
Buy[oil,period]	Tons of raw oil to buy per period
Use[oil,period]	Tons of raw oil to use per period
Manufacture[period]	Tons of final product to manufacture per period
Store[oil,period]	Tons of raw oil to store as inventory at the end of each period

Objective

The objective is to maximize the following profit function, where Revenue, RawCost, and StorageCost are linear functions of Manufacture, Buy, and Store, respectively:

$$\text{Profit} = \text{Revenue} - \text{RawCost} - \text{StorageCost}$$

Constraints

The following constraints are used in this example:

- bounds on decision variables
- for period ∈ PERIODS,

$$\text{Manufacture[period]} = \sum_{\text{oil} \in \text{OILS}} \text{Use[oil,period]}$$

- for period ∈ PERIODS,

$$\sum_{\text{oil} \in \text{VEG}} \text{Use[oil,period]} \leq veg_ub$$

- for period ∈ PERIODS,

$$\sum_{\text{oil} \in \text{NONVEG}} \text{Use[oil,period]} \leq nonveg_ub$$

- for oil ∈ OILS and period ∈ PERIODS,

$$\text{Store[oil,period} - 1] + \text{Buy[oil,period]} = \text{Use[oil,period]} + \text{Store[oil,period]}$$

- for period ∈ PERIODS,

$$hardness_lb \leq \frac{\sum_{\text{oil} \in \text{OILS}} hardness[oil] \cdot \text{Use[oil,period]}}{\text{Manufacture[period]}} \leq hardness_ub$$

Input Data

The following data sets contain the input data that are used in this example:

```
data cost_data;
   input veg1-veg2 oil1-oil3;
   datalines;
110  120  130  110  115
130  130  110   90  115
110  140  130  100   95
120  110  120  120  125
100  120  150  110  105
 90  100  140   80  135
;

data hardness_data;
   input oil $ hardness;
   datalines;
```

```
veg1 8.8
veg2 6.1
oil1 2.0
oil2 4.2
oil3 5.0
;
```

It is possible to store the other (scalar) parameters in an additional data set that contains one observation, with one data set variable per parameter. But the SAS macro language is used instead, with one macro variable per parameter.

```
%let revenue_per_ton = 150;
%let veg_ub = 200;
%let nonveg_ub = 250;
%let store_ub = 1000;
%let storage_cost_per_ton = 5;
%let hardness_lb = 3;
%let hardness_ub = 6;
%let init_storage = 500;
```

PROC OPTMODEL Statements and Output

The first READ DATA statement populates the OILS index set and reads the one-dimensional hardness data:

```
proc optmodel;
   set <str> OILS;
   num hardness {OILS};
   read data hardness_data into OILS=[oil] hardness;
   print hardness;
```

The PRINT statement results in the first section of output, shown in Figure 1.1.

Figure 1.1 *hardness* Parameter

The OPTMODEL Procedure

[1]	hardness
oil1	2.0
oil2	4.2
oil3	5.0
veg1	8.8
veg2	6.1

The second READ DATA statement populates the PERIODS index set and uses the already-populated OILS index set to loop across data set variables when reading the two-dimensional cost data. The PERIODS index set is numeric and is populated by using the automatic variable _N_ from the cost_data data set, rather than by using the month names.

```
set PERIODS;
num cost {OILS, PERIODS};
read data cost_data into PERIODS=[_N_] {oil in OILS}
   <cost[oil,_N_]=col(oil)>;
print cost;
```

The PRINT statement results in the second section of output, shown in Figure 1.2.

Figure 1.2 *cost* Parameter

	cost					
	1	2	3	4	5	6
oil1	130	110	130	120	150	140
oil2	110	90	100	120	110	80
oil3	115	115	95	125	105	135
veg1	110	130	110	120	100	90
veg2	120	130	140	110	120	100

You can declare implicit variables with the IMPVAR statement, instead of defining explicit variables by using the VAR statement with an additional constraint. When you use the IMPVAR statement, PROC OPTMODEL performs an algebraic substitution, thereby reducing the number of variables and constraints passed to the solver.

```
var Buy {OILS, PERIODS} >= 0;
var Use {OILS, PERIODS} >= 0;
impvar Manufacture {period in PERIODS} = sum {oil in OILS} Use[oil,period];
```

The initial and terminal storage constraints for each raw oil are imposed by using the FIX statement to fix the values of the corresponding Store[oil,period] variables. An alternate approach is to use the CON statement to explicitly declare an equality constraint that contains exactly one variable.

```
num last_period = max {period in PERIODS} period;
var Store {OILS, PERIODS union {0}} >= 0 <= &store_ub;
for {oil in OILS} do;
   fix Store[oil,0]           = &init_storage;
   fix Store[oil,last_period] = &init_storage;
end;
```

The following SET statement uses the SAS function SUBSTR together with the colon operator (:) to select the subset of oils whose name starts with “veg”:

```
set VEG = {oil in OILS: substr(oil,1,3) = 'veg'};
```

The following SET statement uses the DIFF operator to declare the non-vegetable oils to be the oils that do not appear in the set VEG:

```
set NONVEG = OILS diff VEG;
```

The following statements declare implicit variables, the objective, and constraints:

```
impvar Revenue =
   sum {period in PERIODS} &revenue_per_ton * Manufacture[period];
impvar RawCost =
   sum {oil in OILS, period in PERIODS} cost[oil,period] * Buy[oil,period];
impvar StorageCost =
   sum {oil in OILS, period in PERIODS}
      &storage_cost_per_ton * Store[oil,period];
max Profit = Revenue - RawCost - StorageCost;

con Veg_ub_con {period in PERIODS}:
   sum {oil in VEG} Use[oil,period] <= &veg_ub;

con Nonveg_ub_con {period in PERIODS}:
   sum {oil in NONVEG} Use[oil,period] <= &nonveg_ub;

con Flow_balance_con {oil in OILS, period in PERIODS}:
   Store[oil,period-1] + Buy[oil,period]
      = Use[oil,period] + Store[oil,period];
```

As expressed on page 6, the hardness of the final product is a ratio of linear functions of the decision variables. To increase algorithmic performance and reliability, the following two CON statements take advantage of the constant limits on hardness to linearize the nonlinear range constraint by clearing the denominator:

```
con Hardness_ub_con {period in PERIODS}:
   sum {oil in OILS} hardness[oil] * Use[oil,period]
   >= &hardness_lb * Manufacture[period];

con Hardness_lb_con {period in PERIODS}:
   sum {oil in OILS} hardness[oil] * Use[oil,period]
   <= &hardness_ub * Manufacture[period];
```

The following EXPAND statement displays the resulting model with all data populated, as shown in Figure 1.3:

```
expand;
```

This optional statement is useful for debugging purposes, to make sure that the model that PROC OPTMODEL creates is what you intended.

Figure 1.3 Output from EXPAND Statement

```
The OPTMODEL Procedure

Var Buy[veg1,1] >= 0
Var Buy[veg1,2] >= 0
Var Buy[veg1,3] >= 0
Var Buy[veg1,4] >= 0
Var Buy[veg1,5] >= 0
Var Buy[veg1,6] >= 0
Var Buy[veg2,1] >= 0
Var Buy[veg2,2] >= 0
Var Buy[veg2,3] >= 0
Var Buy[veg2,4] >= 0
Var Buy[veg2,5] >= 0
Var Buy[veg2,6] >= 0
Var Buy[oil1,1] >= 0
Var Buy[oil1,2] >= 0
Var Buy[oil1,3] >= 0
Var Buy[oil1,4] >= 0
Var Buy[oil1,5] >= 0
Var Buy[oil1,6] >= 0
Var Buy[oil2,1] >= 0
Var Buy[oil2,2] >= 0
Var Buy[oil2,3] >= 0
Var Buy[oil2,4] >= 0
Var Buy[oil2,5] >= 0
Var Buy[oil2,6] >= 0
Var Buy[oil3,1] >= 0
Var Buy[oil3,2] >= 0
Var Buy[oil3,3] >= 0
Var Buy[oil3,4] >= 0
Var Buy[oil3,5] >= 0
Var Buy[oil3,6] >= 0
Var Use[veg1,1] >= 0
Var Use[veg1,2] >= 0
Var Use[veg1,3] >= 0
Var Use[veg1,4] >= 0
Var Use[veg1,5] >= 0
Var Use[veg1,6] >= 0
Var Use[veg2,1] >= 0
Var Use[veg2,2] >= 0
Var Use[veg2,3] >= 0
Var Use[veg2,4] >= 0
Var Use[veg2,5] >= 0
Var Use[veg2,6] >= 0
Var Use[oil1,1] >= 0
Var Use[oil1,2] >= 0
Var Use[oil1,3] >= 0
Var Use[oil1,4] >= 0
Var Use[oil1,5] >= 0
Var Use[oil1,6] >= 0
Var Use[oil2,1] >= 0
Var Use[oil2,2] >= 0
Var Use[oil2,3] >= 0
Var Use[oil2,4] >= 0
Var Use[oil2,5] >= 0
Var Use[oil2,6] >= 0
```

Figure 1.3 *continued*

```
Var Use[oil3,1] >= 0
Var Use[oil3,2] >= 0
Var Use[oil3,3] >= 0
Var Use[oil3,4] >= 0
Var Use[oil3,5] >= 0
Var Use[oil3,6] >= 0
Var Store[veg1,1] >= 0 <= 1000
Var Store[veg1,2] >= 0 <= 1000
Var Store[veg1,3] >= 0 <= 1000
Var Store[veg1,4] >= 0 <= 1000
Var Store[veg1,5] >= 0 <= 1000
Fix Store[veg1,6] = 500
Fix Store[veg1,0] = 500
Var Store[veg2,1] >= 0 <= 1000
Var Store[veg2,2] >= 0 <= 1000
Var Store[veg2,3] >= 0 <= 1000
Var Store[veg2,4] >= 0 <= 1000
Var Store[veg2,5] >= 0 <= 1000
Fix Store[veg2,6] = 500
Fix Store[veg2,0] = 500
Var Store[oil1,1] >= 0 <= 1000
Var Store[oil1,2] >= 0 <= 1000
Var Store[oil1,3] >= 0 <= 1000
Var Store[oil1,4] >= 0 <= 1000
Var Store[oil1,5] >= 0 <= 1000
Fix Store[oil1,6] = 500
Fix Store[oil1,0] = 500
Var Store[oil2,1] >= 0 <= 1000
Var Store[oil2,2] >= 0 <= 1000
Var Store[oil2,3] >= 0 <= 1000
Var Store[oil2,4] >= 0 <= 1000
Var Store[oil2,5] >= 0 <= 1000
Fix Store[oil2,6] = 500
Fix Store[oil2,0] = 500
Var Store[oil3,1] >= 0 <= 1000
Var Store[oil3,2] >= 0 <= 1000
Var Store[oil3,3] >= 0 <= 1000
Var Store[oil3,4] >= 0 <= 1000
Var Store[oil3,5] >= 0 <= 1000
Fix Store[oil3,6] = 500
Fix Store[oil3,0] = 500
Impvar Manufacture[1] = Use[veg1,1] + Use[veg2,1] + Use[oil1,1] + Use[oil2,1] +
Use[oil3,1]
Impvar Manufacture[2] = Use[veg1,2] + Use[veg2,2] + Use[oil1,2] + Use[oil2,2] +
Use[oil3,2]
Impvar Manufacture[3] = Use[veg1,3] + Use[veg2,3] + Use[oil1,3] + Use[oil2,3] +
Use[oil3,3]
Impvar Manufacture[4] = Use[veg1,4] + Use[veg2,4] + Use[oil1,4] + Use[oil2,4] +
Use[oil3,4]
Impvar Manufacture[5] = Use[veg1,5] + Use[veg2,5] + Use[oil1,5] + Use[oil2,5] +
Use[oil3,5]
Impvar Manufacture[6] = Use[veg1,6] + Use[veg2,6] + Use[oil1,6] + Use[oil2,6] +
Use[oil3,6]
Impvar Revenue = 150*Manufacture[1] + 150*Manufacture[2] + 150*Manufacture[3] +
150*Manufacture[4] + 150*Manufacture[5] + 150*Manufacture[6]
Impvar RawCost = 110*Buy[veg1,1] + 130*Buy[veg1,2] + 110*Buy[veg1,3] + 120*
```

Figure 1.3 *continued*

```
Buy[veg1,4] + 100*Buy[veg1,5] + 90*Buy[veg1,6] + 120*Buy[veg2,1] + 130*
Buy[veg2,2] + 140*Buy[veg2,3] + 110*Buy[veg2,4] + 120*Buy[veg2,5] + 100*
Buy[veg2,6] + 130*Buy[oil1,1] + 110*Buy[oil1,2] + 130*Buy[oil1,3] + 120*
Buy[oil1,4] + 150*Buy[oil1,5] + 140*Buy[oil1,6] + 110*Buy[oil2,1] + 90*
Buy[oil2,2] + 100*Buy[oil2,3] + 120*Buy[oil2,4] + 110*Buy[oil2,5] + 80*
Buy[oil2,6] + 115*Buy[oil3,1] + 115*Buy[oil3,2] + 95*Buy[oil3,3] + 125*
Buy[oil3,4] + 105*Buy[oil3,5] + 135*Buy[oil3,6]
Impvar StorageCost = 5*Store[veg1,1] + 5*Store[veg1,2] + 5*Store[veg1,3] + 5*
Store[veg1,4] + 5*Store[veg1,5] + 5*Store[veg1,6] + 5*Store[veg2,1] + 5*
Store[veg2,2] + 5*Store[veg2,3] + 5*Store[veg2,4] + 5*Store[veg2,5] + 5*
Store[veg2,6] + 5*Store[oil1,1] + 5*Store[oil1,2] + 5*Store[oil1,3] + 5*
Store[oil1,4] + 5*Store[oil1,5] + 5*Store[oil1,6] + 5*Store[oil2,1] + 5*
Store[oil2,2] + 5*Store[oil2,3] + 5*Store[oil2,4] + 5*Store[oil2,5] + 5*
Store[oil2,6] + 5*Store[oil3,1] + 5*Store[oil3,2] + 5*Store[oil3,3] + 5*
Store[oil3,4] + 5*Store[oil3,5] + 5*Store[oil3,6]
Maximize Profit=Revenue - RawCost - StorageCost
Constraint Veg_ub_con[1]: Use[veg1,1] + Use[veg2,1] <= 200
Constraint Veg_ub_con[2]: Use[veg1,2] + Use[veg2,2] <= 200
Constraint Veg_ub_con[3]: Use[veg1,3] + Use[veg2,3] <= 200
Constraint Veg_ub_con[4]: Use[veg1,4] + Use[veg2,4] <= 200
Constraint Veg_ub_con[5]: Use[veg1,5] + Use[veg2,5] <= 200
Constraint Veg_ub_con[6]: Use[veg1,6] + Use[veg2,6] <= 200
Constraint Nonveg_ub_con[1]: Use[oil1,1] + Use[oil2,1] + Use[oil3,1] <= 250
Constraint Nonveg_ub_con[2]: Use[oil1,2] + Use[oil2,2] + Use[oil3,2] <= 250
Constraint Nonveg_ub_con[3]: Use[oil1,3] + Use[oil2,3] + Use[oil3,3] <= 250
Constraint Nonveg_ub_con[4]: Use[oil1,4] + Use[oil2,4] + Use[oil3,4] <= 250
Constraint Nonveg_ub_con[5]: Use[oil1,5] + Use[oil2,5] + Use[oil3,5] <= 250
Constraint Nonveg_ub_con[6]: Use[oil1,6] + Use[oil2,6] + Use[oil3,6] <= 250
Constraint Flow_balance_con[veg1,1]: Store[veg1,0] + Buy[veg1,1] - Use[veg1,1] -
Store[veg1,1] = 0
Constraint Flow_balance_con[veg1,2]: Store[veg1,1] + Buy[veg1,2] - Use[veg1,2] -
Store[veg1,2] = 0
Constraint Flow_balance_con[veg1,3]: Store[veg1,2] + Buy[veg1,3] - Use[veg1,3] -
Store[veg1,3] = 0
Constraint Flow_balance_con[veg1,4]: Store[veg1,3] + Buy[veg1,4] - Use[veg1,4] -
Store[veg1,4] = 0
Constraint Flow_balance_con[veg1,5]: Store[veg1,4] + Buy[veg1,5] - Use[veg1,5] -
Store[veg1,5] = 0
Constraint Flow_balance_con[veg1,6]: Store[veg1,5] + Buy[veg1,6] - Use[veg1,6] -
Store[veg1,6] = 0
Constraint Flow_balance_con[veg2,1]: Store[veg2,0] + Buy[veg2,1] - Use[veg2,1] -
Store[veg2,1] = 0
Constraint Flow_balance_con[veg2,2]: Store[veg2,1] + Buy[veg2,2] - Use[veg2,2] -
Store[veg2,2] = 0
Constraint Flow_balance_con[veg2,3]: Store[veg2,2] + Buy[veg2,3] - Use[veg2,3] -
Store[veg2,3] = 0
Constraint Flow_balance_con[veg2,4]: Store[veg2,3] + Buy[veg2,4] - Use[veg2,4] -
Store[veg2,4] = 0
Constraint Flow_balance_con[veg2,5]: Store[veg2,4] + Buy[veg2,5] - Use[veg2,5] -
Store[veg2,5] = 0
Constraint Flow_balance_con[veg2,6]: Store[veg2,5] + Buy[veg2,6] - Use[veg2,6] -
Store[veg2,6] = 0
Constraint Flow_balance_con[oil1,1]: Store[oil1,0] + Buy[oil1,1] - Use[oil1,1] -
Store[oil1,1] = 0
Constraint Flow_balance_con[oil1,2]: Store[oil1,1] + Buy[oil1,2] - Use[oil1,2] -
Store[oil1,2] = 0
```

Figure 1.3 *continued*

```
Constraint Flow_balance_con[oil1,3]: Store[oil1,2] + Buy[oil1,3] - Use[oil1,3] -
Store[oil1,3] = 0
Constraint Flow_balance_con[oil1,4]: Store[oil1,3] + Buy[oil1,4] - Use[oil1,4] -
Store[oil1,4] = 0
Constraint Flow_balance_con[oil1,5]: Store[oil1,4] + Buy[oil1,5] - Use[oil1,5] -
Store[oil1,5] = 0
Constraint Flow_balance_con[oil1,6]: Store[oil1,5] + Buy[oil1,6] - Use[oil1,6] -
Store[oil1,6] = 0
Constraint Flow_balance_con[oil2,1]: Store[oil2,0] + Buy[oil2,1] - Use[oil2,1] -
Store[oil2,1] = 0
Constraint Flow_balance_con[oil2,2]: Store[oil2,1] + Buy[oil2,2] - Use[oil2,2] -
Store[oil2,2] = 0
Constraint Flow_balance_con[oil2,3]: Store[oil2,2] + Buy[oil2,3] - Use[oil2,3] -
Store[oil2,3] = 0
Constraint Flow_balance_con[oil2,4]: Store[oil2,3] + Buy[oil2,4] - Use[oil2,4] -
Store[oil2,4] = 0
Constraint Flow_balance_con[oil2,5]: Store[oil2,4] + Buy[oil2,5] - Use[oil2,5] -
Store[oil2,5] = 0
Constraint Flow_balance_con[oil2,6]: Store[oil2,5] + Buy[oil2,6] - Use[oil2,6] -
Store[oil2,6] = 0
Constraint Flow_balance_con[oil3,1]: Store[oil3,0] + Buy[oil3,1] - Use[oil3,1] -
Store[oil3,1] = 0
Constraint Flow_balance_con[oil3,2]: Store[oil3,1] + Buy[oil3,2] - Use[oil3,2] -
Store[oil3,2] = 0
Constraint Flow_balance_con[oil3,3]: Store[oil3,2] + Buy[oil3,3] - Use[oil3,3] -
Store[oil3,3] = 0
Constraint Flow_balance_con[oil3,4]: Store[oil3,3] + Buy[oil3,4] - Use[oil3,4] -
Store[oil3,4] = 0
Constraint Flow_balance_con[oil3,5]: Store[oil3,4] + Buy[oil3,5] - Use[oil3,5] -
Store[oil3,5] = 0
Constraint Flow_balance_con[oil3,6]: Store[oil3,5] + Buy[oil3,6] - Use[oil3,6] -
Store[oil3,6] = 0
Constraint Hardness_ub_con[1]: 8.8*Use[veg1,1] + 6.1*Use[veg2,1] + 2*Use[oil1,1]
+ 4.2*Use[oil2,1] + 5*Use[oil3,1] - 3*Manufacture[1] >= 0
Constraint Hardness_ub_con[2]: 8.8*Use[veg1,2] + 6.1*Use[veg2,2] + 2*Use[oil1,2]
+ 4.2*Use[oil2,2] + 5*Use[oil3,2] - 3*Manufacture[2] >= 0
Constraint Hardness_ub_con[3]: 8.8*Use[veg1,3] + 6.1*Use[veg2,3] + 2*Use[oil1,3]
+ 4.2*Use[oil2,3] + 5*Use[oil3,3] - 3*Manufacture[3] >= 0
Constraint Hardness_ub_con[4]: 8.8*Use[veg1,4] + 6.1*Use[veg2,4] + 2*Use[oil1,4]
+ 4.2*Use[oil2,4] + 5*Use[oil3,4] - 3*Manufacture[4] >= 0
Constraint Hardness_ub_con[5]: 8.8*Use[veg1,5] + 6.1*Use[veg2,5] + 2*Use[oil1,5]
+ 4.2*Use[oil2,5] + 5*Use[oil3,5] - 3*Manufacture[5] >= 0
Constraint Hardness_ub_con[6]: 8.8*Use[veg1,6] + 6.1*Use[veg2,6] + 2*Use[oil1,6]
+ 4.2*Use[oil2,6] + 5*Use[oil3,6] - 3*Manufacture[6] >= 0
Constraint Hardness_lb_con[1]: 8.8*Use[veg1,1] + 6.1*Use[veg2,1] + 2*Use[oil1,1]
+ 4.2*Use[oil2,1] + 5*Use[oil3,1] - 6*Manufacture[1] <= 0
Constraint Hardness_lb_con[2]: 8.8*Use[veg1,2] + 6.1*Use[veg2,2] + 2*Use[oil1,2]
+ 4.2*Use[oil2,2] + 5*Use[oil3,2] - 6*Manufacture[2] <= 0
Constraint Hardness_lb_con[3]: 8.8*Use[veg1,3] + 6.1*Use[veg2,3] + 2*Use[oil1,3]
+ 4.2*Use[oil2,3] + 5*Use[oil3,3] - 6*Manufacture[3] <= 0
Constraint Hardness_lb_con[4]: 8.8*Use[veg1,4] + 6.1*Use[veg2,4] + 2*Use[oil1,4]
+ 4.2*Use[oil2,4] + 5*Use[oil3,4] - 6*Manufacture[4] <= 0
Constraint Hardness_lb_con[5]: 8.8*Use[veg1,5] + 6.1*Use[veg2,5] + 2*Use[oil1,5]
+ 4.2*Use[oil2,5] + 5*Use[oil3,5] - 6*Manufacture[5] <= 0
Constraint Hardness_lb_con[6]: 8.8*Use[veg1,6] + 6.1*Use[veg2,6] + 2*Use[oil1,6]
+ 4.2*Use[oil2,6] + 5*Use[oil3,6] - 6*Manufacture[6] <= 0
```

By using the `.sol` suffix, the numeric parameter *hardness_sol* computes hardness of the final product from the optimal decision variable values returned by the solver:

```
num hardness_sol {period in PERIODS} =
   (sum {oil in OILS} hardness[oil] * Use[oil,period].sol)
      / Manufacture[period].sol;
```

You can declare *hardness_sol* even before the solver is called. Because the declaration includes an equals sign, the values are automatically updated each time the right-hand side changes. The following statements call the solver and print the solution:

```
solve;
print Buy Use Store Manufacture hardness_sol;
```

Multiple CREATE DATA statements, with the variables of interest grouped according to their index sets, create multiple output data sets (not shown):

```
create data sol_data1 from [oil period] Buy Use Store;
create data sol_data2 from [period] Manufacture;
```

In this example, all variables are real, the objective function is linear, and all constraints are linear. So PROC OPTMODEL automatically recognizes that this model is a linear programming problem, and the first SOLVE statement calls the default linear programming algorithm, which is the dual simplex algorithm. To invoke a non-default algorithm (such as primal simplex, interior point, or network simplex), you can use the ALGORITHM= option in the SOLVE statement:

```
   solve with lp / algorithm=ps;
   print Buy Use Store Manufacture hardness_sol;
   solve with lp / algorithm=ip;
   print Buy Use Store Manufacture hardness_sol;
   solve with lp / algorithm=ns;
   print Buy Use Store Manufacture hardness_sol;
quit;
```

Each algorithm returns an optimal solution with a profit of £107,843, although the optimal solutions differ from each other, as shown in Figure 1.4 through Figure 1.7.

Figure 1.4 shows the output when you use the (default) dual simplex algorithm.

Figure 1.4 Output from Dual Simplex Algorithm

Problem Summary	
Objective Sense	Maximization
Objective Function	Profit
Objective Type	Linear
Number of Variables	95
Bounded Above	0
Bounded Below	60
Bounded Below and Above	25
Free	0
Fixed	10
Number of Constraints	54
Linear LE (<=)	18
Linear EQ (=)	30
Linear GE (>=)	6
Linear Range	0
Constraint Coefficients	210

Performance Information	
Execution Mode	Single-Machine
Number of Threads	1

Solution Summary	
Solver	LP
Algorithm	Dual Simplex
Objective Function	Profit
Solution Status	Optimal
Objective Value	107842.59259
Primal Infeasibility	5.684342E-13
Dual Infeasibility	1.421085E-14
Bound Infeasibility	1.136868E-13
Iterations	70
Presolve Time	0.00
Solution Time	0.00

Figure 1.4 *continued*

[1]	[2]	Buy	Use	Store
oil1	**0**			500.000
oil1	**1**	0.00	0.000	500.000
oil1	**2**	0.00	0.000	500.000
oil1	**3**	0.00	0.000	500.000
oil1	**4**	0.00	0.000	500.000
oil1	**5**	0.00	0.000	500.000
oil1	**6**	0.00	0.000	500.000
oil2	**0**			500.000
oil2	**1**	0.00	250.000	250.000
oil2	**2**	0.00	0.000	250.000
oil2	**3**	0.00	250.000	0.000
oil2	**4**	0.00	0.000	0.000
oil2	**5**	0.00	0.000	0.000
oil2	**6**	750.00	250.000	500.000
oil3	**0**			500.000
oil3	**1**	0.00	-0.000	500.000
oil3	**2**	0.00	250.000	250.000
oil3	**3**	750.00	0.000	1000.000
oil3	**4**	0.00	250.000	750.000
oil3	**5**	0.00	250.000	500.000
oil3	**6**	0.00	0.000	500.000
veg1	**0**			500.000
veg1	**1**	0.00	159.259	340.741
veg1	**2**	0.00	85.185	255.556
veg1	**3**	0.00	85.185	170.370
veg1	**4**	0.00	85.185	85.185
veg1	**5**	0.00	85.185	0.000
veg1	**6**	659.26	159.259	500.000
veg2	**0**			500.000
veg2	**1**	0.00	40.741	459.259
veg2	**2**	0.00	114.815	344.444
veg2	**3**	0.00	114.815	229.630
veg2	**4**	0.00	114.815	114.815
veg2	**5**	0.00	114.815	0.000
veg2	**6**	540.74	40.741	500.000

[1]	Manufacture	hardness_sol
1	450	6.0000
2	450	6.0000
3	450	5.5556
4	450	6.0000
5	450	6.0000
6	450	6.0000

Figure 1.5 shows the output when you use the ALGORITHM=PS option to invoke the primal simplex algorithm.

Figure 1.5 Output from Primal Simplex Algorithm

Problem Summary	
Objective Sense	Maximization
Objective Function	Profit
Objective Type	Linear
Number of Variables	95
Bounded Above	0
Bounded Below	60
Bounded Below and Above	25
Free	0
Fixed	10
Number of Constraints	54
Linear LE (<=)	18
Linear EQ (=)	30
Linear GE (>=)	6
Linear Range	0
Constraint Coefficients	210

Performance Information	
Execution Mode	Single-Machine
Number of Threads	1

Solution Summary	
Solver	LP
Algorithm	Primal Simplex
Objective Function	Profit
Solution Status	Optimal
Objective Value	107842.59259
Primal Infeasibility	1.136868E-13
Dual Infeasibility	3.552714E-15
Bound Infeasibility	0
Iterations	71
Presolve Time	0.00
Solution Time	0.00

Figure 1.5 *continued*

[1]	[2]	Buy	Use	Store
oil1	**0**			500.000
oil1	**1**	0.00	0.000	500.000
oil1	**2**	0.00	0.000	500.000
oil1	**3**	0.00	0.000	500.000
oil1	**4**	0.00	0.000	500.000
oil1	**5**	0.00	0.000	500.000
oil1	**6**	0.00	0.000	500.000
oil2	**0**			500.000
oil2	**1**	0.00	0.000	500.000
oil2	**2**	0.00	0.000	500.000
oil2	**3**	0.00	0.000	500.000
oil2	**4**	0.00	250.000	250.000
oil2	**5**	0.00	250.000	0.000
oil2	**6**	750.00	250.000	500.000
oil3	**0**			500.000
oil3	**1**	0.00	250.000	250.000
oil3	**2**	0.00	250.000	0.000
oil3	**3**	250.00	250.000	0.000
oil3	**4**	0.00	0.000	0.000
oil3	**5**	500.00	0.000	500.000
oil3	**6**	0.00	0.000	500.000
veg1	**0**			500.000
veg1	**1**	0.00	85.185	414.815
veg1	**2**	0.00	85.185	329.630
veg1	**3**	0.00	85.185	244.444
veg1	**4**	0.00	159.259	85.185
veg1	**5**	0.00	85.185	0.000
veg1	**6**	659.26	159.259	500.000
veg2	**0**			500.000
veg2	**1**	0.00	114.815	385.185
veg2	**2**	0.00	114.815	270.370
veg2	**3**	0.00	114.815	155.556
veg2	**4**	0.00	40.741	114.815
veg2	**5**	0.00	114.815	0.000
veg2	**6**	540.74	40.741	500.000

[1]	Manufacture	hardness_sol
1	450	6.0000
2	450	6.0000
3	450	6.0000
4	450	6.0000
5	450	5.5556
6	450	6.0000

Figure 1.6 shows the output when you use the ALGORITHM=IP option to invoke the interior point algorithm.

Figure 1.6 Output from Interior Point Algorithm

Problem Summary	
Objective Sense	Maximization
Objective Function	Profit
Objective Type	Linear
Number of Variables	95
Bounded Above	0
Bounded Below	60
Bounded Below and Above	25
Free	0
Fixed	10
Number of Constraints	54
Linear LE (<=)	18
Linear EQ (=)	30
Linear GE (>=)	6
Linear Range	0
Constraint Coefficients	210

Performance Information	
Execution Mode	Single-Machine
Number of Threads	4

Solution Summary	
Solver	LP
Algorithm	Interior Point
Objective Function	Profit
Solution Status	Optimal
Objective Value	107842.59259
Primal Infeasibility	9.094947E-13
Dual Infeasibility	2.557954E-13
Bound Infeasibility	0
Duality Gap	0
Complementarity	0
Iterations	8
Iterations2	60
Presolve Time	0.01
Solution Time	0.04

Figure 1.6 *continued*

[1]	[2]	Buy	Use	Store
oil1	**0**			500.00
oil1	**1**	0.00	0.000	500.00
oil1	**2**	0.00	0.000	500.00
oil1	**3**	0.00	0.000	500.00
oil1	**4**	0.00	0.000	500.00
oil1	**5**	0.00	0.000	500.00
oil1	**6**	0.00	0.000	500.00
oil2	**0**			500.00
oil2	**1**	0.00	250.000	250.00
oil2	**2**	750.00	250.000	750.00
oil2	**3**	0.00	250.000	500.00
oil2	**4**	0.00	250.000	250.00
oil2	**5**	0.00	250.000	0.00
oil2	**6**	750.00	250.000	500.00
oil3	**0**			500.00
oil3	**1**	0.00	0.000	500.00
oil3	**2**	0.00	0.000	500.00
oil3	**3**	0.00	0.000	500.00
oil3	**4**	0.00	0.000	500.00
oil3	**5**	0.00	0.000	500.00
oil3	**6**	0.00	0.000	500.00
veg1	**0**			500.00
veg1	**1**	0.00	22.222	477.78
veg1	**2**	0.00	159.259	318.52
veg1	**3**	0.00	159.259	159.26
veg1	**4**	0.00	159.259	0.00
veg1	**5**	0.00	0.000	0.00
veg1	**6**	659.26	159.259	500.00
veg2	**0**			500.00
veg2	**1**	0.00	177.778	322.22
veg2	**2**	0.00	40.741	281.48
veg2	**3**	0.00	40.741	240.74
veg2	**4**	0.00	40.741	200.00
veg2	**5**	0.00	200.000	0.00
veg2	**6**	540.74	40.741	500.00

[1]	Manufacture	hardness_sol
1	450	5.1778
2	450	6.0000
3	450	6.0000
4	450	6.0000
5	450	5.0444
6	450	6.0000

Figure 1.7 shows the output when you use the ALGORITHM=NS option to invoke the network simplex algorithm.

Figure 1.7 Output from Network Simplex Algorithm

Problem Summary	
Objective Sense	Maximization
Objective Function	Profit
Objective Type	Linear
Number of Variables	95
Bounded Above	0
Bounded Below	60
Bounded Below and Above	25
Free	0
Fixed	10
Number of Constraints	54
Linear LE (<=)	18
Linear EQ (=)	30
Linear GE (>=)	6
Linear Range	0
Constraint Coefficients	210

Performance Information	
Execution Mode	Single-Machine
Number of Threads	1

Solution Summary	
Solver	LP
Algorithm	Network Simplex
Objective Function	Profit
Solution Status	Optimal
Objective Value	107842.59259
Primal Infeasibility	5.684342E-13
Dual Infeasibility	2.842171E-14
Bound Infeasibility	2.842171E-14
Iterations	38
Iterations2	42
Presolve Time	0.00
Solution Time	0.00

Figure 1.7 *continued*

[1]	[2]	Buy	Use	Store
oil1	**0**			500.000
oil1	**1**	0.00	0.000	500.000
oil1	**2**	0.00	0.000	500.000
oil1	**3**	0.00	0.000	500.000
oil1	**4**	0.00	0.000	500.000
oil1	**5**	0.00	0.000	500.000
oil1	**6**	0.00	0.000	500.000
oil2	**0**			500.000
oil2	**1**	0.00	0.000	500.000
oil2	**2**	250.00	0.000	750.000
oil2	**3**	0.00	250.000	500.000
oil2	**4**	0.00	250.000	250.000
oil2	**5**	0.00	250.000	0.000
oil2	**6**	750.00	250.000	500.000
oil3	**0**			500.000
oil3	**1**	0.00	250.000	250.000
oil3	**2**	0.00	250.000	0.000
oil3	**3**	0.00	0.000	-0.000
oil3	**4**	0.00	-0.000	0.000
oil3	**5**	500.00	0.000	500.000
oil3	**6**	0.00	0.000	500.000
veg1	**0**			500.000
veg1	**1**	0.00	85.185	414.815
veg1	**2**	0.00	85.185	329.630
veg1	**3**	0.00	159.259	170.370
veg1	**4**	0.00	11.111	159.259
veg1	**5**	-0.00	159.259	0.000
veg1	**6**	659.26	159.259	500.000
veg2	**0**			500.000
veg2	**1**	0.00	114.815	385.185
veg2	**2**	0.00	114.815	270.370
veg2	**3**	0.00	40.741	229.630
veg2	**4**	0.00	188.889	40.741
veg2	**5**	0.00	40.741	0.000
veg2	**6**	540.74	40.741	500.000

[1]	Manufacture	hardness_sol
1	450	6.0000
2	450	6.0000
3	450	6.0000
4	450	5.1111
5	450	6.0000
6	450	6.0000

Features Demonstrated

The following features are demonstrated in this example:

- problem type: linear programming
- numeric and string index sets
- reading dense two-dimensional data
- bounds in the VAR statement
- FIX statement
- IMPVAR statement
- MAX aggregation operator
- SUBSTR function
- using a colon (:) to select members of a set
- set operators DIFF and UNION
- linearizing a ratio constraint
- range constraint
- EXPAND statement
- using a variable suffix (such as `.sol`) in the declaration of a numeric parameter
- multiple input and output data sets
- ALGORITHM= option

Chapter 2
Food Manufacture 2: Limiting the Number of Ingredients and Adding Extra Conditions

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Problem Statement

It is wished to impose the following extra conditions on the food manufacture problem:[1]

(1) The food may never be made up of more than three oils in any month.

(2) If an oil is used in a month at least 20 tons must be used.

(3) If either of VEG 1 or VEG 2 is used in a month then OIL 3 must also be used.

Extend the food manufacture model to encompass these restrictions and find the new optimal solution.

Mathematical Programming Formulation

This formulation builds on the formulation used in Chapter 1. This section includes only the new elements of the formulation.

[1] Reproduced with permission of John Wiley & Sons Ltd. (Williams 1999, p. 232).

Parameters

Table 2.1 shows the additional parameters that are used in this example.

Table 2.1 Parameters

Parameter Name	Interpretation
max_num_oils_used	Maximum number of oils used per period
min_oil_used_threshold	Minimum tons of oil used per period if used in that period
Use[oil,period].ub	Upper bound on Use[oil,period]

Variables

Table 2.2 shows the additional variables that are used in this example.

Table 2.2 Variables

Variable Name	Interpretation
IsUsed[oil,period]	1 if Use[oil,period] is positive; 0 otherwise

Objective

The objective is the same as in Chapter 1.

Constraints

The following additional constraints are used in this example:

- for oil $\in$ OILS and period $\in$ PERIODS,

$$\text{Use[oil,period]} \leq \textit{Use[oil,period].ub} \cdot \text{IsUsed[oil,period]}$$

- for period $\in$ PERIODS,

$$\sum_{\text{oil} \in \text{OILS}} \text{IsUsed[oil,period]} \leq \textit{max_num_oils_used}$$

- for oil $\in$ OILS and period $\in$ PERIODS,

$$\text{Use[oil,period]} \geq \textit{min_oil_used_threshold} \cdot \text{IsUsed[oil,period]}$$

- for oil $\in$ {'veg1', 'veg2'} and period $\in$ PERIODS,

$$\text{IsUsed[oil,period]} \leq \text{IsUsed['oil3',period]}$$

Input Data

The following macro variables contain the additional input data that are used in this example:

```
%let max_num_oils_used = 3;
%let min_oil_used_threshold = 20;
```

PROC OPTMODEL Statements and Output

For completeness, all statements are shown. Statements that are new or changed from Chapter 1 are indicated.

```
proc optmodel;
   set <str> OILS;
   num hardness {OILS};
   read data hardness_data into OILS=[oil] hardness;

   set PERIODS;
   num cost {OILS, PERIODS};
   read data cost_data into PERIODS=[_N_] {oil in OILS}
      <cost[oil,_N_]=col(oil)>;

   var Buy {OILS, PERIODS} >= 0;
   var Use {OILS, PERIODS} >= 0;
   impvar Manufacture {period in PERIODS} = sum {oil in OILS} Use[oil,period];

   num last_period = max {period in PERIODS} period;
   var Store {OILS, PERIODS union {0}} >= 0 <= &store_ub;
   for {oil in OILS} do;
      fix Store[oil,0]           = &init_storage;
      fix Store[oil,last_period] = &init_storage;
   end;

   set VEG = {oil in OILS: substr(oil,1,3) = 'veg'};
   set NONVEG = OILS diff VEG;

   impvar Revenue =
      sum {period in PERIODS} &revenue_per_ton * Manufacture[period];
   impvar RawCost =
      sum {oil in OILS, period in PERIODS} cost[oil,period] * Buy[oil,period];
   impvar StorageCost =
      sum {oil in OILS, period in PERIODS}
         &storage_cost_per_ton * Store[oil,period];
   max Profit = Revenue - RawCost - StorageCost;

   con Veg_ub_con {period in PERIODS}:
      sum {oil in VEG} Use[oil,period] <= &veg_ub;

   con Nonveg_ub_con {period in PERIODS}:
      sum {oil in NONVEG} Use[oil,period] <= &nonveg_ub;
```

```
con Flow_balance_con {oil in OILS, period in PERIODS}:
   Store[oil,period-1] + Buy[oil,period]
      = Use[oil,period] + Store[oil,period];

con Hardness_ub_con {period in PERIODS}:
   sum {oil in OILS} hardness[oil] * Use[oil,period]
   >= &hardness_lb * Manufacture[period];

con Hardness_lb_con {period in PERIODS}:
   sum {oil in OILS} hardness[oil] * Use[oil,period]
   <= &hardness_ub * Manufacture[period];
```

The remaining statements are new in this example. The BINARY option in the following VAR statement declares IsUsed to be a binary variable:

```
var IsUsed {OILS, PERIODS} binary;
```

The `.ub` variable suffix imposes an upper bound on the Use variable, in preparation for the subsequent Link constraint. The validity of this upper bound follows from the Veg_ub_con and Nonveg_ub_con constraints.

```
for {period in PERIODS} do;
   for {oil in VEG}    Use[oil,period].ub = &veg_ub;
   for {oil in NONVEG} Use[oil,period].ub = &nonveg_ub;
end;
```

The following Link constraint enforces the rule that Use[oil,period] > 0 implies that IsUsed[oil,period] $= 1$:

```
con Link {oil in OILS, period in PERIODS}:
   Use[oil,period] <= Use[oil,period].ub * IsUsed[oil,period];
```

The following Logical1, Logical2, and Logical3 constraints correspond directly to the three extra conditions in the problem statement:

```
con Logical1 {period in PERIODS}:
   sum {oil in OILS} IsUsed[oil,period] <= &max_num_oils_used;

con Logical2 {oil in OILS, period in PERIODS}:
   Use[oil,period] >= &min_oil_used_threshold * IsUsed[oil,period];

con Logical3 {oil in {'veg1','veg2'}, period in PERIODS}:
   IsUsed[oil,period] <= IsUsed['oil3',period];

num hardness_sol {period in PERIODS} =
   (sum {oil in OILS} hardness[oil] * Use[oil,period].sol)
      / Manufacture[period].sol;
```

Because PROC OPTMODEL automatically recognizes that this model is a mixed integer linear programming problem, the following SOLVE statement calls the MILP solver, as shown in Figure 2.1:

```
solve;
```

Figure 2.1 Summaries from Mixed Integer Linear Programming Solver

The OPTMODEL Procedure

Problem Summary	
Objective Sense	Maximization
Objective Function	Profit
Objective Type	Linear
Number of Variables	125
Bounded Above	0
Bounded Below	30
Bounded Below and Above	85
Free	0
Fixed	10
Binary	30
Integer	0
Number of Constraints	132
Linear LE (<=)	66
Linear EQ (=)	30
Linear GE (>=)	36
Linear Range	0
Constraint Coefficients	384

Performance Information	
Execution Mode	Single-Machine
Number of Threads	4

Solution Summary	
Solver	MILP
Algorithm	Branch and Cut
Objective Function	Profit
Solution Status	Optimal within Relative Gap
Objective Value	100278.7037
Relative Gap	0.0000191931
Absolute Gap	1.9246939569
Primal Infeasibility	2.202682E-13
Bound Infeasibility	2.067033E-13
Integer Infeasibility	8.881784E-16
Best Bound	100280.6284
Nodes	572
Iterations	5445
Presolve Time	0.03
Solution Time	0.13

The following PRINT statement creates the output shown in Figure 2.2:

```
print Buy Use Store IsUsed Manufacture hardness_sol Logical1.body;
```

The `.body` constraint suffix accesses the left-hand side value of the Logical1 constraint. For each period, the solution uses no more than three oils, as shown in Figure 2.2. The following CREATE DATA statements create multiple output data sets, as in Chapter 1:

```
   create data sol_data1 from [oil period] Buy Use Store IsUsed;
   create data sol_data2 from [period] Manufacture;
quit;
```

Figure 2.2 Output from Mixed Integer Linear Programming Solver

[1]	[2]	Buy	Use	Store	IsUsed
oil1	**0**			500.00	
oil1	**1**	0.00	0.000	500.00	0
oil1	**2**	0.00	0.000	500.00	0
oil1	**3**	0.00	0.000	500.00	0
oil1	**4**	0.00	0.000	500.00	0
oil1	**5**	0.00	0.000	500.00	0
oil1	**6**	0.00	0.000	500.00	0
oil2	**0**			500.00	
oil2	**1**	0.00	0.000	500.00	0
oil2	**2**	0.00	-0.000	500.00	-0
oil2	**3**	-0.00	40.000	460.00	1
oil2	**4**	0.00	230.000	230.00	1
oil2	**5**	0.00	230.000	0.00	1
oil2	**6**	730.00	230.000	500.00	1
oil3	**0**			500.00	
oil3	**1**	0.00	250.000	250.00	1
oil3	**2**	0.00	250.000	-0.00	1
oil3	**3**	770.00	210.000	560.00	1
oil3	**4**	0.00	20.000	540.00	1
oil3	**5**	0.00	20.000	520.00	1
oil3	**6**	0.00	20.000	500.00	1
veg1	**0**			500.00	
veg1	**1**	0.00	85.185	414.81	1
veg1	**2**	0.00	85.185	329.63	1
veg1	**3**	0.00	0.000	329.63	0
veg1	**4**	0.00	155.000	174.63	1
veg1	**5**	0.00	155.000	19.63	1
veg1	**6**	480.37	0.000	500.00	0
veg2	**0**			500.00	
veg2	**1**	0.00	114.815	385.19	1
veg2	**2**	0.00	114.815	270.37	1
veg2	**3**	0.00	200.000	70.37	1
veg2	**4**	0.00	0.000	70.37	0
veg2	**5**	0.00	0.000	70.37	0
veg2	**6**	629.63	200.000	500.00	1

Figure 2.2 *continued*

[1]	Manufacture	hardness_sol	Logical1.BODY
1	450	6.0000	3
2	450	6.0000	3
3	450	5.4178	3
4	405	6.0000	3
5	405	6.0000	3
6	450	5.0800	3

Note that the maximum profit of £100,279 is smaller than in Chapter 1. This result is expected because this model contains additional constraints.

Features Demonstrated

The following features are demonstrated in this example:

- problem type: mixed integer linear programming
- numeric and string index sets
- reading dense two-dimensional data
- bounds in the VAR statement
- FIX statement
- IMPVAR statement
- MAX aggregation operator
- SUBSTR function
- using a colon (:) to select members of a set
- set operators DIFF and UNION
- using a variable suffix (such as `.sol`) in the declaration of a numeric parameter
- multiple input and output data sets
- BINARY option
- `.ub` variable suffix
- `.body` constraint suffix

Chapter 3
Factory Planning 1: What to Make, On What Machines, and When

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Problem Statement

An engineering factory makes seven products (PROD 1 to PROD 7) on the following machines: four grinders, two vertical drills, three horizontal drills, one borer, and one planer.[1] Each product yields a certain contribution to profit (defined as £/unit selling price minus cost of raw materials). These quantities (in £/unit) together with the unit production times (hours) required on each process are given below. A dash indicates that a product does not require a process.

	PROD 1	PROD 2	PROD 3	PROD 4	PROD 5	PROD 6	PROD 7
Contribution to profit	10	6	8	4	11	9	3
Grinding	0.5	0.7	—	—	0.3	0.2	0.5
Vertical drilling	0.1	0.2	—	0.3	—	0.6	—
Horizontal drilling	0.2	—	0.8	—	—	—	0.6
Boring	0.05	0.03	—	0.07	0.1	—	0.08
Planing	—	—	0.01	—	0.05	—	0.05

In the present month (January) and the five subsequent months certain machines will be down for maintenance. These machines will be:

January	1 grinder
February	2 horizontal drills
March	1 borer
April	1 vertical drill
May	1 grinder and 1 vertical drill
June	1 planer and 1 horizontal drill

[1] Reproduced with permission of John Wiley & Sons Ltd. (Williams 1999, pp. 233–234).

There are marketing limitations on each product in each month. These are:

	1	2	3	4	5	6	7
January	500	1000	300	300	800	200	100
February	600	500	200	0	400	300	150
March	300	600	0	0	500	400	100
April	200	300	400	500	200	0	100
May	0	100	500	100	1000	300	0
June	500	500	100	300	1100	500	60

It is possible to store up to 100 of each product at a time at a cost of £0.5 per unit per month. There are no stocks at present but it is desired to have a stock of 50 of each type of product at the end of June.

The factory works a 6 day week with two shifts of 8 hours each day.

No sequencing problems need to be considered.

When and what should the factory make in order to maximize the total profit? Recommend any price increases and the value of acquiring any new machines.

N.B. It may be assumed that each month consists of only 24 working days.

Mathematical Programming Formulation

Index Sets and Their Members

The following index sets and their members are used in this example:

- product ∈ PRODUCTS
- machine_type ∈ MACHINE_TYPES
- period ∈ PERIODS

Parameters

Table 3.1 shows the parameters that are used in this example.

Table 3.1 Parameters

Parameter Name	Interpretation
profit[product]	Profit per unit of product
demand[product,period]	Demand for each product per period
num_machines[machine_type]	Total number of machines for each machine type
num_machines_per_period[machine_type,period]	For each machine type, the number of machines available per period
num_machines_down_per_period[machine_type,period]	For each machine type, the number of machines down per period
production_time[product,machine_type]	Production time per unit of product on each machine type
store_ub	Number of units that can be stored for each product per period
storage_cost_per_unit	Storage cost per unit per period
final_storage	Number of units of each product in storage at the end of the last period
num_hours_per_period	Number of working hours per period (month)

Variables

Table 3.2 shows the variables that are used in this example.

Table 3.2 Variables

Variable Name	Interpretation
Make[product,period]	Number of units of each product to make per period
Sell[product,period]	Number of units of each product to sell per period
Store[product,period]	Number of units of each product to store as inventory at the end of each period

Objective

The objective is to maximize the following function, where StorageCost is a linear function of Store:

$$\text{TotalProfit} = \sum_{\text{product} \in \text{PRODUCTS}} \sum_{\text{period} \in \text{PERIODS}} \textit{profit[product]}\ \text{Sell[product,period]} - \text{StorageCost}$$

Constraints

The following constraints are used in this example:

- bounds on decision variables
- for machine_type ∈ MACHINE_TYPES and period ∈ PERIODS,

$$\sum_{\text{product} \in \text{PRODUCTS}} \textit{production_time[product,machine_type]} \cdot \text{Make[product,period]}$$
$$\leq \textit{num_hours_per_period} \cdot \textit{num_machines_per_period[machine_type,period]}$$

- for product ∈ PRODUCTS and period ∈ PERIODS,

$$\begin{aligned} &(\text{if period} - 1 \in \text{PERIODS, then Store[product,period} - 1\text{]; else } 0) \\ &+ \text{Make[product,period]} \\ &= \text{Sell[product,period]} + \text{Store[product,period]} \end{aligned}$$

Input Data

The following data sets and macro variables contain the input data that are used in this example:

```
data product_data;
   input product $ profit;
   datalines;
prod1 10
prod2  6
prod3  8
prod4  4
prod5 11
prod6  9
prod7  3
;

data demand_data;
   input prod1-prod7;
   datalines;
500 1000 300 300  800 200 100
600  500 200   0  400 300 150
300  600   0   0  500 400 100
200  300 400 500  200   0 100
  0  100 500 100 1000 300   0
500  500 100 300 1100 500  60
;

data machine_type_data;
   input machine_type $ num_machines;
```

```
   datalines;
grinder 4
vdrill  2
hdrill  3
borer   1
planer  1
;

data machine_type_period_data;
   input machine_type $ period num_down;
   datalines;
grinder 1 1
hdrill  2 2
borer   3 1
vdrill  4 1
grinder 5 1
vdrill  5 1
planer  6 1
hdrill  6 1
;

data machine_type_product_data;
   input machine_type $ prod1-prod7;
   datalines;
grinder 0.5  0.7  0    0    0.3  0.2 0.5
vdrill  0.1  0.2  0    0.3  0    0.6 0
hdrill  0.2  0    0.8  0    0    0   0.6
borer   0.05 0.03 0    0.07 0.1  0   0.08
planer  0    0    0.01 0    0.05 0   0.05
;

%let store_ub = 100;
%let storage_cost_per_unit = 0.5;
%let final_storage = 50;
%let num_hours_per_period = 24 * 2 * 8;
```

PROC OPTMODEL Statements and Output

This example uses both one-dimensional and dense two-dimensional data, as in Chapter 1 and Chapter 2:

```
proc optmodel;
   set <str> PRODUCTS;
   num profit {PRODUCTS};
   read data product_data into PRODUCTS=[product] profit;

   set PERIODS;
   num demand {PRODUCTS, PERIODS};
   read data demand_data into PERIODS=[_N_]
      {product in PRODUCTS} <demand[product,_N_]=col(product)>;

   set <str> MACHINE_TYPES;
   num num_machines {MACHINE_TYPES};
   read data machine_type_data into MACHINE_TYPES=[machine_type] num_machines;
```

But this problem also has sparse two-dimensional data: for most (machine_type, period) pairs, the number of machines down is 0. In the following statements, the INIT option in the second NUM statement initializes *num_machines_down_per_period* to 0. The read of the sparse data set machine_type_period_data populates only the nonzero values. The subsequent computation of *num_machines_per_period[machine_type,period]* then uses the initial value of *num_machines_down_per_period[machine_type,period]* when no other value has been supplied:

```
num num_machines_per_period {machine_type in MACHINE_TYPES, PERIODS}
   init num_machines[machine_type];
num num_machines_down_per_period {MACHINE_TYPES, PERIODS} init 0;
read data machine_type_period_data into [machine_type period]
   num_machines_down_per_period=num_down;
for {machine_type in MACHINE_TYPES, period in PERIODS}
   num_machines_per_period[machine_type,period] =
   num_machines_per_period[machine_type,period]
 - num_machines_down_per_period[machine_type,period];
print num_machines_per_period;
```

Figure 3.1 shows the resulting values of *num_machines_per_period*.

Figure 3.1 *num_machines_per_period* Parameter

The OPTMODEL Procedure

num_machines_per_period						
	1	**2**	**3**	**4**	**5**	**6**
borer	1	1	0	1	1	1
grinder	3	4	4	4	3	4
hdrill	3	1	3	3	3	2
planer	1	1	1	1	1	0
vdrill	2	2	2	1	1	2

The following statements declare and read dense two-dimensional data:

```
num production_time {PRODUCTS, MACHINE_TYPES};
read data machine_type_product_data into [machine_type]
   {product in PRODUCTS}
   <production_time[product,machine_type]=col(product)>;
```

The following statements are straightforward:

```
var Make {PRODUCTS, PERIODS} >= 0;
var Sell {product in PRODUCTS, period in PERIODS} >= 0
   <= demand[product,period];

num last_period = max {period in PERIODS} period;
var Store {PRODUCTS, PERIODS} >= 0 <= &store_ub;
for {product in PRODUCTS}
   fix Store[product,last_period] = &final_storage;

impvar StorageCost =
   sum {product in PRODUCTS, period in PERIODS}
      &storage_cost_per_unit * Store[product,period];
max TotalProfit =
   sum {product in PRODUCTS, period in PERIODS}
      profit[product] * Sell[product,period]
 - StorageCost;

con Machine_hours_con {machine_type in MACHINE_TYPES, period in PERIODS}:
   sum {product in PRODUCTS}
      production_time[product,machine_type] * Make[product,period]
<= &num_hours_per_period * num_machines_per_period[machine_type,period];
```

The following Flow_balance_con constraint uses an IF-THEN/ELSE expression to handle the boundary conditions: if a previous period exists, the units in storage at the end of the previous period are available to be sold in the current period. (Because ELSE 0 is the default, you could use just an IF-THEN expression instead.)

```
con Flow_balance_con {product in PRODUCTS, period in PERIODS}:
   (if period - 1 in PERIODS then Store[product,period-1] else 0)
 + Make[product,period]
 = Sell[product,period] + Store[product,period];

solve;
print Make Sell Store;
```

The maximum total profit is £93,715, as shown in Figure 3.2.

Figure 3.2 Output from Linear Programming Solver

Problem Summary	
Objective Sense	Maximization
Objective Function	TotalProfit
Objective Type	Linear
Number of Variables	126
Bounded Above	0
Bounded Below	42
Bounded Below and Above	71
Free	0
Fixed	13
Number of Constraints	72
Linear LE (<=)	30
Linear EQ (=)	42
Linear GE (>=)	0
Linear Range	0
Constraint Coefficients	281

Performance Information	
Execution Mode	Single-Machine
Number of Threads	1

Solution Summary	
Solver	LP
Algorithm	Dual Simplex
Objective Function	TotalProfit
Solution Status	Optimal
Objective Value	93715.178571
Primal Infeasibility	0
Dual Infeasibility	0
Bound Infeasibility	0
Iterations	31
Presolve Time	0.00
Solution Time	0.00

Figure 3.2 *continued*

[1]	[2]	Make	Sell	Store
prod1	1	500.00	500.00	0.0
prod1	2	700.00	600.00	100.0
prod1	3	0.00	100.00	0.0
prod1	4	200.00	200.00	0.0
prod1	5	0.00	0.00	0.0
prod1	6	550.00	500.00	50.0
prod2	1	888.57	888.57	0.0
prod2	2	600.00	500.00	100.0
prod2	3	0.00	100.00	0.0
prod2	4	300.00	300.00	0.0
prod2	5	100.00	100.00	0.0
prod2	6	550.00	500.00	50.0
prod3	1	382.50	300.00	82.5
prod3	2	117.50	200.00	0.0
prod3	3	0.00	0.00	0.0
prod3	4	400.00	400.00	0.0
prod3	5	600.00	500.00	100.0
prod3	6	0.00	50.00	50.0
prod4	1	300.00	300.00	0.0
prod4	2	0.00	0.00	0.0
prod4	3	0.00	0.00	0.0
prod4	4	500.00	500.00	0.0
prod4	5	100.00	100.00	0.0
prod4	6	350.00	300.00	50.0
prod5	1	800.00	800.00	0.0
prod5	2	500.00	400.00	100.0
prod5	3	0.00	100.00	0.0
prod5	4	200.00	200.00	0.0
prod5	5	1100.00	1000.00	100.0
prod5	6	0.00	50.00	50.0
prod6	1	200.00	200.00	0.0
prod6	2	300.00	300.00	0.0
prod6	3	400.00	400.00	0.0
prod6	4	0.00	0.00	0.0
prod6	5	300.00	300.00	0.0
prod6	6	550.00	500.00	50.0
prod7	1	0.00	0.00	0.0
prod7	2	250.00	150.00	100.0
prod7	3	0.00	100.00	0.0
prod7	4	100.00	100.00	0.0
prod7	5	100.00	0.00	100.0
prod7	6	0.00	50.00	50.0

You can use the `.dual` constraint suffix to access the optimal dual variables returned by the solver:

```
   print Machine_hours_con.dual;
   create data sol_data1 from [product period] Make Sell Store;
quit;
```

These values, shown in Figure 3.3, suggest the change in optimal objective value if the factory acquires an additional machine in that period. For this test instance, it turns out that the positive dual variables all correspond to machines that are down.

Figure 3.3 Optimal Dual Variables

	Machine_hours_con.DUAL					
	1	2	3	4	5	6
borer	0.0000	0.0000	200.0000	0.0000	0.0000	0.0000
grinder	8.5714	0.0000	0.0000	0.0000	0.0000	0.0000
hdrill	0.0000	0.6250	0.0000	0.0000	0.0000	0.0000
planer	0.0000	0.0000	0.0000	0.0000	0.0000	800.0000
vdrill	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Features Demonstrated

The following features are demonstrated in this example:

- problem type: linear programming
- numeric and string index sets
- reading dense two-dimensional data
- reading sparse two-dimensional data
- INIT option
- bounds in the VAR statement
- FIX statement
- IMPVAR statement
- MAX aggregation operator
- IF-THEN/ELSE expression
- `.dual` constraint suffix

Chapter 4
Factory Planning 2: When Should Machines Be Down for Maintenance

Contents

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Problem Statement

Instead of stipulating when each machine is down for maintenance in the factory planning problem, it is desired to find the best month for each machine to be down.[1]

Each machine must be down for maintenance in one month of the six apart from the grinding machines, only two of which need be down in any six months.

Extend the model to allow it to make these extra decisions. How much is the extra flexibility of allowing down times to be chosen worth?

Mathematical Programming Formulation

This formulation builds on the formulation used in Chapter 3. This section includes only the new elements of the formulation.

[1] Reproduced with permission of John Wiley & Sons Ltd. (Williams 1999, p. 234).

Parameters

Table 4.1 shows the additional parameters that are used in this example.

Table 4.1 Parameters

Parameter Name	Interpretation
num_machines_needing_maintenance[machine_type]	For each machine type, the number of machines that need maintenance

Variables

Table 4.2 shows the additional variables that are used in this example.

Table 4.2 Variables

Variable Name	Interpretation
NumMachinesDown[machine_type,period]	For each machine type, the number of machines down per period

Objective

The objective is the same as in Chapter 3.

Constraints

The following additional constraints are used in this example:

- for machine_type $\in$ MACHINE_TYPES and period $\in$ PERIODS,

$$\sum_{\text{product} \in \text{PRODUCTS}} \textit{production_time[product,machine_type]} \cdot \text{Make[product,period]}$$
$$\leq \textit{num_hours_per_period} \cdot (\textit{num_machines[machine_type]} - \text{NumMachinesDown[machine_type,period]})$$

- for machine_type $\in$ MACHINE_TYPES,

$$\sum_{\text{period} \in \text{PERIODS}} \text{NumMachinesDown[machine_type,period]} = \textit{num_machines_needing_maintenance[machine_type]}$$

Input Data

Ignore machine_type_period_data from Chapter 3, and replace machine_type_data as follows:

```
data machine_type_data;
   input machine_type $ num_machines num_machines_needing_maintenance;
   datalines;
grinder 4 2
vdrill  2 2
hdrill  3 3
borer   1 1
planer  1 1
;
```

PROC OPTMODEL Statements and Output

For completeness, all statements are shown. Statements that are new or changed from Chapter 3 are indicated.

```
proc optmodel;
   set <str> PRODUCTS;
   num profit {PRODUCTS};
   read data product_data into PRODUCTS=[product] profit;

   set PERIODS;
   num demand {PRODUCTS, PERIODS};
   read data demand_data into PERIODS=[_N_]
      {product in PRODUCTS} <demand[product,_N_]=col(product)>;

   set <str> MACHINE_TYPES;
   num num_machines {MACHINE_TYPES};
```

The following statements declare and populate the *num_machines_needing_maintenance* parameter:

```
   num num_machines_needing_maintenance {MACHINE_TYPES};
   read data machine_type_data into MACHINE_TYPES=[machine_type]
      num_machines num_machines_needing_maintenance;
```

The following statements are the same as in Chapter 3:

```
   num production_time {PRODUCTS, MACHINE_TYPES};
   read data machine_type_product_data into [machine_type]
      {product in PRODUCTS}
      <production_time[product,machine_type]=col(product)>;

   var Make {PRODUCTS, PERIODS} >= 0;
   var Sell {product in PRODUCTS, period in PERIODS} >= 0
      <= demand[product,period];

   num last_period = max {period in PERIODS} period;
   var Store {PRODUCTS, PERIODS} >= 0 <= &store_ub;
```

```
for {product in PRODUCTS}
   fix Store[product,last_period] = &final_storage;

impvar StorageCost =
   sum {product in PRODUCTS, period in PERIODS}
      &storage_cost_per_unit * Store[product,period];
max TotalProfit =
   sum {product in PRODUCTS, period in PERIODS}
      profit[product] * Sell[product,period]
 - StorageCost;
```

Most of the remaining statements are new or modified from Chapter 3. The INTEGER option in the following VAR statement declares NumMachinesDown to be an integer variable:

```
var NumMachinesDown {MACHINE_TYPES, PERIODS} >= 0 integer;

con Machine_hours_con {machine_type in MACHINE_TYPES, period in PERIODS}:
   sum {product in PRODUCTS}
      production_time[product,machine_type] * Make[product,period]
<= &num_hours_per_period *
   (num_machines[machine_type] - NumMachinesDown[machine_type,period]);

con Maintenance_con {machine_type in MACHINE_TYPES}:
   sum {period in PERIODS} NumMachinesDown[machine_type,period]
 = num_machines_needing_maintenance[machine_type];

con Flow_balance_con {product in PRODUCTS, period in PERIODS}:
   (if period - 1 in PERIODS then Store[product,period-1] else 0)
 + Make[product,period]
 = Sell[product,period] + Store[product,period];
```

Because the problem contains integer variables, the SOLVE statement automatically invokes the MILP solver:

```
solve;
print Make best4. Sell best4. Store best4.;
print NumMachinesDown best4.;
create data sol_data1 from [product period] Make Sell Store;
create data sol_data2 from [machine_type period] NumMachinesDown;
quit;
```

The solver determines when machines should be down and obtains a total profit of £108,855, as shown in Figure 4.1. This objective value represents an increase of £15,140 from the optimal objective in Chapter 3.

Figure 4.1 Output from Mixed Integer Linear Programming Solver

The OPTMODEL Procedure

Problem Summary	
Objective Sense	Maximization
Objective Function	TotalProfit
Objective Type	Linear
Number of Variables	156
Bounded Above	0
Bounded Below	72
Bounded Below and Above	71
Free	0
Fixed	13
Binary	0
Integer	30
Number of Constraints	77
Linear LE (<=)	30
Linear EQ (=)	47
Linear GE (>=)	0
Linear Range	0
Constraint Coefficients	341

Performance Information	
Execution Mode	Single-Machine
Number of Threads	4

Solution Summary	
Solver	MILP
Algorithm	Branch and Cut
Objective Function	TotalProfit
Solution Status	Optimal
Objective Value	108855.00605
Relative Gap	0
Absolute Gap	0
Primal Infeasibility	1.719513E-12
Bound Infeasibility	4.320339E-16
Integer Infeasibility	9.2010167E-6
Best Bound	108855.00605
Nodes	1
Iterations	315
Presolve Time	0.01
Solution Time	0.04

Figure 4.1 *continued*

[1]	[2]	Make	Sell	Store
prod1	**1**	500	500	0
prod1	**2**	600	600	0
prod1	**3**	400	300	100
prod1	**4**	0	100	0
prod1	**5**	0	0	0
prod1	**6**	550	500	50
prod2	**1**	1000	1000	0
prod2	**2**	500	500	0
prod2	**3**	700	600	100
prod2	**4**	0	100	0
prod2	**5**	100	100	0
prod2	**6**	550	500	50
prod3	**1**	300	300	0
prod3	**2**	200	200	0
prod3	**3**	100	0	100
prod3	**4**	0	100	0
prod3	**5**	500	500	0
prod3	**6**	150	100	50
prod4	**1**	300	300	0
prod4	**2**	0	0	0
prod4	**3**	100	0	100
prod4	**4**	0	100	0
prod4	**5**	100	100	0
prod4	**6**	350	300	50
prod5	**1**	800	800	0
prod5	**2**	400	400	0
prod5	**3**	600	500	100
prod5	**4**	0	100	0
prod5	**5**	1000	1000	0
prod5	**6**	1150	1100	50
prod6	**1**	200	200	0
prod6	**2**	300	300	0
prod6	**3**	400	400	0
prod6	**4**	0	0	0
prod6	**5**	300	300	0
prod6	**6**	550	500	50
prod7	**1**	100	100	0
prod7	**2**	150	150	0
prod7	**3**	200	100	100
prod7	**4**	0	100	0
prod7	**5**	0	0	0
prod7	**6**	110	60	50

Figure 4.1 *continued*

	NumMachinesDown					
	1	**2**	**3**	**4**	**5**	**6**
borer	0	0	0	1	0	0
grinder	0	0	0	0	2	0
hdrill	1	2	0	0	0	0
planer	0	0	0	1	0	0
vdrill	0	0	0	2	0	0

As expected, the optimal numbers of machines down differ from the *num_machines_down_per_period* parameter values in Chapter 3.

Features Demonstrated

The following features are demonstrated in this example:

- problem type: mixed integer linear programming
- numeric and string index sets
- reading dense two-dimensional data
- bounds in the VAR statement
- FIX statement
- IMPVAR statement
- MAX aggregation operator
- IF-THEN/ELSE expression
- multiple input and output data sets
- INTEGER option

Chapter 5
Manpower Planning: How to Recruit, Retrain, Make Redundant, or Overman

Contents

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Problem Statement

A company is undergoing a number of changes which will affect its manpower requirements in future years.[1] Owing to the installation of new machinery, fewer unskilled but more skilled and semi-skilled workers will be required. In addition to this a downturn in trade is expected in the next year which will reduce the need for workers in all categories. The estimated manpower requirements for the next three years are as follows:

	Unskilled	Semi-skilled	Skilled
Current strength	2000	1500	1000
Year 1	1000	1400	1000
Year 2	500	2000	1500
Year 3	0	2500	2000

The company wishes to decide its policy with regard to the following over the next three years:

(1) Recruitment

(2) Retraining

(3) Redundancy

(4) Short-time working.

There is a natural wastage of labour. A fairly large number of workers leave during their first year. After this the rate is much smaller. Taking this into account, the wastage rates can be taken as below:

[1] Reproduced with permission of John Wiley & Sons Ltd. (Williams 1999, pp. 234–236).

	Unskilled	Semi-skilled	Skilled
Less than one year's service	25%	20%	10%
More than one year's service	10%	5%	5%

There has been no recent recruitment and all workers in the current labour force have been employed for more than one year.

Recruitment

It is possible to recruit a limited number of workers from outside. In any one year the numbers which can be recruited in each category are:

Unskilled	Semi-skilled	Skilled
500	800	500

Retraining

It is possible to retrain up to 200 unskilled workers per year to make them semi-skilled. This costs £400 per worker. The retraining of semi-skilled workers to make them skilled is limited to no more than one quarter of the skilled labour at the time as some training is done on the job. To retrain a semi-skilled worker in this way costs £500.

Downgrading of workers to a lower skill is possible but 50% of such workers leave, although it costs the company nothing. (This wastage is additional to the 'natural wastage' described above.)

Redundancy

The redundancy payment to an unskilled worker is £200 and to a semi-skilled or skilled worker £500.

Overmanning

It is possible to employ up to 150 more workers over the whole company than are needed but the extra costs per employee per year are:

Unskilled	Semi-skilled	Skilled
£1500	£2000	£3000

Short-time Working

Up to 50 workers in each category of skill can be put on short-time working. The cost of this (per employee per year) is:

Unskilled	Semi-skilled	Skilled
£500	£400	£400

An employee on short-time working meets the production requirements of half an employee.

The company's declared objective is to minimize redundancy. How should they operate in order to do this?

If their policy were to minimize costs, how much extra would this save? Deduce the cost of saving each type of job each year.

Mathematical Programming Formulation

Index Sets and Their Members

The following index sets and their members are used in this example:

- worker $\in$ WORKERS
- period $\in$ PERIODS0
- period $\in$ PERIODS $=$ PERIODS0 $\setminus \{0\}$
- $(i, j) \in$ RETRAIN_PAIRS: worker i retrained as worker j
- $(i, j) \in$ DOWNGRADE_PAIRS: worker i downgraded to worker j

Parameters

Table 5.1 shows the parameters that are used in this example.

Table 5.1 Parameters

Parameter Name	Interpretation
waste_new[worker]	Fraction of workers who leave during their first year
waste_old[worker]	Fraction of workers who leave after their first year
recruit_ub[worker]	Upper bound on number of workers who are recruited from outside
redundancy_cost[worker]	Cost per worker made redundant
overmanning_cost[worker]	Cost per excess worker
shorttime_ub[worker]	Upper bound on number of short-time workers
shorttime_cost[worker]	Cost per short-time worker
demand[worker,period]	Manpower requirements
retrain_ub[i,j]	Upper bound on number of workers who can be retrained from *i* to *j* per year
retrain_cost[i,j]	Cost to retrain worker *i* as worker *j*
semiskill_retrain_frac_ub	Upper bound on fraction of semi-skilled workers who can be retrained to skilled
downgrade_leave_frac	Fraction of downgraded workers who leave
overmanning_ub	Upper bound on number of excess workers
shorttime_frac	Fraction of production requirements of a full-time employee that is met by each short-time employee

Variables

Table 5.2 shows the variables that are used in this example.

Table 5.2 Variables

Variable Name	Interpretation
NumWorkers[worker,period]	Number of workers per period
NumRecruits[worker,period]	Number of recruited workers per period
NumRedundant[worker,period]	Number of workers made redundant per period
NumShortTime[worker,period]	Number of short-time workers per period
NumExcess[worker,period]	Number of excess workers per period
NumRetrain[i,j,period]	Number of workers retrained from *i* to *j* per period
NumDowngrade[i,j,period]	Number of workers downgraded from *i* to *j* per period

Objectives

One objective is to minimize the following function:

$$\text{Redundancy} = \sum_{\text{worker} \in \text{WORKERS}} \sum_{\text{period} \in \text{PERIODS}} \text{NumRedundant[worker,period]}$$

A second objective is to minimize the following function:

$$\begin{aligned}\text{Cost} = & \sum \sum \textit{redundancy_cost[worker]} \cdot \text{NumRedundant[worker,period]} \\ & + \sum \sum \textit{shorttime_cost[worker]} \cdot \text{NumShorttime[worker,period]} \\ & + \sum \sum \textit{overmanning_cost[worker]} \cdot \text{NumExcess[worker,period]} \\ & + \sum_{\text{(i,j)} \in \text{RETRAIN_PAIRS}} \sum_{\text{period} \in \text{PERIODS}} \textit{retrain_cost[i,j]} \cdot \text{NumRetrain[i,j,period]}\end{aligned}$$

Constraints

The following constraints are used in this example:

- bounds on variables
- for worker $\in$ WORKERS and period $\in$ PERIODS,

$$\begin{aligned}& \text{NumWorkers[worker,period]} \\ & - (1 - \textit{shorttime_frac}) \cdot \text{NumShortTime[worker,period]} \\ & - \text{NumExcess[worker,period]} \\ & = \textit{demand[worker,period]}\end{aligned}$$

- for worker $\in$ WORKERS and period $\in$ PERIODS,

$$\begin{aligned}& \text{NumWorkers[worker,period]} \\ & = (1 - \textit{waste_old[worker]}) \cdot \text{NumWorkers[worker,period} - 1] \\ & + (1 - \textit{waste_new[worker]}) \cdot \text{NumRecruits[worker,period]} \\ & + (1 - \textit{waste_old[worker]}) \cdot \sum_{\text{(i,worker)} \in \text{RETRAIN_PAIRS}} \text{NumRetrain[i,worker,period]} \\ & + (1 - \textit{downgrade_leave_frac}) \cdot \sum_{\text{(i,worker)} \in \text{DOWNGRADE_PAIRS}} \text{NumDowngrade[i,worker,period]} \\ & - \sum_{\text{(worker,j)} \in \text{RETRAIN_PAIRS}} \text{NumRetrain[worker,j,period]} \\ & - \sum_{\text{(worker,j)} \in \text{DOWNGRADE_PAIRS}} \text{NumDowngrade[worker,j,period]} \\ & - \text{NumRedundant[worker,period]}\end{aligned}$$

- for period ∈ PERIODS,

  NumRetrain['semiskilled','skilled',period] $\leq$ *semiskill_retrain_frac_ub* · NumWorkers['skilled',period]

- for period ∈ PERIODS,

$$\sum_{\text{worker} \in \text{WORKERS}} \text{NumExcess[worker,period]} \leq \textit{overmanning_ub}$$

Input Data

The following data sets and macro variables contain the input data that are used in this example:

```
data demand_data;
   input period unskilled semiskilled skilled;
   datalines;
0 2000 1500 1000
1 1000 1400 1000
2  500 2000 1500
3    0 2500 2000
;

data worker_data;
   input worker $12. waste_new waste_old recruit_ub redundancy_cost
      overmanning_cost shorttime_ub shorttime_cost;
   datalines;
unskilled   0.25 0.10 500 200 1500 50 500
semiskilled 0.20 0.05 800 500 2000 50 400
skilled     0.10 0.05 500 500 3000 50 400
;

data retrain_data;
   input worker1 $12. worker2 $12. retrain_ub retrain_cost;
   datalines;
unskilled   semiskilled 200 400
semiskilled skilled       . 500
;

data downgrade_data;
   input worker1 $12. worker2 $12.;
   datalines;
semiskilled unskilled
skilled     semiskilled
skilled     unskilled
;

%let semiskill_retrain_frac_ub = 0.25;
%let downgrade_leave_frac = 0.5;
%let overmanning_ub = 150;
%let shorttime_frac = 0.5;
```

PROC OPTMODEL Statements and Output

The first several index sets are one-dimensional, as in all the previous examples:

```
proc optmodel;
   set <str> WORKERS;
   num waste_new {WORKERS};
   num waste_old {WORKERS};
   num recruit_ub {WORKERS};
   num redundancy_cost {WORKERS};
   num overmanning_cost {WORKERS};
   num shorttime_ub {WORKERS};
   num shorttime_cost {WORKERS};
   read data worker_data into WORKERS=[worker]
      waste_new waste_old recruit_ub redundancy_cost overmanning_cost
      shorttime_ub shorttime_cost;

   set PERIODS0;
   num demand {WORKERS, PERIODS0};
   read data demand_data into PERIODS0=[period]
      {worker in WORKERS} <demand[worker,period]=col(worker)>;

   var NumWorkers {WORKERS, PERIODS0} >= 0;
   for {worker in WORKERS} fix NumWorkers[worker,0] = demand[worker,0];

   set PERIODS = PERIODS0 diff {0};
   var NumRecruits {worker in WORKERS, PERIODS} >= 0 <= recruit_ub[worker];
   var NumRedundant {WORKERS, PERIODS} >= 0;
   var NumShortTime {worker in WORKERS, PERIODS} >= 0 <= shorttime_ub[worker];
   var NumExcess {WORKERS, PERIODS} >= 0;
```

Both RETRAIN_PAIRS and DOWNGRADE_PAIRS are two-dimensional index sets, declared by using the optional <STR,STR> specification in the SET statement so that these sets contain pairs of strings. In general, a set can consist of tuples of any length and any combination of NUM and STR scalar-types.

```
   set <str,str> RETRAIN_PAIRS;
   num retrain_ub {RETRAIN_PAIRS};
   num retrain_cost {RETRAIN_PAIRS};
   read data retrain_data into RETRAIN_PAIRS=[worker1 worker2]
      retrain_ub retrain_cost;

   var NumRetrain {RETRAIN_PAIRS, PERIODS} >= 0;
   for {<i,j> in RETRAIN_PAIRS: retrain_ub[i,j] ne .}
      for {period in PERIODS} NumRetrain[i,j,period].ub = retrain_ub[i,j];

   set <str,str> DOWNGRADE_PAIRS;
   read data downgrade_data into DOWNGRADE_PAIRS=[worker1 worker2];
   var NumDowngrade {DOWNGRADE_PAIRS, PERIODS} >= 0;

   con Demand_con {worker in WORKERS, period in PERIODS}:
      NumWorkers[worker,period]
    - (1 - &shorttime_frac) * NumShortTime[worker,period]
    - NumExcess[worker,period]
    = demand[worker,period];
```

The following Flow_balance_con constraint uses an implicit slice to express a few of the summations compactly:

```
con Flow_balance_con {worker in WORKERS, period in PERIODS}:
   NumWorkers[worker,period]
 = (1 - waste_old[worker]) * NumWorkers[worker,period-1]
 + (1 - waste_new[worker]) * NumRecruits[worker,period]
 + (1 - waste_old[worker]) *
      sum {<i,(worker)> in RETRAIN_PAIRS} NumRetrain[i,worker,period]
 + (1 - &downgrade_leave_frac) *
      sum {<i,(worker)> in DOWNGRADE_PAIRS} NumDowngrade[i,worker,period]
 - sum {<(worker),j> in RETRAIN_PAIRS} NumRetrain[worker,j,period]
 - sum {<(worker),j> in DOWNGRADE_PAIRS} NumDowngrade[worker,j,period]
 - NumRedundant[worker,period];
```

For example,

```
<i,(worker)> in RETRAIN_PAIRS
```

is equivalent to

```
i in slice(<*,worker>,RETRAIN_PAIRS)
```

which is equivalent to

```
i in WORKERS: <i,worker> in RETRAIN_PAIRS
```

The remaining two constraints are straightforward:

```
con Semiskill_retrain_con {period in PERIODS}:
   NumRetrain['semiskilled','skilled',period]
<= &semiskill_retrain_frac_ub * NumWorkers['skilled',period];

con Overmanning_con {period in PERIODS}:
   sum {worker in WORKERS} NumExcess[worker,period] <= &overmanning_ub;
```

This example uses two objectives, Redundancy and Cost, declared in the following MIN statements:

```
min Redundancy =
   sum {worker in WORKERS, period in PERIODS} NumRedundant[worker,period];
min Cost =
   sum {worker in WORKERS, period in PERIODS} (
      redundancy_cost[worker] * NumRedundant[worker,period]
    + shorttime_cost[worker] * NumShorttime[worker,period]
    + overmanning_cost[worker] * NumExcess[worker,period])
 + sum {<i,j> in RETRAIN_PAIRS, period in PERIODS}
      retrain_cost[i,j] * NumRetrain[i,j,period];
```

The LP solver is called twice, and each SOLVE statement includes the OBJ option to specify which objective to optimize. The first PRINT statement after each SOLVE statement reports the values of both objectives even though only one objective is optimized at a time:

```
solve obj Redundancy;
print Redundancy Cost;
print NumWorkers NumRecruits NumRedundant NumShortTime NumExcess;
print NumRetrain;
```

```
   print NumDowngrade;
   create data sol_data1 from [worker period]
      NumWorkers NumRecruits NumRedundant NumShortTime NumExcess;
   create data sol_data2 from [worker1 worker2 period] NumRetrain NumDowngrade;

   solve obj Cost;
   print Redundancy Cost;
   print NumWorkers NumRecruits NumRedundant NumShortTime NumExcess;
   print NumRetrain;
   print NumDowngrade;
   create data sol_data3 from [worker period]
      NumWorkers NumRecruits NumRedundant NumShortTime NumExcess;
   create data sol_data4 from [worker1 worker2 period] NumRetrain NumDowngrade;
quit;
```

Figure 5.1 shows the output that results from the first SOLVE statement.

Figure 5.1 Output from First SOLVE Statement, Minimizing Redundancy

The OPTMODEL Procedure

Problem Summary	
Objective Sense	Minimization
Objective Function	Redundancy
Objective Type	Linear
Number of Variables	63
Bounded Above	0
Bounded Below	39
Bounded Below and Above	21
Free	0
Fixed	3
Number of Constraints	24
Linear LE (<=)	6
Linear EQ (=)	18
Linear GE (>=)	0
Linear Range	0
Constraint Coefficients	108

Performance Information	
Execution Mode	Single-Machine
Number of Threads	1

Figure 5.1 *continued*

Solution Summary	
Solver	LP
Algorithm	Dual Simplex
Objective Function	Redundancy
Solution Status	Optimal
Objective Value	841.796875
Primal Infeasibility	1.421085E-14
Dual Infeasibility	0
Bound Infeasibility	0
Iterations	13
Presolve Time	0.00
Solution Time	0.00

Redundancy	Cost
841.8	1462048

[1]	[2]	NumWorkers	NumRecruits	NumRedundant	NumShortTime	NumExcess
semiskilled	**0**	1500				
semiskilled	**1**	1443	0.00	0.00	50	17.969
semiskilled	**2**	2000	682.20	0.00	0	0.000
semiskilled	**3**	2500	645.72	0.00	0	0.000
skilled	**0**	1000				
skilled	**1**	1025	0.00	0.00	50	0.000
skilled	**2**	1525	500.00	0.00	50	0.000
skilled	**3**	2000	500.00	0.00	0	0.000
unskilled	**0**	2000				
unskilled	**1**	1157	0.00	442.97	50	132.031
unskilled	**2**	675	0.00	166.33	50	150.000
unskilled	**3**	175	0.00	232.50	50	150.000

[1]	[2]	[3]	NumRetrain
semiskilled	**skilled**	**1**	256.25
semiskilled	**skilled**	**2**	106.58
semiskilled	**skilled**	**3**	106.58
unskilled	**semiskilled**	**1**	200.00
unskilled	**semiskilled**	**2**	200.00
unskilled	**semiskilled**	**3**	200.00

Figure 5.1 *continued*

[1]	[2]	[3]	NumDowngrade
semiskilled	unskilled	1	0.00
semiskilled	unskilled	2	0.00
semiskilled	unskilled	3	0.00
skilled	semiskilled	1	168.44
skilled	semiskilled	2	0.00
skilled	semiskilled	3	0.00
skilled	unskilled	1	0.00
skilled	unskilled	2	0.00
skilled	unskilled	3	0.00

Figure 5.2 shows the output that results from the second SOLVE statement.

Figure 5.2 Output from Second SOLVE Statement, Minimizing Cost

Problem Summary	
Objective Sense	Minimization
Objective Function	Cost
Objective Type	Linear
Number of Variables	63
Bounded Above	0
Bounded Below	39
Bounded Below and Above	21
Free	0
Fixed	3
Number of Constraints	24
Linear LE (<=)	6
Linear EQ (=)	18
Linear GE (>=)	0
Linear Range	0
Constraint Coefficients	108

Performance Information	
Execution Mode	Single-Machine
Number of Threads	1

Figure 5.2 *continued*

Solution Summary	
Solver	LP
Algorithm	Dual Simplex
Objective Function	Cost
Solution Status	Optimal
Objective Value	498677.28532
Primal Infeasibility	2.842171E-14
Dual Infeasibility	0
Bound Infeasibility	0
Iterations	8
Presolve Time	0.00
Solution Time	0.00

Redundancy	Cost
1423.7	498677

[1]	[2]	NumWorkers	NumRecruits	NumRedundant	NumShortTime	NumExcess
semiskilled	**0**	1500				
semiskilled	**1**	1400	0.000	0.00	0	0
semiskilled	**2**	2000	800.000	0.00	0	0
semiskilled	**3**	2500	800.000	0.00	0	0
skilled	**0**	1000				
skilled	**1**	1000	55.556	0.00	0	0
skilled	**2**	1500	500.000	0.00	0	0
skilled	**3**	2000	500.000	0.00	0	0
unskilled	**0**	2000				
unskilled	**1**	1000	0.000	812.50	0	0
unskilled	**2**	500	0.000	257.62	0	0
unskilled	**3**	0	0.000	353.60	0	0

[1]	[2]	[3]	NumRetrain
semiskilled	**skilled**	**1**	0.000
semiskilled	**skilled**	**2**	105.263
semiskilled	**skilled**	**3**	131.579
unskilled	**semiskilled**	**1**	0.000
unskilled	**semiskilled**	**2**	142.382
unskilled	**semiskilled**	**3**	96.399

Figure 5.2 *continued*

[1]	[2]	[3]	NumDowngrade
semiskilled	**unskilled**	**1**	25
semiskilled	**unskilled**	**2**	0
semiskilled	**unskilled**	**3**	0
skilled	**semiskilled**	**1**	0
skilled	**semiskilled**	**2**	0
skilled	**semiskilled**	**3**	0
skilled	**unskilled**	**1**	0
skilled	**unskilled**	**2**	0
skilled	**unskilled**	**3**	0

Features Demonstrated

The following features are demonstrated in this example:

- problem type: linear programming
- numeric and string index sets
- reading dense two-dimensional data
- reading sparse two-dimensional data
- sets of tuples
- bounds in the VAR statement
- `.ub` variable suffix
- FIX statement
- using a colon (:) to select members of a set
- set operator DIFF
- SLICE expression
- implicit slice
- multiple objectives and the OBJ option
- multiple input and output data sets

Chapter 6
Refinery Optimization: How to Run an Oil Refinery

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Problem Statement

An oil refinery purchases two crude oils (crude 1 and crude 2).[1] These crude oils are put through four processes: distillation, reforming, cracking, and blending, to produce petrols and fuels which are sold.

Distillation

Distillation separates each crude oil into fractions known as *light naphtha*, *medium naphtha*, *heavy naphtha*, *light oil*, *heavy oil* and *residuum* according to their boiling points. Light, medium and heavy naphthas have octane numbers of 90, 80 and 70 respectively. The fractions into which one barrel of each type of crude splits are given in the table:

	Light naphtha	Medium naphtha	Heavy naphtha	Light oil	Heavy oil	Residuum
Crude 1	0.1	0.2	0.2	0.12	0.2	0.13
Crude 2	0.15	0.25	0.18	0.08	0.19	0.12

N.B. There is a small amount of wastage in distillation.

[1] Reproduced with permission of John Wiley & Sons Ltd. (Williams 1999, pp. 236–238).

Reforming

The naphthas can be used immediately for blending into different grades of petrol or can go through a process known as reforming. Reforming produces a product known as reformed gasoline with an octane number of 115. The yields of reformed gasoline from each barrel of the different naphthas are given below:

1 barrel of light naphtha yields 0.6 barrels of reformed gasoline;
1 barrel of medium naphtha yields 0.52 barrels of reformed gasoline;
1 barrel of heavy naphtha yields 0.45 barrels of reformed gasoline.

Cracking

The oils (light and heavy) can either be used directly for blending into *jet fuel* or *fuel oil* or be put through a process known as catalytic cracking. The catalytic cracker produces *cracked oil* and *cracked gasoline*. Cracked gasoline has an octane number of 105.

1 barrel of light oil yields 0.68 barrels of cracked oil and 0.28 barrels of cracked gasoline;
1 barrel of heavy oil yields 0.75 barrels of cracked oil and 0.2 barrels of cracked gasoline.

Cracked oil is used for blending *fuel oil* and *jet fuel*; cracked gasoline is used for blending *petrol*.

Residuum can be used for either producing *lube-oil* or blending into *jet fuel* and *fuel oil*:

1 barrel of residuum yields 0.5 barrels of lube-oil.

Blending

Petrols (Motor Fuel)

There are two sorts of petrol, *regular* and *premium*, obtained by blending the naphtha, reformed gasoline and cracked gasoline. The only stipulations concerning them are that regular must have an octane number of at least 84 and that premium must have an octane number of at least 94. It is assumed that octane numbers blend linearly by volume.

Jet Fuel

The stipulation concerning jet fuel is that its vapour pressure must not exceed 1 kilogram per square centimetre. The vapour pressures for light, heavy and cracked oils and residuum are 1.0, 0.6, 1.5 and 0.05 kilograms per square centimetre respectively. It may again be assumed that vapour pressures blend linearly by volume.

Fuel Oil

To produce fuel oil, light oil, cracked oil, heavy oil and residuum must be blended in the ratio 10:4:3:1.

There are availability and capacity limitations on the quantities and processes used:

(a) The daily availability of crude 1 is 20,000 barrels.

(b) The daily availability of crude 2 is 30,000 barrels.

(c) At most 45,000 barrels of crude can be distilled per day.

(d) At most 10,000 barrels of naphtha can be reformed per day.

(e) At most 8000 barrels of oil can be cracked per day.

(f) The daily production of lube oil must be between 500 and 1000 barrels.

(g) Premium motor fuel production must be at least 40% of regular motor fuel production.

The profit contributions from the sale of the final products are (in pence per barrel)

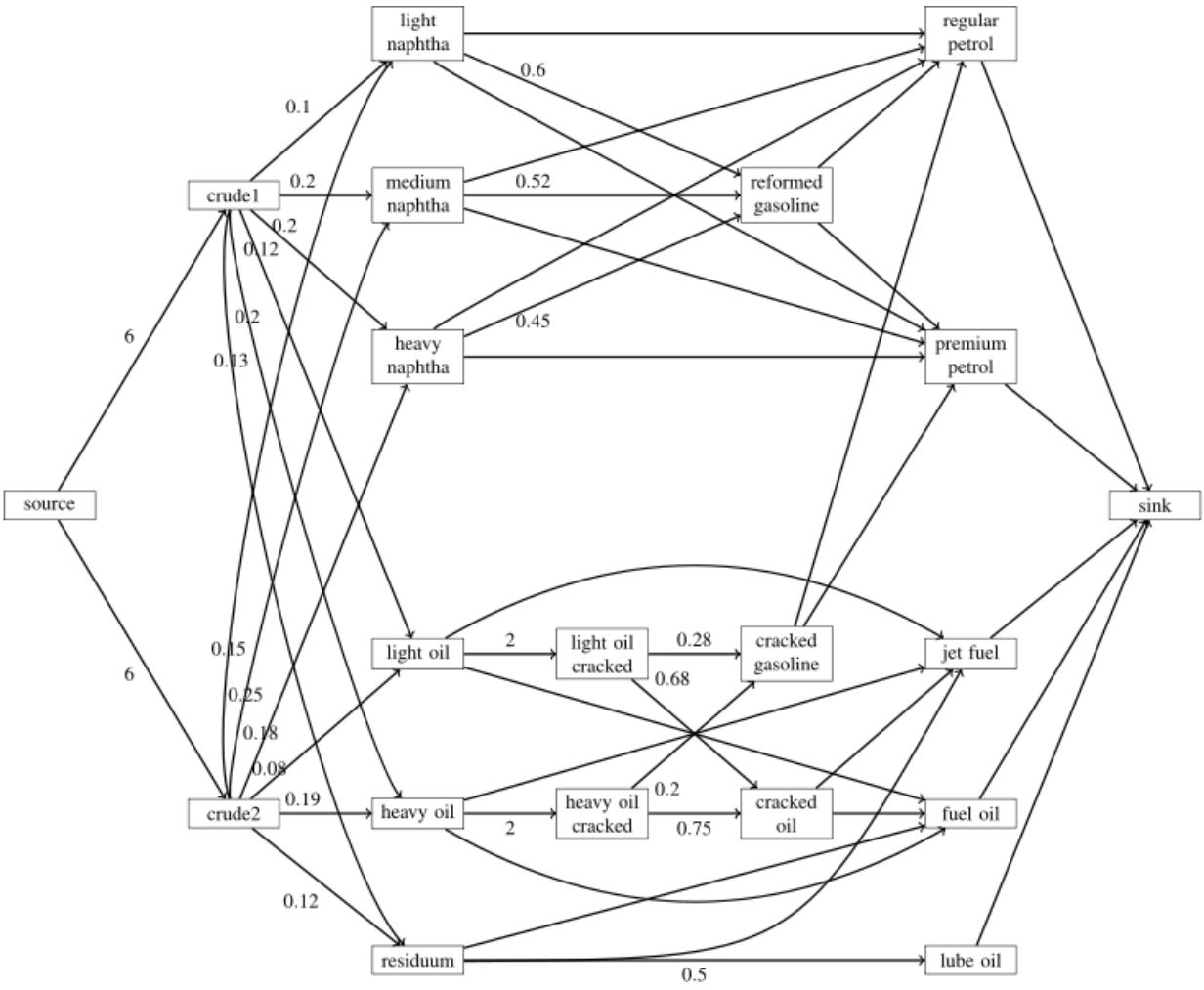
Premium petrol	700
Regular petrol	600
Jet fuel	400
Fuel oil	350
Lube-oil	150

How should the operations of the refinery be planned in order to maximize total profit?

Mathematical Programming Formulation

The problem is represented as a generalized network flow problem with side constraints. Each node corresponds to a material, and each arc represents conversion of one material to another via one of the four processes, as shown in Figure 6.1. The arc multiplier values 6 and 2, together with the Distillation and Cracking constraints specified later, are used to split the flow into equal parts at the head nodes of the corresponding arcs.

Figure 6.1 Generalized Network with Arc Multipliers



Index Sets and Their Members

The following index sets and their members are used in this example:

- $i \in$ NODES
- $(i, j) \in$ ARCS
- product $\in$ FINAL_PRODUCTS
- crude $\in$ CRUDES
- oil $\in$ OILS
- oil $\in$ CRACKED_OILS
- petrol $\in$ PETROLS

Parameters

Table 6.1 shows the parameters that are used in this example.

Table 6.1 Parameters

Parameter Name	Interpretation
arc_mult[arc]	Arc multiplier (default value of 1)
profit[product]	Profit from sale of final product (in pounds per barrel)
octane[i]	Octane per node
octane_lb[petrol]	Lower bound on octane
vapour_pressure[i]	Vapour pressure per node
fuel_oil_coefficient[i]	Fuel oil blending coefficient per node
sum_fuel_oil_coefficient	$\sum_{(i,\text{'fuel_oil'})\in\text{ARCS}} fuel_oil_coefficient[i]$
vapour_pressure_ub	Upper bound on vapour pressure
crude_total_ub	Upper bound on number of barrels of crude distilled per day
naphtha_ub	Upper bound on number of barrels of naphtha reformed per day
cracked_oil_ub	Upper bound on number of barrels of oil cracked per day
lube_oil_lb	Lower bound on number of barrels of lube oil produced per day
lube_oil_ub	Upper bound on number of barrels of lube oil produced per day
premium_ratio	Lower bound on ratio of premium motor fuel production to regular motor fuel production

Variables

Table 6.2 shows the variables that are used in this example.

Table 6.2 Variables

Variable Name	Interpretation
Flow[i,j]	Flow across arc (i, j)
CrudeDistilled[crude]	Barrels of crude oil distilled
OilCracked[oil]	Barrels of oil cracked

Objective

The objective is to maximize the following function:

$$\text{TotalProfit} = \sum_{i\in\text{FINAL_PRODUCTS}} profit[i] \cdot \text{Flow[i,'sink']}$$

Constraints

The following constraints are used in this example:

- bounds on variables
- for $i \in$ NODES \ {'source', 'sink'},

$$\sum_{(i,j)\in \text{ARCS}} \text{Flow[i,j]} = \sum_{(j,i)\in \text{ARCS}} \textit{arc_mult[j,i]} \cdot \text{Flow[j,i]}$$

- for $(i, j) \in$ ARCS such that $i \in$ CRUDES,

$$\text{Flow[i,j]} = \text{CrudeDistilled[i]}$$

- for $(i, j) \in$ ARCS such that $i \in$ CRACKED_OILS,

$$\text{Flow[i,j]} = \text{OilCracked[i]}$$

- for petrol $\in$ PETROLS,

$$\frac{\sum_{(i,\text{petrol})\in \text{ARCS}} \textit{octane[i]} \cdot \textit{arc_mult[i,petrol]} \cdot \text{Flow[i,petrol]}}{\sum_{(i,\text{petrol})\in \text{ARCS}} \textit{arc_mult[i,petrol]} \cdot \text{Flow[i,petrol]}} \geq \textit{octane_lb[petrol]}$$

- $$\frac{\sum_{(i,\text{'jet_fuel'})\in \text{ARCS}} \textit{vapour_pressure[i]} \cdot \textit{arc_mult[i,'jet_fuel']} \cdot \text{Flow[i,'jet_fuel']}}{\sum_{(i,\text{'jet_fuel'})\in \text{ARCS}} \textit{arc_mult[i,'jet_fuel']} \cdot \text{Flow[i,'jet_fuel']}} \leq \textit{vapour_pressure_ub}$$
- for $(i,$ 'fuel_oil') $\in$ ARCS,

$$\frac{\text{Flow[i,'fuel_oil']}}{\sum_{(j,\text{'fuel_oil'})\in \text{ARCS}} \text{Flow[j,'fuel_oil']}} = \frac{\textit{fuel_oil_coefficient[i]}}{\textit{sum_fuel_oil_coefficient}}$$

- $$\sum_{i\in \text{CRUDES}} \text{CrudeDistilled[i]} \leq \textit{crude_total_ub}$$
- $$\sum_{\substack{(i,\text{'reformed_gasoline'})\in \text{ARCS:} \\ \text{index}(i,\text{'naphtha'})>0}} \text{Flow[i,'reformed_gasoline']} \leq \textit{naphtha_ub}$$
- $$\sum_{(i,\text{'cracked_oil'})\in \text{ARCS}} \text{Flow[i,'cracked_oil']} \leq \textit{cracked_oil_ub}$$
- $\textit{lube_oil_lb} \leq \text{Flow['lube_oil','sink']} \leq \textit{lube_oil_ub}$
- $$\frac{\sum_{(\text{'premium_petrol'},j)\in \text{ARCS}} \text{Flow['premium_petrol',j]}}{\sum_{(\text{'regular_petrol'},j)\in \text{ARCS}} \text{Flow['regular_petrol',j]}} \geq \textit{premium_ratio}$$

Input Data

The following data sets and macro variables contain the input data that are used in this example:

```
data crude_data;
   input crude $ crude_ub;
   datalines;
crude1 20000
crude2 30000
;

data arc_data;
   input i $18. j $18. multiplier;
   datalines;
source             crude1            6
source             crude2            6
crude1             light_naphtha     0.1
crude1             medium_naphtha    0.2
crude1             heavy_naphtha     0.2
crude1             light_oil         0.12
crude1             heavy_oil         0.2
crude1             residuum          0.13
crude2             light_naphtha     0.15
crude2             medium_naphtha    0.25
crude2             heavy_naphtha     0.18
crude2             light_oil         0.08
crude2             heavy_oil         0.19
crude2             residuum          0.12
light_naphtha      regular_petrol    .
light_naphtha      premium_petrol    .
medium_naphtha     regular_petrol    .
medium_naphtha     premium_petrol    .
heavy_naphtha      regular_petrol    .
heavy_naphtha      premium_petrol    .
light_naphtha      reformed_gasoline 0.6
medium_naphtha     reformed_gasoline 0.52
heavy_naphtha      reformed_gasoline 0.45
light_oil          jet_fuel          .
light_oil          fuel_oil          .
heavy_oil          jet_fuel          .
heavy_oil          fuel_oil          .
light_oil          light_oil_cracked 2
light_oil_cracked  cracked_oil       0.68
light_oil_cracked  cracked_gasoline  0.28
heavy_oil          heavy_oil_cracked 2
heavy_oil_cracked  cracked_oil       0.75
heavy_oil_cracked  cracked_gasoline  0.2
cracked_oil        jet_fuel          .
cracked_oil        fuel_oil          .
reformed_gasoline  regular_petrol    .
reformed_gasoline  premium_petrol    .
cracked_gasoline   regular_petrol    .
```

```
cracked_gasoline  premium_petrol    .
residuum          lube_oil          0.5
residuum          jet_fuel          .
residuum          fuel_oil          .
;

data octane_data;
   input i $18. octane;
   datalines;
light_naphtha      90
medium_naphtha     80
heavy_naphtha      70
reformed_gasoline 115
cracked_gasoline  105
;

data petrol_data;
   input petrol $15. octane_lb;
   datalines;
regular_petrol 84
premium_petrol 94
;

data vapour_pressure_data;
   input oil $12. vapour_pressure;
   datalines;
light_oil   1.0
heavy_oil   0.6
cracked_oil 1.5
residuum    0.05
;

data fuel_oil_ratio_data;
   input oil $12. coefficient;
   datalines;
light_oil   10
cracked_oil  4
heavy_oil    3
residuum     1
;

data final_product_data;
   input product $15. profit;
   datalines;
premium_petrol 700
regular_petrol 600
jet_fuel       400
fuel_oil       350
lube_oil       150
;

%let vapour_pressure_ub = 1;
%let crude_total_ub = 45000;
%let naphtha_ub = 10000;
```

```
%let cracked_oil_ub = 8000;
%let lube_oil_lb = 500;
%let lube_oil_ub = 1000;
%let premium_ratio = 0.40;
```

PROC OPTMODEL Statements and Output

The NOMISS option in the first READ DATA statement ensures that only nonmissing values of the variable multiplier in the arc_data data set populate the *arc_mult* parameter. Because *arc_mult* is declared with an initial value of 1, parameters with no value default to 1.

```
proc optmodel;
   set <str,str> ARCS;
   num arc_mult {ARCS} init 1;
   read data arc_data nomiss into ARCS=[i j] arc_mult=multiplier;
   var Flow {ARCS} >= 0;

   set <str> FINAL_PRODUCTS;
   num profit {FINAL_PRODUCTS};
   read data final_product_data into FINAL_PRODUCTS=[product] profit;
```

The following FOR loop converts *profit* from pence per barrel to pounds per barrel, without altering the input data set:

```
   for {product in FINAL_PRODUCTS} profit[product] = profit[product] / 100;
```

Most arcs appear in the arc_data data set, but the following assignment statement uses the set operators UNION and CROSS to augment the ARCS set with an arc from each final product to the sink node:

```
   ARCS = ARCS union (FINAL_PRODUCTS cross {'sink'});

   set NODES = union {<i,j> in ARCS} {i,j};

   max TotalProfit = sum {i in FINAL_PRODUCTS} profit[i] * Flow[i,'sink'];

   con Flow_balance {i in NODES diff {'source','sink'}}:
      sum {<(i),j> in ARCS} Flow[i,j]
         = sum {<j,(i)> in ARCS} arc_mult[j,i] * Flow[j,i];

   set <str> CRUDES;
   var CrudeDistilled {CRUDES} >= 0;
```

Because the decision variable CrudeDistilled automatically contains the `.ub` suffix, you can populate *CrudeDistilled.ub* directly by using the following READ DATA statement, without having to declare an additional parameter to store this upper bound:

```
   read data crude_data into CRUDES=[crude] CrudeDistilled.ub=crude_ub;
   con Distillation {<i,j> in ARCS: i in CRUDES}:
      Flow[i,j] = CrudeDistilled[i];
```

The SETOF operator, used together with the concatenation operator (||) in the following statements, enables you to create one index set from another:

```
set OILS = {'light_oil','heavy_oil'};
set CRACKED_OILS = setof {i in OILS} i||'_cracked';
var OilCracked {CRACKED_OILS} >= 0;
con Cracking {<i,j> in ARCS: i in CRACKED_OILS}:
   Flow[i,j] = OilCracked[i];

num octane {NODES} init .;
read data octane_data nomiss into [i] octane;

set <str> PETROLS;
num octane_lb {PETROLS};
read data petrol_data into PETROLS=[petrol] octane_lb;

num vapour_pressure {NODES} init .;
read data vapour_pressure_data nomiss into [oil] vapour_pressure;
```

As expressed on page 70, both octane numbers and vapour pressures of the blended fuels are ratios of linear functions of the decision variables. To increase algorithmic performance and reliability, the following two CON statements linearize the nonlinear ratio constraints by clearing the denominators:

```
con Blending_petrol {petrol in PETROLS}:
   sum {<i,(petrol)> in ARCS}
      octane[i] * arc_mult[i,petrol] * Flow[i,petrol]
>= octane_lb[petrol] *
   sum {<i,(petrol)> in ARCS} arc_mult[i,petrol] * Flow[i,petrol];

con Blending_jet_fuel:
   sum {<i,'jet_fuel'> in ARCS}
      vapour_pressure[i] * arc_mult[i,'jet_fuel'] * Flow[i,'jet_fuel']
<= &vapour_pressure_ub *
   sum {<i,'jet_fuel'> in ARCS} arc_mult[i,'jet_fuel'] * Flow[i,'jet_fuel'];
```

Similarly, the following CON statement linearizes the fuel oil ratio constraints by clearing the denominators:

```
num fuel_oil_coefficient {NODES} init 0;
read data fuel_oil_ratio_data nomiss into [oil]
   fuel_oil_coefficient=coefficient;
num sum_fuel_oil_coefficient
   = sum {<i,'fuel_oil'> in ARCS} fuel_oil_coefficient[i];
con Blending_fuel_oil {<i,'fuel_oil'> in ARCS}:
   sum_fuel_oil_coefficient * Flow[i,'fuel_oil']
= fuel_oil_coefficient[i] * sum {<j,'fuel_oil'> in ARCS} Flow[j,'fuel_oil'];

con Crude_total_ub_con:
   sum {i in CRUDES} CrudeDistilled[i] <= &crude_total_ub;
```

The following CON statement uses the SAS function INDEX together with the colon operator (:) to select the subset of arcs whose tail node contains “naphtha” in its name:

```
con Naphtha_ub_con:
   sum {<i,'reformed_gasoline'> in ARCS: index(i,'naphtha') > 0}
      Flow[i,'reformed_gasoline']
<= &naphtha_ub;

con Cracked_oil_ub_con:
   sum {<i,'cracked_oil'> in ARCS} Flow[i,'cracked_oil'] <= &cracked_oil_ub;

con Lube_oil_range_con:
   &lube_oil_lb <= Flow['lube_oil','sink'] <= &lube_oil_ub;
```

As expressed on page 70, the premium ratio constraint involves a ratio of linear functions of the decision variables. The following CON statement linearizes the nonlinear ratio constraint by clearing the denominator:

```
con Premium_ratio_con:
   sum {<'premium_petrol',j> in ARCS} Flow['premium_petrol',j]
>= &premium_ratio *
   sum {<'regular_petrol',j> in ARCS} Flow['regular_petrol',j];

num octane_sol {petrol in PETROLS} =
   (sum {<i,(petrol)> in ARCS}
      octane[i] * arc_mult[i,petrol] * Flow[i,petrol].sol) /
   (sum {<i,(petrol)> in ARCS} arc_mult[i,petrol] * Flow[i,petrol].sol);

num vapour_pressure_sol =
   (sum {<i,'jet_fuel'> in ARCS} vapour_pressure[i] *
      arc_mult[i,'jet_fuel'] * Flow[i,'jet_fuel'].sol)
   / (sum {<i,'jet_fuel'> in ARCS} arc_mult[i,'jet_fuel'] *
      Flow[i,'jet_fuel'].sol);

num fuel_oil_ratio_sol {<i,'fuel_oil'> in ARCS} =
   (arc_mult[i,'fuel_oil'] * Flow[i,'fuel_oil'].sol) /
   (sum {<j,'fuel_oil'> in ARCS} arc_mult[j,'fuel_oil'] *
      Flow[j,'fuel_oil'].sol);

solve;
print CrudeDistilled;
print OilCracked Flow;
print octane_sol octane_lb;
```

Although the previous PRINT statements print all values of the given parameters, the following two PRINT statements use an index set to print a specified subset of the values:

```
   print {<i,'jet_fuel'> in ARCS} vapour_pressure vapour_pressure_sol;
   print {<i,'fuel_oil'> in ARCS} fuel_oil_coefficient fuel_oil_ratio_sol;
   create data sol_data1 from [i j] Flow;
quit;
```

Figure 6.2 shows the output from the linear programming solver. For this test instance, it turns out that the optimal solution contains no fuel oil.

Figure 6.2 Output from Linear Programming Solver

The OPTMODEL Procedure

Problem Summary	
Objective Sense	Maximization
Objective Function	TotalProfit
Objective Type	Linear
Number of Variables	51
Bounded Above	0
Bounded Below	49
Bounded Below and Above	2
Free	0
Fixed	0
Number of Constraints	46
Linear LE (<=)	4
Linear EQ (=)	38
Linear GE (>=)	3
Linear Range	1
Constraint Coefficients	158

Performance Information	
Execution Mode	Single-Machine
Number of Threads	1

Solution Summary	
Solver	LP
Algorithm	Dual Simplex
Objective Function	TotalProfit
Solution Status	Optimal
Objective Value	211365.13477
Primal Infeasibility	2.910383E-11
Dual Infeasibility	0
Bound Infeasibility	0
Iterations	21
Presolve Time	0.00
Solution Time	0.00

[1]	CrudeDistilled
crude1	15000
crude2	30000

[1]	OilCracked
heavy_oil_cracked	3800
light_oil_cracked	4200

Figure 6.2 *continued*

[1]	[2]	Flow
cracked_gasoline	premium_petrol	0.0
cracked_gasoline	regular_petrol	1936.0
cracked_oil	fuel_oil	0.0
cracked_oil	jet_fuel	5706.0
crude1	heavy_naphtha	15000.0
crude1	heavy_oil	15000.0
crude1	light_naphtha	15000.0
crude1	light_oil	15000.0
crude1	medium_naphtha	15000.0
crude1	residuum	15000.0
crude2	heavy_naphtha	30000.0
crude2	heavy_oil	30000.0
crude2	light_naphtha	30000.0
crude2	light_oil	30000.0
crude2	medium_naphtha	30000.0
crude2	residuum	30000.0
fuel_oil	sink	0.0
heavy_naphtha	premium_petrol	1677.8
heavy_naphtha	reformed_gasoline	5406.9
heavy_naphtha	regular_petrol	1315.3
heavy_oil	fuel_oil	0.0
heavy_oil	heavy_oil_cracked	3800.0
heavy_oil	jet_fuel	4900.0
heavy_oil_cracked	cracked_gasoline	3800.0
heavy_oil_cracked	cracked_oil	3800.0
jet_fuel	sink	15156.0
light_naphtha	premium_petrol	2706.9
light_naphtha	reformed_gasoline	0.0
light_naphtha	regular_petrol	3293.1
light_oil	fuel_oil	0.0
light_oil	jet_fuel	0.0
light_oil	light_oil_cracked	4200.0
light_oil_cracked	cracked_gasoline	4200.0
light_oil_cracked	cracked_oil	4200.0
lube_oil	sink	500.0
medium_naphtha	premium_petrol	0.0
medium_naphtha	reformed_gasoline	0.0
medium_naphtha	regular_petrol	10500.0
premium_petrol	sink	6817.8
reformed_gasoline	premium_petrol	2433.1
reformed_gasoline	regular_petrol	0.0
regular_petrol	sink	17044.4
residuum	fuel_oil	0.0
residuum	jet_fuel	4550.0
residuum	lube_oil	1000.0
source	crude1	15000.0
source	crude2	30000.0

Figure 6.2 *continued*

[1]	octane_sol	octane_lb
premium_petrol	94	94
regular_petrol	84	84

[1]	vapour_pressure
cracked_oil	1.50
heavy_oil	0.60
light_oil	1.00
residuum	0.05

vapour_pressure_sol
0.77372

[1]	fuel_oil_coefficient	fuel_oil_ratio_sol
cracked_oil	4	.
heavy_oil	3	.
light_oil	10	.
residuum	1	.

Features Demonstrated

The following features are demonstrated in this example:

- problem type: linear programming (generalized network flow with side constraints)
- numeric and string index sets
- reading sparse two-dimensional data
- NOMISS option
- sets of tuples
- bounds in the VAR statement
- **`.ub`** variable suffix
- using a colon (:) to select members of a set
- set operators UNION, DIFF, CROSS, and SETOF
- linearizing a ratio constraint
- range constraint
- INDEX function
- implicit slice
- using a variable suffix (such as **`.sol`**) in the declaration of a numeric parameter
- index set in the PRINT statement

Chapter 7
Mining: Which Pits to Work and When to Close Them Down

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Problem Statement

A mining company is going to continue operating in a certain area for the next five years.[1] There are four mines in this area but it can operate at most three in any one year. Although a mine may not operate in a certain year it is still necessary to keep it 'open', in the sense that royalties are payable, should it be operated in a future year. Clearly if a mine is not going to be worked again it can be closed down permanently and no more royalties need be paid. The yearly royalties payable on each mine kept 'open' are

Mine 1	£5 million
Mine 2	£4 million
Mine 3	£4 million
Mine 4	£5 million

There is an upper limit to the amount of ore which can be extracted from each mine in a year. These upper limits are:

Mine 1	2×10^6 tons
Mine 2	2.5×10^6 tons
Mine 3	1.3×10^6 tons
Mine 4	3×10^6 tons

[1] Reproduced with permission of John Wiley & Sons Ltd. (Williams 1999, pp. 238–239).

The ore from the different mines is of varying quality. This quality is measured on a scale so that blending ores together results in a linear combination of the quality measurements, e.g. if equal quantities of two ores were combined the resultant ore would have a quality measurement half way between that of the ingredient ores. Measured in these units the qualities of the ores from the mines are given below:

Mine 1	1.0
Mine 2	0.7
Mine 3	1.5
Mine 4	0.5

In each year it is necessary to combine the total outputs from each mine to produce a blended ore of exactly some stipulated quality. For each year these qualities are

Year 1	0.9
Year 2	0.8
Year 3	1.2
Year 4	0.6
Year 5	1.0

The final blended ore sells for £10 per ton each year. Revenue and expenditure for future years must be discounted at a rate of 10% per annum.

Which mines should be operated each year and how much should they produce?

Mathematical Programming Formulation

Index Sets and Their Members

The following index sets and their members are used in this example:

- mine ∈ MINES
- year ∈ YEARS

Parameters

Table 7.1 shows the parameters that are used in this example.

Table 7.1 Parameters

Parameter Name	Interpretation
cost[mine]	Yearly royalties payable on each mine kept open
extract_ub[mine]	Upper bound on amount of ore extracted from each mine in a year
quality[mine]	Quality of ore from each mine
quality_required[year]	Required quality of blended ore per year
max_num_worked_per_year	Maximum number of mines worked per year
revenue_per_ton	Revenue per ton of blended ore
discount_rate	Annual discount rate for future years
Extract[mine,year].ub	Upper bound on Extract[mine,year]
quality_sol[year]	Quality of blended ore per year

Variables

Table 7.2 shows the variables that are used in this example.

Table 7.2 Variables

Variable Name	Interpretation
IsOpen[mine,year]	1 if mine is open in that year; 0 otherwise
IsWorked[mine,year]	1 if mine is worked in that year; 0 otherwise
Extract[mine,year]	Amount of ore extracted per mine per year
ExtractedPerYear[year]	Amount of ore extracted per year

Objective

The objective is to maximize the following function, where TotalRevenue is a linear function of ExtractedPerYear, and TotalCost is a linear function of IsOpen:

$$\text{TotalProfit} = \text{TotalRevenue} - \text{TotalCost}$$

Constraints

The following constraints are used in this example:

- bounds on variables

- for year $\in$ YEARS,

$$\text{ExtractedPerYear[year]} = \sum_{\text{mine} \in \text{MINES}} \text{Extract[mine,year]}$$

- for mine $\in$ MINES and year $\in$ YEARS,

$$\text{Extract[mine,year]} \leq \textit{Extract[mine,year].ub} \cdot \text{IsWorked[mine,year]}$$

- for year $\in$ YEARS,

$$\sum_{\text{mine} \in \text{MINES}} \text{IsWorked[mine,year]} \leq \textit{max_num_worked_per_year}$$

- for mine $\in$ MINES and year $\in$ YEARS,

$$\text{IsWorked[mine,year]} \leq \text{IsOpen[mine,year]}$$

- for mine $\in$ MINES and year $\in$ YEARS $\setminus \{1\}$,

$$\text{IsOpen[mine,year]} \leq \text{IsOpen[mine,year} - 1\text{]}$$

- for year $\in$ YEARS,

$$\frac{\sum_{\text{mine} \in \text{MINES}} \textit{quality[mine]} \cdot \text{Extract[mine,year]}}{\text{ExtractedPerYear[year]}} = \textit{quality_required[year]}$$

Input Data

The following data sets and macro variables contain the input data that are used in this example:

```
data mine_data;
   input mine $ cost extract_ub quality;
   datalines;
mine1 5 2   1.0
mine2 4 2.5 0.7
mine3 4 1.3 1.5
mine4 5 3   0.5
;

data year_data;
   input year quality_required;
   datalines;
1 0.9
2 0.8
3 1.2
4 0.6
5 1.0
;

%let max_num_worked_per_year = 3;
%let revenue_per_ton = 10;
%let discount_rate = 0.10;
```

PROC OPTMODEL Statements and Output

The first several PROC OPTMODEL statements are straightforward declarations of index sets, parameters, and variables:

```
proc optmodel;
   set <str> MINES;
   num cost {MINES};
   num extract_ub {MINES};
   num quality {MINES};
   read data mine_data into MINES=[mine] cost extract_ub quality;

   set YEARS;
   num quality_required {YEARS};
   read data year_data into YEARS=[year] quality_required;

   var IsOpen {MINES, YEARS} binary;
   var IsWorked {MINES, YEARS} binary;
   var Extract {mine in MINES, YEARS} >= 0 <= extract_ub[mine];
```

The following IMPVAR statement declares ExtractedPerYear as an implicit variable. Using the IMPVAR statement is an alternative to using a VAR statement and an additional constraint.

```
   impvar ExtractedPerYear {year in YEARS}
      = sum {mine in MINES} Extract[mine,year];
```

The following NUM statement uses a formula to compute the discount (shown in Figure 7.1) that is used in the objective function:

```
   num discount {year in YEARS} = 1 / (1 + &discount_rate)^(year - 1);
   print discount;

   impvar TotalRevenue =
      &revenue_per_ton * sum {year in YEARS} discount[year] *
         ExtractedPerYear[year];
   impvar TotalCost =
      sum {mine in MINES, year in YEARS} discount[year] * cost[mine] *
         IsOpen[mine,year];
   max TotalProfit = TotalRevenue - TotalCost;
```

Figure 7.1 *discount* Parameter

The OPTMODEL Procedure

[1]	discount
1	1.00000
2	0.90909
3	0.82645
4	0.75131
5	0.68301

The following Link constraint enforces the rule that Extract[mine,year] > 0 implies that

IsWorked[mine,year] = 1 (as in Chapter 2):

```
con Link {mine in MINES, year in YEARS}:
   Extract[mine,year] <= Extract[mine,year].ub * IsWorked[mine,year];
```

The following Cardinality constraint enforces the limit on number of mines worked per year:

```
con Cardinality {year in YEARS}:
   sum {mine in MINES} IsWorked[mine,year] <= &max_num_worked_per_year;
```

The following Worked_implies_open constraint enforces the rule that IsWorked[mine,year] = 1 implies that IsOpen[mine,year] = 1:

```
con Worked_implies_open {mine in MINES, year in YEARS}:
   IsWorked[mine,year] <= IsOpen[mine,year];
```

The following Continuity constraint enforces the rule that IsOpen[mine,year] = 1 implies that IsOpen[mine,year − 1] = 1:

```
con Continuity {mine in MINES, year in YEARS diff {1}}:
   IsOpen[mine,year] <= IsOpen[mine,year-1];
```

As expressed on page 84, the quality of the blended ore for each year is a ratio of linear functions of the decision variables. The following CON statement linearizes the nonlinear ratio constraint by clearing the denominator:

```
con Quality_con {year in YEARS}:
   sum {mine in MINES} quality[mine] * Extract[mine,year]
 = quality_required[year] * ExtractedPerYear[year];
```

By using the `.sol` suffix, the numeric parameter *quality_sol* computes the quality of the blended ore from the optimal decision variable values that are returned by the solver:

```
num quality_sol {year in YEARS} =
   (sum {mine in MINES} quality[mine] * Extract[mine,year].sol) /
      ExtractedPerYear[year].sol;

solve;
print IsOpen IsWorked Extract;
print ExtractedPerYear quality_sol quality_required;
create data sol_data1 from [mine year] IsOpen IsWorked Extract;
create data sol_data2 from [year] ExtractedPerYear;
quit;
```

Figure 7.2 shows the output from the mixed integer linear programming solver. The values of *quality_sol* and *quality_required* agree, as enforced by the Quality_con constraint.

Figure 7.2 Output from Mixed Integer Linear Programming Solver

Problem Summary	
Objective Sense	Maximization
Objective Function	TotalProfit
Objective Type	Linear
Number of Variables	60
Bounded Above	0
Bounded Below	0
Bounded Below and Above	60
Free	0
Fixed	0
Binary	40
Integer	0
Number of Constraints	66
Linear LE (<=)	61
Linear EQ (=)	5
Linear GE (>=)	0
Linear Range	0
Constraint Coefficients	151

Performance Information	
Execution Mode	Single-Machine
Number of Threads	4

Solution Summary	
Solver	MILP
Algorithm	Branch and Cut
Objective Function	TotalProfit
Solution Status	Optimal
Objective Value	146.86197857
Relative Gap	0
Absolute Gap	0
Primal Infeasibility	3.108624E-15
Bound Infeasibility	3.108624E-15
Integer Infeasibility	1.5384592E-6
Best Bound	146.86197857
Nodes	1
Iterations	83
Presolve Time	0.03
Solution Time	0.05

Figure 7.2 *continued*

[1]	[2]	IsOpen	IsWorked	Extract
mine1	**1**	1	1	2.0000
mine1	**2**	1	0	0.0000
mine1	**3**	1	1	1.9500
mine1	**4**	1	1	0.1250
mine1	**5**	1	1	2.0000
mine2	**1**	1	0	0.0000
mine2	**2**	1	1	2.5000
mine2	**3**	1	0	0.0000
mine2	**4**	1	1	2.5000
mine2	**5**	1	1	2.1667
mine3	**1**	1	1	1.3000
mine3	**2**	1	1	1.3000
mine3	**3**	1	1	1.3000
mine3	**4**	1	0	0.0000
mine3	**5**	1	1	1.3000
mine4	**1**	1	1	2.4500
mine4	**2**	1	1	2.2000
mine4	**3**	1	0	0.0000
mine4	**4**	1	1	3.0000
mine4	**5**	0	0	0.0000

[1]	ExtractedPerYear	quality_sol	quality_required
1	5.7500	0.9	0.9
2	6.0000	0.8	0.8
3	3.2500	1.2	1.2
4	5.6250	0.6	0.6
5	5.4667	1.0	1.0

Features Demonstrated

The following features are demonstrated in this example:

- problem type: mixed integer linear programming
- numeric and string index sets
- calculated numeric parameter
- bounds in the VAR statement
- IMPVAR statement
- **`.ub`** variable suffix
- set operator DIFF
- linearizing a ratio constraint
- using a variable suffix (such as **`.sol`**) in the declaration of a numeric parameter
- multiple input and output data sets

Chapter 8
Farm Planning: How Much to Grow and Rear

Contents

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Problem Statement

A farmer wishes to plan production on his 200 acre farm over the next five years.[1]

At present he has a herd of 120 cows. This is made up of 20 heifers and 100 milk-producing cows. Each heifer needs 2/3 acre to support it and each dairy cow 1 acre. A dairy cow produces an average of 1.1 calves per year. Half of these calves will be bullocks which are sold almost immediately for an average of £30 each. The remaining heifers can be either sold almost immediately for £40 or reared to become milk-producing cows at two years old. It is intended that all dairy cows be sold at 12 years old for an average of £120 each, although there will probably be an annual loss of 5% per year among heifers and 2% among dairy cows. At present there are 10 cows of each age from newborn to 11 years old. The decision of how many heifers to sell in the current year has already been taken and implemented.

The milk from a cow yields an annual revenue of £370. A maximum of 130 cows can be housed at the present time. To provide accommodation for each cow beyond this number will entail a capital outlay of £200 per cow. Each milk-producing cow requires 0.6 tons of grain and 0.7 tons of sugar beet per year. Grain and sugar beet can both be grown on the farm. Each acre yields 1.5 tons of sugar beet. Only 80 acres are suitable for growing grain. They can be divided into four groups whose yields are as follows:

group 1	20 acres	1.1 tons per acre
group 2	30 acres	0.9 tons per acre
group 3	20 acres	0.8 tons per acre
group 4	10 acres	0.65 tons per acre

Grain can be bought for £90 per ton and sold for £75 per ton. Sugar beet can be bought for £70 per ton and sold for £58 per ton.

[1] Reproduced with permission of John Wiley & Sons Ltd. (Williams 1999, pp. 239–240).

The labour requirements are:

each heifer	10 hours per year
each milk-producing cow	42 hours per year
each acre put to grain	4 hours per year
each acre put to sugar beet	14 hours per year

Other costs are:

each heifer	£50 per year
each milk-producing cow	£100 per year
each acre put to grain	£15 per year
each acre put to sugar beet	£10 per year

Labour costs for the farm are at present £4000 per year and provide 5500 hours labour. Any labour needed above this will cost £1.20 per hour.

How should the farmer operate over the next five years to maximize profit? Any capital expenditure would be financed by a 10 year loan at 15% annual interest. The interest and capital repayment would be paid in 10 equally sized yearly instalments. In no year can the cash flow be negative. Lastly, the farmer would not wish to reduce the total number of dairy cows at the end of the five year period by more than 50% nor increase the number by more than 75%.

Mathematical Programming Formulation

Index Sets and Their Members

The following index sets and their members are used in this example:

- age $\in$ AGES
- year $\in$ YEARS
- year $\in$ YEARS0 $= \{0\} \cup$ YEARS
- group $\in$ GROUPS

Parameters

Table 8.1 shows the parameters that are used in this example.

Table 8.1 Parameters

Parameter Name	Interpretation
init_num_cows[age]	Initial number of cows of each age
acres_needed[age]	Number of acres needed to support a cow of each age
annual_loss[age]	Annual percentage loss per cow of each age
bullock_yield[age]	Number of bullocks yielded per year per cow of each age
heifer_yield[age]	Number of heifers yielded per year per cow of each age
milk_revenue[age]	Milk revenue per year per cow of each age
grain_req[age]	Tons of grain required per year per cow of each age
sugar_beet_req[age]	Tons of sugar beet required per year per cow of each age
cow_labour_req[age]	Number of labour hours required per year per cow of each age
cow_other_costs[age]	Other costs per year per cow of each age
acres[group]	Number of acres per group
grain_yield[group]	Tons of grain yielded per acre by each group
yearly_loan_payment	Amount of yearly loan payment per capital outlay
num_years	Number of years in planning horizon
num_acres	Number of acres in farm
bullock_revenue	Revenue for selling a bullock
heifer_revenue	Revenue for selling a heifer
dairy_cow_selling_age	Age by which a dairy cow is sold
dairy_cow_selling_revenue	Revenue for selling a dairy cow
max_num_cows	Maximum number of cows that can be housed at present
sugar_beet_yield	Tons of sugar beet yielded per acre
grain_cost	Cost per ton of grain bought
grain_revenue	Revenue per ton of grain sold
grain_labour_req	Number of labour hours required per year per acre of grain
grain_other_costs	Other costs per year per acre of grain
sugar_beet_cost	Cost per ton of sugar beet bought
sugar_beet_revenue	Revenue per ton of sugar beet sold
sugar_beet_labour_req	Number of labour hours required per year per acre of sugar beet
sugar_beet_other_costs	Other costs per year per acre of sugar beet
nominal_labour_cost	Present labour costs for farm per year
nominal_labour_hours	Maximum number of labour hours available without incurring additional labour costs
excess_labour_cost	Cost per hour of additional labour
capital_outlay_unit	Capital outlay per additional cow beyond *max_num_cows*
num_loan_years	Number of years for loan
annual_interest_rate	Annual interest rate for loan
max_decrease_ratio	Maximum relative decrease in total number of cows at end of planning horizon
max_increase_ratio	Maximum relative increase in total number of cows at end of planning horizon

Variables

Table 8.2 shows the variables that are used in this example.

Table 8.2 Variables

Variable Name	Interpretation
NumCows[age,year]	Number of cows of each age per year
NumBullocksSold[year]	Number of bullocks sold per year
NumHeifersSold[year]	Number of heifers sold per year
GrainAcres[group,year]	Acres of grain per group and year
GrainBought[year]	Tons of grain bought per year
GrainSold[year]	Tons of grain sold per year
SugarBeetAcres[year]	Acres of sugar beet per year
SugarBeetBought[year]	Tons of sugar beet bought per year
SugarBeetSold[year]	Tons of sugar beet sold per year
NumExcessLabourHours[year]	Number of additional labour hours per year
CapitalOutlay[year]	Number of capital outlays per year
GrainGrown[group,year]	Tons of grain grown per group and year
SugarBeetGrown[year]	Tons of sugar beet grown per year
Revenue[year]	Revenue per year
Cost[year]	Cost per year
Profit[year]	Profit per year

Objective

The objective is to maximize the following function, where Profit[year] = Revenue[year] − Cost[year] and Revenue and Cost are linear functions of other variables:

$$\text{TotalProfit} = \sum_{\text{year} \in \text{YEARS}} \left(\text{Profit[year]} - \textit{yearly_loan_payment} \cdot (\textit{num_years} - 1 + \text{year}) \cdot \text{CapitalOutlay[year]} \right)$$

Constraints

The following constraints are used in this example:

- bounds on variables

- for year $\in$ YEARS,

$$\sum_{\text{age}\in\text{AGES}} \textit{acres_needed[age]} \cdot \text{NumCows[age,year]}$$
$$+ \sum_{\text{group}\in\text{GROUPS}} \text{GrainAcres[group,year]} + \text{SugarBeetAcres[year]}$$
$$\leq \textit{num_acres}$$

- for age $\in$ AGES $\setminus$ $\{\textit{dairy_cow_selling_age}\}$ and year $\in$ YEARS0 $\setminus$ $\{\textit{num_years}\}$,

$$\text{NumCows[age} + 1, \text{year} + 1] = (1 - \textit{annual_loss[age]}) \cdot \text{NumCows[age,year]}$$

- for year $\in$ YEARS,

$$\text{NumBullocksSold[year]} = \sum_{\text{age}\in\text{AGES}} \textit{bullock_yield[age]} \cdot \text{NumCows[age,year]}$$

- for year $\in$ YEARS,

$$\text{NumCows[0,year]} = \sum_{\text{age}\in\text{AGES}} \textit{heifer_yield[age]} \cdot \text{NumCows[age,year]} - \text{NumHeifersSold[year]}$$

- for year $\in$ YEARS,

$$\sum_{\text{age}\in\text{AGES}} \text{NumCows[age,year]} \leq \textit{max_num_cows} + \sum_{\substack{\text{y}\in\text{YEARS:} \\ \text{y}\leq\text{year}}} \text{CapitalOutlay[y]}$$

- for group $\in$ GROUPS and year $\in$ YEARS,

$$\text{GrainGrown[group,year]} = \textit{grain_yield[group]} \cdot \text{GrainAcres[group,year]}$$

- for year $\in$ YEARS,

$$\sum_{\text{age}\in\text{AGES}} \textit{grain_req[age]} \cdot \text{NumCows[age,year]}$$
$$\leq \sum_{\text{group}\in\text{GROUPS}} \text{GrainGrown[group,year]} + \text{GrainBought[year]} - \text{GrainSold[year]}$$

- for year $\in$ YEARS,

$$\text{SugarBeetGrown[year]} = \textit{sugar_beet_yield} \cdot \text{SugarBeetAcres[year]}$$

- for year $\in$ YEARS,

$$\sum_{\text{age}\in\text{AGES}} \textit{sugar_beet_req[age]} \cdot \text{NumCows[age,year]}$$
$$\leq \text{SugarBeetGrown[year]} + \text{SugarBeetBought[year]} - \text{SugarBeetSold[year]}$$

- for year $\in$ YEARS,

$$\sum_{\text{age}\in\text{AGES}} cow_labour_req[age] \cdot \text{NumCows[age,year]}$$
$$+ \sum_{\text{group}\in\text{GROUPS}} grain_labour_req \cdot \text{GrainAcres[group,year]}$$
$$+ sugar_beet_labour_req \cdot \text{SugarBeetAcres[year]}$$
$$\leq nominal_labour_hours + \text{NumExcessLabourHours[year]}$$

- for year $\in$ YEARS,

$$\text{Profit[year]} \geq 0$$

- $1 - max_decrease_ratio \leq \dfrac{\sum_{\substack{\text{age}\in\text{AGES:}\\ \text{age}\geq 2}} \text{NumCows[age,num_years]}}{\sum_{\substack{\text{age}\in\text{AGES:}\\ \text{age}\geq 2}} init_num_cows[age]} \leq 1 + max_increase_ratio$

Input Data

The following data sets and macro variables contain the input data that are used in this example:

```
data cow_data;
   do age = 0 to 11;
      init_num_cows = 10;
      if age < 2 then do;
         acres_needed = 2/3;
         annual_loss = 0.05;
         bullock_yield = 0;
         heifer_yield = 0;
         milk_revenue = 0;
         grain_req = 0;
         sugar_beet_req = 0;
         labour_req = 10;
         other_costs = 50;
      end;
      else do;
         acres_needed = 1;
         annual_loss = 0.02;
         bullock_yield = 1.1/2;
         heifer_yield = 1.1/2;
         milk_revenue = 370;
         grain_req = 0.6;
         sugar_beet_req = 0.7;
         labour_req = 42;
         other_costs = 100;
      end;
      output;
   end;
run;
```

```
data grain_data;
   input group $ acres yield;
   datalines;
group1 20 1.1
group2 30 0.9
group3 20 0.8
group4 10 0.65
;

%let num_years = 5;
%let num_acres = 200;
%let bullock_revenue = 30;
%let heifer_revenue = 40;
%let dairy_cow_selling_age = 12;
%let dairy_cow_selling_revenue = 120;
%let max_num_cows = 130;
%let sugar_beet_yield = 1.5;
%let grain_cost = 90;
%let grain_revenue = 75;
%let grain_labour_req = 4;
%let grain_other_costs = 15;
%let sugar_beet_cost = 70;
%let sugar_beet_revenue = 58;
%let sugar_beet_labour_req = 14;
%let sugar_beet_other_costs = 10;
%let nominal_labour_cost = 4000;
%let nominal_labour_hours = 5500;
%let excess_labour_cost = 1.2;
%let capital_outlay_unit = 200;
%let num_loan_years = 10;
%let annual_interest_rate = 0.15;
%let max_decrease_ratio = 0.50;
%let max_increase_ratio = 0.75;
```

PROC OPTMODEL Statements and Output

The first several PROC OPTMODEL statements are straightforward declarations of index sets, parameters, and variables:

```
proc optmodel;
   set AGES;
   num init_num_cows {AGES};
   num acres_needed {AGES};
   num annual_loss {AGES};
   num bullock_yield {AGES};
   num heifer_yield {AGES};
   num milk_revenue {AGES};
   num grain_req {AGES};
   num sugar_beet_req {AGES};
   num cow_labour_req {AGES};
   num cow_other_costs {AGES};
```

```
read data cow_data into AGES=[age]
   init_num_cows acres_needed annual_loss bullock_yield heifer_yield
   milk_revenue grain_req sugar_beet_req cow_labour_req=labour_req
   cow_other_costs=other_costs;

num num_years = &num_years;
set YEARS = 1..num_years;
set YEARS0 = {0} union YEARS;

var NumCows {AGES union {&dairy_cow_selling_age}, YEARS0} >= 0;
for {age in AGES} fix NumCows[age,0] = init_num_cows[age];
fix NumCows[&dairy_cow_selling_age,0] = 0;

var NumBullocksSold {YEARS} >= 0;
var NumHeifersSold {YEARS} >= 0;

set <str> GROUPS;
num acres {GROUPS};
num grain_yield {GROUPS};
var GrainAcres {GROUPS, YEARS} >= 0;
```

In Chapter 6, the READ DATA statement was used to populate the `.ub` suffix for a one-dimensional variable. The following READ DATA statement populates `.ub` for the two-dimensional variable GrainAcres:

```
read data grain_data into GROUPS=[group]
   {year in YEARS} <GrainAcres[group,year].ub=acres>
   grain_yield=yield;
var GrainBought {YEARS} >= 0;
var GrainSold {YEARS} >= 0;

var SugarBeetAcres {YEARS} >= 0;
var SugarBeetBought {YEARS} >= 0;
var SugarBeetSold {YEARS} >= 0;

var NumExcessLabourHours {YEARS} >= 0;
var CapitalOutlay {YEARS} >= 0;
```

The following NUM statement uses the FINANCE function to calculate the yearly loan payment per capital outlay:

```
num yearly_loan_payment =
   -finance('pmt', &annual_interest_rate, &num_loan_years,
      &capital_outlay_unit);
print yearly_loan_payment;
```

The resulting value of *yearly_loan_payment* (shown in Figure 8.1) differs from the value given in Williams (1999), perhaps because of rounding in intermediate calculations by Williams. The formula

$$yearly_loan_payment = \frac{annual_interest_rate \cdot capital_outlay_unit}{1 - (1 + annual_interest_rate)^{-num_loan_years}}$$

yields the same value as the FINANCE function. For the given input data, it turns out that the optimal solution has no capital outlay and agrees with the solution reported in Williams (1999).

Figure 8.1 *yearly_loan_payment* Parameter

The OPTMODEL Procedure

yearly_loan_payment
39.85

The following IMPVAR statements declare Revenue, Cost, and Profit as linear functions of the decision variables:

```
impvar Revenue {year in YEARS} =
   &bullock_revenue * NumBullocksSold[year]
 + &heifer_revenue * NumHeifersSold[year]
 + &dairy_cow_selling_revenue * NumCows[&dairy_cow_selling_age,year]
 + sum {age in AGES} milk_revenue[age] * NumCows[age,year]
 + &grain_revenue * GrainSold[year]
 + &sugar_beet_revenue * SugarBeetSold[year]
;
impvar Cost {year in YEARS} =
   &grain_cost * GrainBought[year]
 + &sugar_beet_cost * SugarBeetBought[year]
 + &nominal_labour_cost
 + &excess_labour_cost * NumExcessLabourHours[year]
 + sum {age in AGES} cow_other_costs[age] * NumCows[age,year]
 + sum {group in GROUPS} &grain_other_costs * GrainAcres[group,year]
 + &sugar_beet_other_costs * SugarBeetAcres[year]
 + sum {y in YEARS: y <= year} yearly_loan_payment * CapitalOutlay[y]
;
impvar Profit {year in YEARS} = Revenue[year] - Cost[year];
```

The following objective declaration accounts for loan repayments beyond the planning horizon, as described in Williams (1999):

```
max TotalProfit =
   sum {year in YEARS} (Profit[year]
      - yearly_loan_payment * (num_years - 1 + year) * CapitalOutlay[year]);
```

The following model declaration statements are straightforward:

```
con Num_acres_con {year in YEARS}:
   sum {age in AGES} acres_needed[age] * NumCows[age,year]
 + sum {group in GROUPS} GrainAcres[group,year]
 + SugarBeetAcres[year]
<= &num_acres;

con Aging {age in AGES diff {&dairy_cow_selling_age},
           year in YEARS0 diff {num_years}}:
   NumCows[age+1,year+1] = (1 - annual_loss[age]) * NumCows[age,year];

con NumBullocksSold_def {year in YEARS}:
   NumBullocksSold[year]
 = sum {age in AGES} bullock_yield[age] * NumCows[age,year];

con NumHeifersSold_def {year in YEARS}:
```

```
   NumCows[0,year]
 = sum {age in AGES} heifer_yield[age] * NumCows[age,year]
 - NumHeifersSold[year];

con Max_num_cows_def {year in YEARS}:
   sum {age in AGES} NumCows[age,year]
<= &max_num_cows + sum {y in YEARS: y <= year} CapitalOutlay[y];

impvar GrainGrown {group in GROUPS, year in YEARS} =
   grain_yield[group] * GrainAcres[group,year];
con Grain_req_def {year in YEARS}:
   sum {age in AGES} grain_req[age] * NumCows[age,year]
<= sum {group in GROUPS} GrainGrown[group,year]
 + GrainBought[year] - GrainSold[year];

impvar SugarBeetGrown {year in YEARS} =
   &sugar_beet_yield * SugarBeetAcres[year];
con Sugar_beet_req_def {year in YEARS}:
   sum {age in AGES} sugar_beet_req[age] * NumCows[age,year]
<= SugarBeetGrown[year] + SugarBeetBought[year] - SugarBeetSold[year];

con Labour_req_def {year in YEARS}:
   sum {age in AGES} cow_labour_req[age] * NumCows[age,year]
 + sum {group in GROUPS} &grain_labour_req * GrainAcres[group,year]
 + &sugar_beet_labour_req * SugarBeetAcres[year]
<= &nominal_labour_hours + NumExcessLabourHours[year];

con Cash_flow {year in YEARS}:
   Profit[year] >= 0;
```

The following CON statement declares a range constraint:

```
con Final_dairy_cows_range:
   1 - &max_decrease_ratio
<= (sum {age in AGES: age >= 2} NumCows[age,num_years])
 / (sum {age in AGES: age >= 2} init_num_cows[age])
<= 1 + &max_increase_ratio;

solve;

print NumCows NumBullocksSold NumHeifersSold CapitalOutlay
   NumExcessLabourHours Revenue Cost Profit;
print GrainAcres;
print GrainGrown;
print GrainBought GrainSold SugarBeetAcres SugarBeetGrown SugarBeetBought
   SugarBeetSold;
print Num_acres_con.body Max_num_cows_def.body Final_dairy_cows_range.body;
```

The following CREATE DATA statement writes dense two-dimensional data to a data set:

```
create data sol_data1 from [age]=AGES
   {year in YEARS} <col('NumCows_year'||year)=NumCows[age,year].sol>;
```

You could instead use the following CREATE DATA statement to interchange the roles of *age* and *year*:

```
create data sol_data1 from [year]=YEARS
   {age in AGES} <col('NumCows_age'||age)=NumCows[age,year].sol>;
```

The following CREATE DATA statement writes sparse two-dimensional data to a data set:

```
create data sol_data2 from [group year] GrainAcres GrainGrown;
```

The final CREATE DATA statement writes one-dimensional data to a data set, as in previous examples:

```
create data sol_data3 from [year]
   NumBullocksSold NumHeifersSold CapitalOutlay NumExcessLabourHours
   Revenue Cost Profit GrainBought GrainSold
   SugarBeetAcres SugarBeetGrown SugarBeetBought SugarBeetSold;
quit;
```

Figure 8.2 shows the output from the linear programming solver.

Figure 8.2 Output from Linear Programming Solver

Problem Summary	
Objective Sense	Maximization
Objective Function	TotalProfit
Objective Type	Linear
Number of Variables	143
Bounded Above	0
Bounded Below	110
Bounded Below and Above	20
Free	0
Fixed	13
Number of Constraints	101
Linear LE (<=)	25
Linear EQ (=)	70
Linear GE (>=)	5
Linear Range	1
Constraint Coefficients	780

Performance Information	
Execution Mode	Single-Machine
Number of Threads	1

Figure 8.2 *continued*

Solution Summary	
Solver	LP
Algorithm	Dual Simplex
Objective Function	TotalProfit
Solution Status	Optimal
Objective Value	121719.17286
Primal Infeasibility	1.818989E-12
Dual Infeasibility	0
Bound Infeasibility	0
Iterations	55
Presolve Time	0.00
Solution Time	0.02

NumCows						
	0	**1**	**2**	**3**	**4**	**5**
0	10.0000	22.8000	11.5844	0.0000	0.0000	0.0000
1	10.0000	9.5000	21.6600	11.0052	0.0000	0.0000
2	10.0000	9.5000	9.0250	20.5770	10.4549	0.0000
3	10.0000	9.8000	9.3100	8.8445	20.1655	10.2458
4	10.0000	9.8000	9.6040	9.1238	8.6676	19.7622
5	10.0000	9.8000	9.6040	9.4119	8.9413	8.4943
6	10.0000	9.8000	9.6040	9.4119	9.2237	8.7625
7	10.0000	9.8000	9.6040	9.4119	9.2237	9.0392
8	10.0000	9.8000	9.6040	9.4119	9.2237	9.0392
9	10.0000	9.8000	9.6040	9.4119	9.2237	9.0392
10	10.0000	9.8000	9.6040	9.4119	9.2237	9.0392
11	10.0000	9.8000	9.6040	9.4119	9.2237	9.0392
12	0.0000	9.8000	9.6040	9.4119	9.2237	9.0392

[1]	NumBullocksSold	NumHeifersSold	CapitalOutlay	NumExcessLabourHours	Revenue	Cost	Profit
1	53.735	30.935	0	0	41495	19588	21906
2	52.342	40.757	0	0	41153	19265	21889
3	57.436	57.436	0	0	45212	19396	25816
4	56.964	56.964	0	0	45860	19034	26826
5	50.853	50.853	0	0	42717	17434	25283

GrainAcres					
	1	**2**	**3**	**4**	**5**
group1	20.0000	20.0000	20.0000	20.0000	20.0000
group2	0.0000	0.0000	3.1342	0.0000	0.0000
group3	0.0000	0.0000	0.0000	0.0000	0.0000
group4	0.0000	0.0000	0.0000	0.0000	0.0000

Figure 8.2 *continued*

	GrainGrown				
	1	2	3	4	5
group1	22.0000	22.0000	22.0000	22.0000	22.0000
group2	0.0000	0.0000	2.8207	0.0000	0.0000
group3	0.0000	0.0000	0.0000	0.0000	0.0000
group4	0.0000	0.0000	0.0000	0.0000	0.0000

[1]	GrainBought	GrainSold	SugarBeetAcres	SugarBeetGrown	SugarBeetBought	SugarBeetSold
1	36.620	0	60.767	91.150	0	22.760
2	35.100	0	62.670	94.005	0	27.388
3	37.837	0	65.100	97.650	0	24.550
4	40.143	0	76.429	114.643	0	42.143
5	33.476	0	87.539	131.309	0	66.586

[1]	Num_acres_con.BODY	Max_num_cows_def.BODY
1	200	130.000
2	200	128.411
3	200	115.434
4	200	103.571
5	200	92.461

Final_dairy_cows_range.BODY
0.92461

Features Demonstrated

The following features are demonstrated in this example:

- problem type: linear programming
- numeric and string index sets
- bounds in the VAR statement
- FIX statement
- IMPVAR statement
- `.ub` variable suffix
- set operators UNION and DIFF
- using a colon (:) to select members of a set
- FINANCE function
- range constraint
- multiple input and output data sets
- writing dense two-dimensional data
- writing sparse two-dimensional data

Chapter 9
Economic Planning: How Should an Economy Grow

Contents

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Problem Statement

An economy consists of three industries: coal, steel and transport.[1] Each unit produced by one of the industries (a unit will be taken as £1's worth of value of production) requires inputs from possibly its own industry as well as other industries. The required inputs as well as the manpower requirements (also measured in £) are given in Table 9.1. There is a time lag in the economy so that output in year $t + 1$ requires an input in year t.

Output from an industry may also be used to build productive capacity for itself or other industries in future years. The inputs required to give unit increases (capacity for £1's worth of extra production) in productive capacity are given in Table 9.2. Input from an industry in year t results in a (permanent) increase in productive capacity in year $t + 2$.

Table 9.1

	Outputs (year $t + 1$), production		
Inputs (year t)	**Coal**	**Steel**	**Transport**
Coal	0.1	0.5	0.4
Steel	0.1	0.1	0.2
Transport	0.2	0.1	0.2
Manpower	0.6	0.3	0.2

[1] Reproduced with permission of John Wiley & Sons Ltd. (Williams 1999, pp. 240–241).

Table 9.2

	Outputs (year $t+2$), productive capacity		
Inputs (year *t*)	**Coal**	**Steel**	**Transport**
Coal	0.0	0.7	0.9
Steel	0.1	0.1	0.2
Transport	0.2	0.1	0.2
Manpower	0.4	0.2	0.1

Table 9.3

	Year 0	
	Stocks	**Productive capacity**
Coal	150	300
Steel	80	350
Transport	100	280

Stocks of goods may be held from year to year. At present (year 0) the stocks and productive capacities (per year) are given in Table 9.3 (in £m). There is a limited yearly manpower capacity of £470m.

It is wished to investigate different possible growth patterns for the economy over the next five years. In particular it is desirable to know the growth patterns which would result from pursuing the following objectives:

(i) Maximizing total productive capacity at the end of the five years while meeting an exogenous consumption requirement of £60m of coal, £60m of steel, and £30m of transport in every year (apart from year 0).

(ii) Maximizing total production (rather than productive capacity) in the fourth and fifth years, but ignoring exogenous demand in each year.

(iii) Maximizing the total manpower requirement (ignoring the manpower capacity limitation) over the period while meeting the yearly exogenous demands of (i).

Mathematical Programming Formulation

Index Sets and Their Members

The following index sets and their members are used in this example:

- year $\in$ YEARS

- year $\in$ YEARS0 $= \{0\} \cup$ YEARS
- $i, j \in$ INDUSTRIES
- $i \in$ INPUTS

Parameters

Table 9.4 shows the parameters that are used in this example.

Table 9.4 Parameters

Parameter Name	Interpretation
num_years	Number of years in planning horizon
init_stocks[i]	Initial stocks per industry (in £m)
init_productive_capacity[i]	Initial productive capacity per industry (in £m)
demand[i]	Yearly exogenous demand per industry (in £m) for years 1 through *num_years*
final_demand[i]	Yearly exogenous demand per industry (in £m) for years *num_years* + 1 and later
production_coeff[i,j]	Production coefficient for input *i* and industry *j*
productive_capacity_coeff[i,j]	Productive capacity coefficient for input *i* and industry *j*
manpower_capacity	Yearly manpower capacity (in £m)

Variables

Table 9.5 shows the variables that are used in this example.

Table 9.5 Variables

Variable Name	Interpretation
StaticProduction[i]	Static production of industry *i*
Production[i,year]	Production of industry *i* per year
Stock[i,year]	Stock level of industry *i* at beginning of each year
ExtraCapacity[i,year]	Incremental productive capacity of industry *i* available per year
ProductiveCapacity[i,year]	Permanent productive capacity of industry *i* available per year

Objectives

The first objective is to maximize the following function:

$$\text{TotalProductiveCapacity} = \sum_{i \in \text{INDUSTRIES}} \text{ProductiveCapacity[i,num_years]}$$

The second objective is to maximize the following function:

$$\text{TotalProduction} = \sum_{i \in \text{INDUSTRIES}} \sum_{\text{year}=4}^{5} \text{Production[i,year]}$$

The third objective is to maximize the following function:

$$\text{NumDamagedCarsRepaired[i,day]} \leq \left(\textit{repair_capacity[i]} + \sum_{\substack{\text{expansion} \in \text{EXPANSIONS:} \\ \textit{expansion_depot[expansion]}=i}} \textit{expansion_amount[expansion]} \cdot \text{ExpandCapacity[expansion]} \right)$$

Constraints

The following constraints are used in this example:

- bounds on variables
- for $i \in$ INDUSTRIES,

  $$\text{StaticProduction[i]} = \textit{demand[i]} + \sum_{j \in \text{INDUSTRIES}} \textit{production_coeff[i,j]} \cdot \text{StaticProduction[j]}$$

- for $i \in$ INDUSTRIES and year $\in \{1, \ldots, \textit{num_years} + 1\}$,

  $$\text{ProductiveCapacity[i,year]} = \textit{init_productive_capacity[i]} + \sum_{y=2}^{\text{year}} \text{ExtraCapacity[i,y]}$$

- for $i \in$ INDUSTRIES and year $\in$ YEARS0,

  $$\begin{aligned} &\text{Stock[i,year]} + \text{Production[i,year]} \\ &= (\text{if year } \in \text{YEARS then } \textit{demand[i]} \text{ else } 0) \\ &+ \sum_{j \in \text{INDUSTRIES}} \textit{production_coeff[i,j]} \cdot \text{Production[j,year} + 1] \\ &+ \sum_{j \in \text{INDUSTRIES}} \textit{productive_capacity_coeff[i,j]} \cdot \text{ExtraCapacity[j,year} + 2] \\ &+ \text{Stock[i,year} + 1] \end{aligned}$$

- for year $\in \{1, \ldots, \textit{num_years} + 1\}$,

  $$\begin{aligned} &\sum_{j \in \text{INDUSTRIES}} \textit{production_coeff['manpower',j]} \cdot \text{Production[j,year]} \\ &+ \sum_{j \in \text{INDUSTRIES}} \textit{productive_capacity_coeff['manpower',j]} \cdot \text{ExtraCapacity[j,year} + 1] \\ &\leq \textit{manpower_capacity} \end{aligned}$$

- for $i \in$ INDUSTRIES and year $\in \{1, \ldots, \textit{num_years} + 1\}$,

  $$\text{Production[i,year]} \leq \text{ProductiveCapacity[i,year]}$$

Input Data

The following data sets and macro variables contain the input data that are used in this example:

```
data industry_data;
   input industry $9. init_stocks init_productive_capacity demand;
   datalines;
coal      150 300 60
steel      80 350 60
transport 100 280 30
;

data production_data;
   input input $9. coal steel transport;
   datalines;
coal      0.1 0.5 0.4
steel     0.1 0.1 0.2
transport 0.2 0.1 0.2
manpower  0.6 0.3 0.2
;

data productive_capacity_data;
   input input $9. coal steel transport;
   datalines;
coal      0.0 0.7 0.9
steel     0.1 0.1 0.2
transport 0.2 0.1 0.2
manpower  0.4 0.2 0.1
;

%let manpower_capacity = 470;
%let num_years = 5;
```

PROC OPTMODEL Statements and Output

The following PROC OPTMODEL statements declare index sets and parameters and then read the input data:

```
proc optmodel;
   num num_years = &num_years;
   set YEARS = 1..num_years;
   set YEARS0 = {0} union YEARS;

   set <str> INDUSTRIES;
   num init_stocks {INDUSTRIES};
   num init_productive_capacity {INDUSTRIES};
   num demand {INDUSTRIES};
   read data industry_data into INDUSTRIES=[industry]
      init_stocks init_productive_capacity demand;
```

```
set <str> INPUTS;
num production_coeff {INPUTS, INDUSTRIES};
read data production_data into INPUTS=[input]
   {j in INDUSTRIES} <production_coeff[input,j]=col(j)>;

num productive_capacity_coeff {INPUTS, INDUSTRIES};
read data productive_capacity_data into INPUTS=[input]
   {j in INDUSTRIES} <productive_capacity_coeff[input,j]=col(j)>;
```

The following PROC OPTMODEL statements declare variables, a constant objective, and constraints for the static Leontief input-output model that is described on page 282 of Williams (1999):

```
var StaticProduction {INDUSTRIES} >= 0;
min Zero = 0;
con Static_con {i in INDUSTRIES}:
   StaticProduction[i]
 = demand[i] + sum {j in INDUSTRIES} production_coeff[i,j] *
   StaticProduction[j];
```

The following SOLVE statement invokes the linear programming solver to solve this system of equations:

```
solve;
print StaticProduction;
```

Figure 9.1 shows the output from the linear programming solver for the static model.

Figure 9.1 Output from Linear Programming Solver for Static Leontief Input-Output Model

The OPTMODEL Procedure

Problem Summary	
Objective Sense	Minimization
Objective Function	Zero
Objective Type	Constant
Number of Variables	3
Bounded Above	0
Bounded Below	3
Bounded Below and Above	0
Free	0
Fixed	0
Number of Constraints	3
Linear LE (<=)	0
Linear EQ (=)	3
Linear GE (>=)	0
Linear Range	0
Constraint Coefficients	9

Performance Information	
Execution Mode	Single-Machine
Number of Threads	1

Figure 9.1 *continued*

Solution Summary	
Solver	LP
Algorithm	Dual Simplex
Objective Function	Zero
Solution Status	Optimal
Objective Value	0
Primal Infeasibility	1.421085E-14
Dual Infeasibility	0
Bound Infeasibility	0
Iterations	0
Presolve Time	0.00
Solution Time	0.00

[1]	StaticProduction
coal	166.397
steel	105.668
transport	92.308

The following NUM and assignment statements declare the *final_demand* parameter and use the **.sol** variable suffix to populate *final_demand* with the solution from the static model:

```
num final_demand {INDUSTRIES};
for {i in INDUSTRIES} final_demand[i] = StaticProduction[i].sol;
```

The following statements declare variables, implicit variables, objectives, and constraints to be used in all three parts of the business problem:

```
var Production {INDUSTRIES, 0..num_years+1} >= 0;
var Stock {INDUSTRIES, 0..num_years+1} >= 0;
var ExtraCapacity {INDUSTRIES, 1..num_years+2} >= 0;
impvar ProductiveCapacity {i in INDUSTRIES, year in 1..num_years+1} =
   init_productive_capacity[i] + sum {y in 2..year} ExtraCapacity[i,y];
for {i in INDUSTRIES} do;
   Production[i,0].ub = 0;
   Stock[i,0].lb = init_stocks[i];
   Stock[i,0].ub = init_stocks[i];
end;

max TotalProductiveCapacity =
   sum {i in INDUSTRIES} ProductiveCapacity[i,num_years];
max TotalProduction =
   sum {i in INDUSTRIES, year in 4..5} Production[i,year];
max TotalManpower =
   sum {i in INDUSTRIES, year in YEARS} (
      production_coeff['manpower',i] * Production[i,year+1]
    + productive_capacity_coeff['manpower',i] * ExtraCapacity[i,year+2]);
```

```
con Continuity_con {i in INDUSTRIES, year in YEARS0}:
   Stock[i,year] + Production[i,year]
 = (if year in YEARS then demand[i] else 0)
 + sum {j in INDUSTRIES} (
      production_coeff[i,j] * Production[j,year+1]
    + productive_capacity_coeff[i,j] * ExtraCapacity[j,year+2])
 + Stock[i,year+1];

con Manpower_con {year in 1..num_years+1}:
   sum {j in INDUSTRIES} (
      production_coeff['manpower',j] * Production[j,year]
    + productive_capacity_coeff['manpower',j] * ExtraCapacity[j,year+1])
 <= &manpower_capacity;

con Capacity_con {i in INDUSTRIES, year in 1..num_years+1}:
   Production[i,year] <= ProductiveCapacity[i,year];

for {i in INDUSTRIES}
   Production[i,num_years+1].lb = final_demand[i];

for {i in INDUSTRIES, year in num_years+1..num_years+2}
   ExtraCapacity[i,year].ub = 0;
```

The following PROBLEM statement specifies the variables, objective, and constraints for Problem1:

```
problem Problem1 include
   Production Stock ExtraCapacity
   TotalProductiveCapacity
   Continuity_con Manpower_con Capacity_con;
```

Because Problem1 and Problem2 have the same variables and constraints, the following PROBLEM statement uses the FROM option to copy these common parts from Problem1 and uses the INCLUDE option to specify only the new objective:

```
problem Problem2 from Problem1 include
   TotalProduction;
```

The following PROBLEM statement specifies the variables, objective, and constraints for Problem3 (omitting the Manpower_con constraint):

```
problem Problem3 include
   Production Stock ExtraCapacity
   TotalManpower
   Continuity_con Capacity_con;
```

Note that implicit variables are not included in the PROBLEM statements. Including them would yield an ERROR message.

The following USE PROBLEM statement switches the focus to the desired problem:

```
use problem Problem1;
solve;
print Production Stock ExtraCapacity ProductiveCapacity Manpower_con.body;
```

Figure 9.2 shows the output from the linear programming solver for Problem1.

Figure 9.2 Output from Linear Programming Solver for Problem1

Problem Summary	
Objective Sense	Maximization
Objective Function	TotalProductiveCapacity
Objective Type	Linear
Number of Variables	63
Bounded Above	0
Bounded Below	51
Bounded Below and Above	0
Free	0
Fixed	12
Number of Constraints	42
Linear LE (<=)	24
Linear EQ (=)	18
Linear GE (>=)	0
Linear Range	0
Constraint Coefficients	255

Performance Information	
Execution Mode	Single-Machine
Number of Threads	1

Solution Summary	
Solver	LP
Algorithm	Dual Simplex
Objective Function	TotalProductiveCapacity
Solution Status	Optimal
Objective Value	2141.8751967
Primal Infeasibility	7.673862E-13
Dual Infeasibility	0
Bound Infeasibility	7.105427E-15
Iterations	38
Presolve Time	0.00
Solution Time	0.00

Figure 9.2 *continued*

[1]	[2]	Production	Stock	ExtraCapacity	ProductiveCapacity
coal	0	0.000	150.0000		
coal	1	260.403	0.0000	0.0	300.0
coal	2	293.406	0.0000	0.0	300.0
coal	3	300.000	0.0000	0.0	300.0
coal	4	17.949	148.4480	189.2	489.2
coal	5	166.397	0.0000	1022.7	1511.9
coal	6	166.397	0.0000	0.0	1511.9
coal	7			0.0	
steel	0	0.000	80.0000		
steel	1	135.342	12.2811	0.0	350.0
steel	2	181.660	0.0000	0.0	350.0
steel	3	193.090	0.0000	0.0	350.0
steel	4	105.668	0.0000	0.0	350.0
steel	5	105.668	0.0000	0.0	350.0
steel	6	105.668	0.0000	0.0	350.0
steel	7			0.0	
transport	0	0.000	100.0000		
transport	1	140.722	6.2408	0.0	280.0
transport	2	200.580	0.0000	0.0	280.0
transport	3	267.152	0.0000	0.0	280.0
transport	4	92.308	0.0000	0.0	280.0
transport	5	92.308	0.0000	0.0	280.0
transport	6	92.308	-0.0000	0.0	280.0
transport	7			0.0	

[1]	Manpower_con.BODY
1	224.99
2	270.66
3	367.04
4	470.00
5	150.00
6	150.00

For Problem2, the right-hand side of each Continuity_con constraint changes to 0:

```
use problem Problem2;
for {i in INDUSTRIES, year in YEARS} do;
   Continuity_con[i,year].lb = 0;
   Continuity_con[i,year].ub = 0;
end;
solve;
print Production Stock ExtraCapacity ProductiveCapacity Manpower_con.body;
```

Figure 9.3 shows the output from the linear programming solver for Problem2.

Figure 9.3 Output from Linear Programming Solver for Problem2

Problem Summary	
Objective Sense	Maximization
Objective Function	TotalProduction
Objective Type	Linear
Number of Variables	63
Bounded Above	0
Bounded Below	51
Bounded Below and Above	0
Free	0
Fixed	12
Number of Constraints	42
Linear LE (<=)	24
Linear EQ (=)	18
Linear GE (>=)	0
Linear Range	0
Constraint Coefficients	255

Performance Information	
Execution Mode	Single-Machine
Number of Threads	1

Solution Summary	
Solver	LP
Algorithm	Dual Simplex
Objective Function	TotalProduction
Solution Status	Optimal
Objective Value	2618.5791147
Primal Infeasibility	2.273737E-13
Dual Infeasibility	0
Bound Infeasibility	0
Iterations	46
Presolve Time	0.00
Solution Time	0.00

Figure 9.3 *continued*

[1]	[2]	Production	Stock	ExtraCapacity	ProductiveCapacity
coal	0	0.000	150.000		
coal	1	184.818	31.629	0.0000	300.00
coal	2	430.505	16.372	130.5047	430.50
coal	3	430.505	0.000	0.0000	430.50
coal	4	430.505	0.000	0.0000	430.50
coal	5	430.505	0.000	0.0000	430.50
coal	6	166.397	324.108	0.0000	430.50
coal	7			0.0000	
steel	0	0.000	80.000		
steel	1	86.730	11.532	0.0000	350.00
steel	2	155.337	0.000	0.0000	350.00
steel	3	182.867	0.000	0.0000	350.00
steel	4	359.402	0.000	9.4023	359.40
steel	5	359.402	176.535	0.0000	359.40
steel	6	105.668	490.269	0.0000	359.40
steel	7			0.0000	
transport	0	0.000	100.000		
transport	1	141.312	0.000	0.0000	280.00
transport	2	198.388	0.000	0.0000	280.00
transport	3	225.918	0.000	0.0000	280.00
transport	4	519.383	0.000	239.3826	519.38
transport	5	519.383	293.465	0.0000	519.38
transport	6	92.308	750.540	0.0000	519.38
transport	7			0.0000	

[1]	Manpower_con.BODY
1	217.37
2	344.58
3	384.17
4	470.00
5	470.00
6	150.00

For Problem3, the right-hand side of each Continuity_con[i,year] constraint changes back to *demand[i]*:

```
   use problem Problem3;
   for {i in INDUSTRIES, year in YEARS} do;
      Continuity_con[i,year].lb = demand[i];
      Continuity_con[i,year].ub = demand[i];
   end;
   solve;
   print Production Stock ExtraCapacity ProductiveCapacity Manpower_con.body;
quit;
```

Figure 9.4 shows the output from the linear programming solver for Problem3.

Figure 9.4 Output from Linear Programming Solver for Problem3

Problem Summary	
Objective Sense	Maximization
Objective Function	TotalManpower
Objective Type	Linear
Number of Variables	63
Bounded Above	0
Bounded Below	51
Bounded Below and Above	0
Free	0
Fixed	12
Number of Constraints	36
Linear LE (<=)	18
Linear EQ (=)	18
Linear GE (>=)	0
Linear Range	0
Constraint Coefficients	219

Performance Information	
Execution Mode	Single-Machine
Number of Threads	1

Solution Summary	
Solver	LP
Algorithm	Dual Simplex
Objective Function	TotalManpower
Solution Status	Optimal
Objective Value	2450.0266228
Primal Infeasibility	2.344791E-12
Dual Infeasibility	0
Bound Infeasibility	0
Iterations	50
Presolve Time	0.00
Solution Time	0.02

Figure 9.4 *continued*

[1]	[2]	Production	Stock	ExtraCapacity	ProductiveCapacity
coal	**0**	0.00	150.0000		
coal	**1**	251.79	0.0000	0.0000	300.00
coal	**2**	316.02	0.0000	16.0152	316.02
coal	**3**	319.83	0.0000	3.8168	319.83
coal	**4**	366.35	0.0000	46.5177	366.35
coal	**5**	859.36	0.0000	493.0099	859.36
coal	**6**	859.36	460.2080	0.0000	859.36
coal	**7**			0.0000	
steel	**0**	0.00	80.0000		
steel	**1**	134.79	11.0280	0.0000	350.00
steel	**2**	175.04	0.0000	0.0000	350.00
steel	**3**	224.06	0.0000	0.0000	350.00
steel	**4**	223.14	0.0000	0.0000	350.00
steel	**5**	220.04	0.0000	0.0000	350.00
steel	**6**	350.00	0.0000	0.0000	350.00
steel	**7**			0.0000	
transport	**0**	0.00	100.0000		
transport	**1**	143.56	4.2472	0.0000	280.00
transport	**2**	181.68	0.0000	0.0000	280.00
transport	**3**	280.00	0.0000	0.0000	280.00
transport	**4**	279.07	0.0000	0.0000	280.00
transport	**5**	275.98	0.0000	0.0000	280.00
transport	**6**	195.54	0.0000	0.0000	280.00
transport	**7**			0.0000	

[1]	Manpower_con.BODY
1	226.63
2	279.98
3	333.73
4	539.77
5	636.82
6	659.72

Features Demonstrated

The following features are demonstrated in this example:

- problem type: linear programming (dynamic Leontief input-output model)
- numeric and string index sets
- set operator UNION
- multiple input data sets
- reading dense two-dimensional data
- IMPVAR statement
- **`.lb`** and **`.ub`** variable suffixes
- IF-THEN/ELSE expression
- using the **`.lb`** and **`.ub`** constraint suffixes to modify the right-hand side of a constraint
- multiple objectives
- PROBLEM and USE PROBLEM statements
- **`.body`** constraint suffix

Chapter 10
Decentralization: How to Disperse Offices from the Capital

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Problem Statement

A large company wishes to move some of its departments out of London.[1] There are benefits to be derived from doing this (cheaper housing, government incentives, easier recruitment, etc.) which have been costed. Also, however, there will be greater costs of communication between departments. These have also been costed for all possible locations of each department.

Where should each department be located so as to minimize overall yearly cost?

The company comprises five departments (A, B, C, D, E). The possible cities for relocation are Bristol and Brighton, or a department may be kept in London. None of these cities (including London) may be the location for more than three of the departments.

Benefits to be derived from each relocation are given below (in thousands of pounds per year):

	A	B	C	D	E
Bristol	10	15	10	20	5
Brighton	10	20	15	15	15

Communication costs are of the form $C_{ik}D_{jl}$ where C_{ik} is the quantity of communication between departments i and k per year and D_{jl} is the cost per unit of communication between cities j and l. C_{ik} and D_{jl} are given by the tables below:

[1] Reproduced with permission of John Wiley & Sons Ltd. (Williams 1999, p. 242).

	Quantities of communication C_{ik} (in thousands of units)				
	A	B	C	D	E
A		0.0	1.0	1.5	0.0
B			1.4	1.2	0.0
C				0.0	2.0
D					0.7

	Costs per unit of communication D_{jl} (in £)		
	Bristol	Brighton	London
Bristol	5	14	13
Brighton		5	9
London			10

Mathematical Programming Formulation

Index Sets and Their Members

The following index sets and their members are used in this example:

- dept, $i, k \in$ DEPTS
- city, $j, l \in$ CITIES
- $(i, j, k, l) \in$ IJKL $= \{i \in \text{DEPTS},\ j \in \text{CITIES},\ k \in \text{DEPTS},\ l \in \text{CITIES} : i < k\}$

Parameters

Table 10.1 shows the parameters that are used in this example.

Table 10.1 Parameters

Parameter Name	Interpretation
benefit[i,j]	Yearly benefit derived from relocating department i to city j
comm[i,k]	Quantity of communication between departments i and k per year
cost[j,l]	Cost per unit of communication between cities j and l
max_num_depts	Upper bound on number of departments per city

Variables

Table 10.2 shows the variables that are used in this example.

Table 10.2 Variables

Variable Name	Interpretation
Assign[i,j]	1 if department i is assigned to city j; 0 otherwise
Product[i,j,k,l]	Assign[i,j] · Assign[k,l]
TotalBenefit	Total yearly benefit of all relocations
TotalCost	Total yearly communication costs of all relocations

Objective

The objective is to maximize the following quadratic function:

$$\text{NetBenefit} = \sum_{i \in \text{DEPTS}} \sum_{j \in \text{CITIES}} benefit[i,j] \cdot \text{Assign[i,j]} - \sum_{(i,j,k,l) \in \text{IJKL}} comm[i,k] \cdot cost[j,l] \cdot \text{Product[i,j,k,l]}$$

where

$$\text{Product[i,j,k,l]} = \text{Assign[i,j]} \cdot \text{Assign[k,l]}$$

Constraints

The following constraints are used in this example:

- bounds on variables
- $\text{TotalBenefit} = \sum_{i \in \text{DEPTS}} \sum_{j \in \text{CITIES}} benefit[i,j] \cdot \text{Assign[i,j]}$
- $\text{TotalCost} = \sum_{(i,j,k,l) \in \text{IJKL}} comm[i,k] \cdot cost[j,l] \cdot \text{Product[i,j,k,l]}$
- for dept $\in$ DEPTS,

  $$\sum_{\text{city} \in \text{CITIES}} \text{Assign[dept,city]} = 1$$

- for city $\in$ CITIES,

  $$\sum_{\text{dept} \in \text{DEPTS}} \text{Assign[dept,city]} \leq max_num_depts$$

- for $(i, j, k, l) \in$ IJKL,

  $$\text{Assign[i,j]} + \text{Assign[k,l]} - 1 \leq \text{Product[i,j,k,l]}$$

- for $(i, j, k, l) \in$ IJKL,

  $$\text{Product[i,j,k,l]} \leq \text{Assign[i,j]}$$

- for $(i, j, k, l) \in$ IJKL,

  $$\text{Product[i,j,k,l]} \leq \text{Assign[k,l]}$$

- for $i \in$ DEPTS and $k \in$ DEPTS and $l \in$ CITIES such that $i < k$,

  $$\sum_{(i,j,k,l) \in \text{IJKL}} \text{Product[i,j,k,l]} = \text{Assign[k,l]}$$

- for $k \in$ DEPTS and $i \in$ DEPTS and $j \in$ CITIES such that $i < k$,

  $$\sum_{(i,j,k,l) \in \text{IJKL}} \text{Product[i,j,k,l]} = \text{Assign[i,j]}$$

Input Data

The following data sets and macro variables contain the input data that are used in this example:

```
data dept_data;
   input dept $ @@;
   datalines;
A B C D E
;

data city_data;
   input city $;
   datalines;
Bristol
Brighton
London
;

data benefit_data;
   input city $ A B C D E;
   datalines;
Bristol  10 15 10 20  5
Brighton 10 20 15 15 15
;

data comm_data;
   input i $ j $ comm;
   datalines;
```

```
A B 0.0
A C 1.0
A D 1.5
A E 0.0
B C 1.4
B D 1.2
B E 0.0
C D 0.0
C E 2.0
D E 0.7
;

data cost_data;
   input i $ j $ cost;
   datalines;
Bristol  Bristol   5
Bristol  Brighton 14
Bristol  London   13
Brighton Brighton  5
Brighton London    9
London   London   10
;

%let max_num_depts = 3;
```

PROC OPTMODEL Statements and Output

The first two READ DATA statements populate the DEPTS and CITIES index sets:

```
proc optmodel;
   set <str> DEPTS;
   read data dept_data into DEPTS=[dept];

   set <str> CITIES;
   read data city_data into CITIES=[city];
```

The following READ DATA statement reads dense two-dimensional data, as in previous examples, with an initial value of 0 for *benefit* to account for the possibility of staying in London:

```
   num benefit {DEPTS, CITIES} init 0;
   read data benefit_data into [city] {dept in DEPTS}
      <benefit[dept,city]=col(dept)>;
   print benefit;
```

Note that this READ DATA statement does not repopulate CITIES by using the following:

```
   CITIES=[city]
```

Doing so would have removed London from the CITIES index set, because London does not appear in the benefit_data data set. Figure 10.1 shows the resulting values of *benefit*.

Figure 10.1 *benefit* Parameter

The OPTMODEL Procedure

	benefit		
	Brighton	Bristol	London
A	10	10	0
B	20	15	0
C	15	10	0
D	15	20	0
E	15	5	0

The following NUM and assignment statements read an upper triangular matrix and use these values to populate the lower triangular part. The INIT option initializes the parameter *comm* to a missing value, and the body of the FOR loop replaces each missing value with the value obtained by reflection across the main diagonal.

```
num comm {DEPTS, DEPTS} init .;
read data comm_data into [i j] comm;
for {i in DEPTS, j in DEPTS} do;
   if i = j then comm[i,j] = 0;
   else if comm[i,j] = . then comm[i,j] = comm[j,i];
end;
print comm;
```

Figure 10.2 shows the resulting values of *comm*.

Figure 10.2 *comm* Parameter

	comm				
	A	B	C	D	E
A	0.0	0.0	1.0	1.5	0.0
B	0.0	0.0	1.4	1.2	0.0
C	1.0	1.4	0.0	0.0	2.0
D	1.5	1.2	0.0	0.0	0.7
E	0.0	0.0	2.0	0.7	0.0

Similar statements are used to populate the *cost* parameter, but instead the main diagonal is read from the cost_data data set:

```
num cost {CITIES, CITIES} init .;
read data cost_data into [i j] cost;
for {i in CITIES, j in CITIES: cost[i,j] = .}
   cost[i,j] = cost[j,i];
print cost;
```

Figure 10.3 shows the resulting values of *cost*.

Figure 10.3 *cost* Parameter

	cost		
	Brighton	Bristol	London
Brighton	5	14	9
Bristol	14	5	13
London	9	13	10

The following declarations are straightforward:

```
var Assign {DEPTS, CITIES} binary;

set IJKL = {i in DEPTS, j in CITIES, k in DEPTS, l in CITIES: i < k};
var Product {IJKL} binary;

impvar TotalBenefit
   = sum {i in DEPTS, j in CITIES} benefit[i,j] * Assign[i,j];
impvar TotalCost
   = sum {<i,j,k,l> in IJKL} comm[i,k] * cost[j,l] * Product[i,j,k,l];
max NetBenefit = TotalBenefit - TotalCost;

con Assign_dept {dept in DEPTS}:
   sum {city in CITIES} Assign[dept,city] = 1;

con Cardinality {city in CITIES}:
   sum {dept in DEPTS} Assign[dept,city] <= &max_num_depts;
```

The following CON statement enforces the rule that Assign[i,j] $=$ 1 and Assign[k,l] $=$ 1 together imply Product[i,j,k,l] $=$ 1:

```
con Product_def {<i,j,k,l> in IJKL}:
   Assign[i,j] + Assign[k,l] - 1 <= Product[i,j,k,l];
```

The following two CON statements enforce the converse rule that Product[i,j,k,l] $=$ 1 implies both Assign[i,j] $=$ 1 and Assign[k,l] $=$ 1:

```
con Product_def2 {<i,j,k,l> in IJKL}:
   Product[i,j,k,l] <= Assign[i,j];
con Product_def3 {<i,j,k,l> in IJKL}:
   Product[i,j,k,l] <= Assign[k,l];
```

Because of the maximization objective in this problem, these two constraints could be omitted and would still be satisfied by an optimal solution. But including them can sometimes reduce the number of simplex iterations and branch-and-bound nodes.

```
solve;

print TotalBenefit TotalCost;
print Assign;
```

Figure 10.4 shows the output from the mixed integer linear programming solver.

Figure 10.4 Output from Mixed Integer Linear Programming Solver

Problem Summary	
Objective Sense	Maximization
Objective Function	NetBenefit
Objective Type	Linear
Number of Variables	105
Bounded Above	0
Bounded Below	0
Bounded Below and Above	105
Free	0
Fixed	0
Binary	105
Integer	0
Number of Constraints	278
Linear LE (<=)	273
Linear EQ (=)	5
Linear GE (>=)	0
Linear Range	0
Constraint Coefficients	660

Performance Information	
Execution Mode	Single-Machine
Number of Threads	4

Solution Summary	
Solver	MILP
Algorithm	Branch and Cut
Objective Function	NetBenefit
Solution Status	Optimal
Objective Value	14.9
Relative Gap	0
Absolute Gap	0
Primal Infeasibility	5.351275E-14
Bound Infeasibility	2.848295E-14
Integer Infeasibility	5.817569E-14
Best Bound	14.9
Nodes	1
Iterations	277
Presolve Time	0.03
Solution Time	0.08

Figure 10.4 *continued*

TotalBenefit	TotalCost
80	65.1

	Assign		
	Brighton	Bristol	London
A	0	1	-0
B	1	0	-0
C	1	0	0
D	0	1	0
E	1	0	0

An alternative "compact linearization" formulation involves fewer constraints (Liberti 2007). The following DROP statement removes three families of constraints from the original formulation:

```
   drop Product_def Product_def2 Product_def3;
```

The following two CON statements declare two new constraints that enforce the desired relationship between the Product and Assign variables:

```
   con Product_def4 {i in DEPTS, k in DEPTS, l in CITIES: i < k}:
      sum {<(i),j,(k),(l)> in IJKL} Product[i,j,k,l] = Assign[k,l];
   con Product_def5 {k in DEPTS, i in DEPTS, j in CITIES: i < k}:
      sum {<(i),(j),(k),l> in IJKL} Product[i,j,k,l] = Assign[i,j];

   solve;

   print TotalBenefit TotalCost;
   print Assign;
quit;
```

Figure 10.5 shows the output from the mixed integer linear programming solver for the compact linearization formulation.

Figure 10.5 Output from Mixed Integer Linear Programming Solver (Compact Linearization)

Problem Summary	
Objective Sense	Maximization
Objective Function	NetBenefit
Objective Type	Linear
Number of Variables	105
Bounded Above	0
Bounded Below	0
Bounded Below and Above	105
Free	0
Fixed	0
Binary	105
Integer	0
Number of Constraints	68
Linear LE (<=)	3
Linear EQ (=)	65
Linear GE (>=)	0
Linear Range	0
Constraint Coefficients	270

Performance Information	
Execution Mode	Single-Machine
Number of Threads	4

Solution Summary	
Solver	MILP
Algorithm	Branch and Cut
Objective Function	NetBenefit
Solution Status	Optimal
Objective Value	14.9
Relative Gap	0
Absolute Gap	0
Primal Infeasibility	3.330669E-16
Bound Infeasibility	0
Integer Infeasibility	3.330669E-16
Best Bound	14.9
Nodes	1
Iterations	215
Presolve Time	0.02
Solution Time	0.02

Figure 10.5 *continued*

TotalBenefit	TotalCost
80	65.1

	Assign		
	Brighton	Bristol	London
A	0	1	0
B	1	0	0
C	1	0	0
D	0	1	0
E	1	0	0

As expected, this solution agrees with the solution reported in Figure 10.4 for the original formulation.

In both formulations, the Product variable can be relaxed to be nonnegative instead of binary. The integrality of Assign, together with the various Product_def* constraints, automatically implies integrality of Product. For real-world problems, you should try both ways to determine which alternative performs better in specific cases.

Similarly, the compact formulation is weaker but contains fewer constraints than the original formulation. The net effect on total solve time is difficult to predict. For real-world problems, you should try both formulations.

Features Demonstrated

The following features are demonstrated in this example:

- problem type: mixed integer linear programming (quadratic assignment problem)
- numeric and string index sets
- reading dense two-dimensional data
- reading sparse (upper triangular) two-dimensional data
- multiple input data sets
- set of tuples
- INIT option
- IMPVAR statement
- product of two binary variables
- DROP statement
- compact linearization

Chapter 11
Curve Fitting: Fitting a Curve to a Set of Data Points

Contents

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Problem Statement

A quantity y is known to depend upon another quantity x.[1] A set of corresponding values has been collected for x and y and is presented in Table 11.

x	0.0	0.5	1.0	1.5	1.9	2.5	3.0	3.5	4.0	4.5
y	1.0	0.9	0.7	1.5	2.0	2.4	3.2	2.0	2.7	3.5
x	5.0	5.5	6.0	6.6	7.0	7.6	8.5	9.0	10.0	
y	1.0	4.0	3.6	2.7	5.7	4.6	6.0	6.8	7.3	

(1) Fit the 'best' straight line $y = bx + a$ to this set of data points. The objective is to minimize the sum of *absolute deviations* of each observed value of y from the value predicted by the linear relationship.

(2) Fit the 'best' straight line where the objective is to minimize the *maximum deviation* of all the observed values of y from the value predicted by the linear relationship.

(3) Fit the 'best' quadratic curve $y = cx^2 + bx + a$ to this set of data points using the same objectives as in (1) and (2).

[1] Reproduced with permission of John Wiley & Sons Ltd. (Williams 1999, pp. 242–243).

Mathematical Programming Formulation

Index Sets and Their Members

The following index set and its members are used in this example:

- $i \in$ POINTS

Parameters

Table 11.1 shows the parameters that are used in this example.

Table 11.1 Parameters

Parameter Name	Interpretation
x[i]	x coordinate of input data point i
y[i]	y coordinate of input data point i
order	Order of polynomial regression curve
sum_abs_dev	Sum of absolute deviations between predicted and observed values
max_abs_dev	Maximum of absolute deviations between predicted and observed values

Variables

Table 11.2 shows the variables that are used in this example.

Table 11.2 Variables

Variable Name	Interpretation
Beta[k]	Regression coefficient of order k
Estimate[i]	Predicted value of y for data point i
Surplus[i]	$\max\{\text{Estimate[i]} - y[i], 0\}$
Slack[i]	$\max\{y[i] - \text{Estimate[i]}, 0\}$
MinMax	$\max_{i \in \text{POINTS}} \lvert\text{Estimate[i]} - y[i]\rvert$

Objectives

The first objective is to minimize the following (nonlinear, nondifferentiable) L_1 norm function:

$$\text{Objective1} = \sum_{i \in \text{POINTS}} |\text{Estimate[i]} - y[i]|$$

The second objective is to minimize the following (nonlinear, nondifferentiable) L_∞ norm function:

$$\text{Objective2} = \max_{i \in \text{POINTS}} |\text{Estimate[i]} - y[i]|$$

Constraints

The following constraints are used in this example:

- bounds on variables
- for $i \in$ POINTS,

$$\text{Estimate[i]} = \text{Beta[0]} + \sum_{k=1}^{order} \text{Beta[k]} \cdot x[i]^k$$

- for $i \in$ POINTS,

$$\text{Estimate[i]} - \text{Surplus[i]} + \text{Slack[i]} = y[i]$$

- for $i \in$ POINTS,

$$\text{MinMax} \geq \text{Surplus[i]} + \text{Slack[i]}$$

Input Data

The following data set contains the input data that are used in this example:

```
data xy_data;
   input x y;
   datalines;
 0.0 1.0
 0.5 0.9
 1.0 0.7
 1.5 1.5
 1.9 2.0
 2.5 2.4
 3.0 3.2
 3.5 2.0
 4.0 2.7
 4.5 3.5
 5.0 1.0
 5.5 4.0
 6.0 3.6
 6.6 2.7
 7.0 5.7
 7.6 4.6
```

```
 8.5 6.0
 9.0 6.8
10.0 7.3
;
```

PROC OPTMODEL Statements and Output

The following PROC OPTMODEL statements declare an index set and parameters and then read the input data:

```
proc optmodel;
   set POINTS;
   num x {POINTS};
   num y {POINTS};
   read data xy_data into POINTS=[_N_] x y;
```

The following NUM statement declares the *order* parameter, which is later set to 1 for Problems (1) and (2) and set to 2 for Problem (3):

```
   num order;
   var Beta {0..order};
   impvar Estimate {i in POINTS}
      = Beta[0] + sum {k in 1..order} Beta[k] * x[i]^k;
```

The following statements encode the linearization of the L_1 norm:

```
   var Surplus {POINTS} >= 0;
   var Slack {POINTS} >= 0;
   min Objective1 = sum {i in POINTS} (Surplus[i] + Slack[i]);
   con Abs_dev_con {i in POINTS}:
      Estimate[i] - Surplus[i] + Slack[i] = y[i];
```

The following statements (which are not used but are shown here for comparison) encode an alternative linearization of the L_1 norm that requires half as many variables but twice as many constraints:

```
   var AbsDeviation {POINTS} >= 0;
   min Objective1 = sum {i in POINTS} AbsDeviation[i];
   con Abs_dev_con1 {i in POINTS}:
      AbsDeviation[i] >= Estimate[i] - y[i];
   con Abs_dev_con2 {i in POINTS}:
      AbsDeviation[i] >= y[i] - Estimate[i];
```

The following additional declarations encode the linearization of the L_∞ norm:

```
   var MinMax;
   min Objective2 = MinMax;
   con MinMax_con {i in POINTS}:
      MinMax >= Surplus[i] + Slack[i];
```

The following statements (which are not used but match the formulation in Williams (1999)) encode an alternative linearization of the L_∞ norm that requires the same number of variables but twice as many constraints:

```
var MinMax;
min Objective2 = MinMax;
con MinMax_con1 {i in POINTS}:
   MinMax >= Surplus[i];
con MinMax_con2 {i in POINTS}:
   MinMax >= Slack[i];
```

The following NUM statements use the ABS function, the `.sol` variable suffix, and the MAX aggregation operator to compute the two norms from the optimal solution:

```
num sum_abs_dev = sum {i in POINTS} abs(Estimate[i].sol - y[i]);
num max_abs_dev = max {i in POINTS} abs(Estimate[i].sol - y[i]);
```

The following PROBLEM statement specifies the variables, objective, and constraints for L_1 minimization:

```
problem L1 include
   Beta Surplus Slack
   Objective1
   Abs_dev_con;
```

The following PROBLEM statement specifies the additional variables, objective, and constraints for L_∞ minimization:

```
problem Linf from L1 include
   MinMax
   Objective2
   MinMax_con;
```

The following statements specify a straight line fit, switch the focus to problem L1, solve Problem (1), print the results, and store the *y*-values that are predicted by the optimal solution:

```
order = 1;
use problem L1;
solve;
print sum_abs_dev max_abs_dev;
print Beta;
print x y Estimate Surplus Slack;
create data sol_data1 from [POINTS] x y Estimate;
```

Figure 11.1 shows the output from the linear programming solver for Problem (1).

Figure 11.1 Output from Linear Programming Solver for Problem (1)

The OPTMODEL Procedure

Problem Summary	
Objective Sense	Minimization
Objective Function	Objective1
Objective Type	Linear
Number of Variables	40
Bounded Above	0
Bounded Below	38
Bounded Below and Above	0
Free	2
Fixed	0
Number of Constraints	19
Linear LE (<=)	0
Linear EQ (=)	19
Linear GE (>=)	0
Linear Range	0
Constraint Coefficients	75

Performance Information	
Execution Mode	Single-Machine
Number of Threads	1

Solution Summary	
Solver	LP
Algorithm	Dual Simplex
Objective Function	Objective1
Solution Status	Optimal
Objective Value	11.46625
Primal Infeasibility	1.776357E-15
Dual Infeasibility	0
Bound Infeasibility	0
Iterations	23
Presolve Time	0.00
Solution Time	0.00

sum_abs_dev	max_abs_dev
11.466	2.7688

[1]	Beta
0	0.58125
1	0.63750

Figure 11.1 *continued*

[1]	x	y	Estimate	Surplus	Slack
1	0.0	1.0	0.58125	0.00000	0.41875
2	0.5	0.9	0.90000	0.00000	0.00000
3	1.0	0.7	1.21875	0.51875	0.00000
4	1.5	1.5	1.53750	0.03750	0.00000
5	1.9	2.0	1.79250	0.00000	0.20750
6	2.5	2.4	2.17500	0.00000	0.22500
7	3.0	3.2	2.49375	0.00000	0.70625
8	3.5	2.0	2.81250	0.81250	0.00000
9	4.0	2.7	3.13125	0.43125	0.00000
10	4.5	3.5	3.45000	0.00000	0.05000
11	5.0	1.0	3.76875	2.76875	0.00000
12	5.5	4.0	4.08750	0.08750	0.00000
13	6.0	3.6	4.40625	0.80625	0.00000
14	6.6	2.7	4.78875	2.08875	0.00000
15	7.0	5.7	5.04375	0.00000	0.65625
16	7.6	4.6	5.42625	0.82625	0.00000
17	8.5	6.0	6.00000	0.00000	0.00000
18	9.0	6.8	6.31875	0.00000	0.48125
19	10.0	7.3	6.95625	0.00000	0.34375

The following statements switch the focus to problem Linf, solve Problem (2), print the results, and store the *y*-values that are predicted by the optimal solution:

```
use problem Linf;
solve;
print sum_abs_dev max_abs_dev;
print Beta;
print x y Estimate Surplus Slack;
create data sol_data2 from [POINTS] x y Estimate;
```

Figure 11.2 shows the output from the linear programming solver for Problem (2).

Figure 11.2 Output from Linear Programming Solver for Problem (2)

Problem Summary	
Objective Sense	Minimization
Objective Function	Objective2
Objective Type	Linear
Number of Variables	41
Bounded Above	0
Bounded Below	38
Bounded Below and Above	0
Free	3
Fixed	0
Number of Constraints	38
Linear LE (<=)	0
Linear EQ (=)	19
Linear GE (>=)	19
Linear Range	0
Constraint Coefficients	132

Performance Information	
Execution Mode	Single-Machine
Number of Threads	1

Solution Summary	
Solver	LP
Algorithm	Dual Simplex
Objective Function	Objective2
Solution Status	Optimal
Objective Value	1.725
Primal Infeasibility	2.220446E-15
Dual Infeasibility	0
Bound Infeasibility	0
Iterations	26
Presolve Time	0.00
Solution Time	0.00

sum_abs_dev	max_abs_dev
19.95	1.725

[1]	Beta
0	-0.400
1	0.625

Figure 11.2 *continued*

[1]	x	y	Estimate	Surplus	Slack
1	0.0	1.0	-0.4000	0.000	1.4000
2	0.5	0.9	-0.0875	0.000	0.9875
3	1.0	0.7	0.2250	0.000	0.4750
4	1.5	1.5	0.5375	0.000	0.9625
5	1.9	2.0	0.7875	0.000	1.2125
6	2.5	2.4	1.1625	0.000	1.2375
7	3.0	3.2	1.4750	0.000	1.7250
8	3.5	2.0	1.7875	0.000	0.2125
9	4.0	2.7	2.1000	0.000	0.6000
10	4.5	3.5	2.4125	0.000	1.0875
11	5.0	1.0	2.7250	1.725	0.0000
12	5.5	4.0	3.0375	0.000	0.9625
13	6.0	3.6	3.3500	0.000	0.2500
14	6.6	2.7	3.7250	1.025	0.0000
15	7.0	5.7	3.9750	0.000	1.7250
16	7.6	4.6	4.3500	0.000	0.2500
17	8.5	6.0	4.9125	0.000	1.0875
18	9.0	6.8	5.2250	0.000	1.5750
19	10.0	7.3	5.8500	0.000	1.4500

The following statements specify a quadratic curve fit, solve both parts of Problem (3), print the results, and store the y-values that are predicted by the optimal solutions:

```
   order = 2;
   use problem L1;
   solve;
   print sum_abs_dev max_abs_dev;
   print Beta;
   print x y Estimate Surplus Slack;
   create data sol_data3 from [POINTS] x y Estimate;

   use problem Linf;
   solve;
   print sum_abs_dev max_abs_dev;
   print Beta;
   print x y Estimate Surplus Slack;
   create data sol_data4 from [POINTS] x y Estimate;
quit;
```

Figure 11.3 shows the output from the linear programming solver for the first part of Problem (3).

Figure 11.3 Output from Linear Programming Solver for First Part of Problem (3)

Problem Summary	
Objective Sense	Minimization
Objective Function	Objective1
Objective Type	Linear
Number of Variables	41
Bounded Above	0
Bounded Below	38
Bounded Below and Above	0
Free	3
Fixed	0
Number of Constraints	19
Linear LE (<=)	0
Linear EQ (=)	19
Linear GE (>=)	0
Linear Range	0
Constraint Coefficients	93

Performance Information	
Execution Mode	Single-Machine
Number of Threads	1

Solution Summary	
Solver	LP
Algorithm	Dual Simplex
Objective Function	Objective1
Solution Status	Optimal
Objective Value	10.458964706
Primal Infeasibility	7.993606E-15
Dual Infeasibility	0
Bound Infeasibility	0
Iterations	20
Presolve Time	0.00
Solution Time	0.00

sum_abs_dev	max_abs_dev
10.459	2.298

[1]	Beta
0	0.982353
1	0.294510
2	0.033725

Figure 11.3 *continued*

[1]	x	y	Estimate	Surplus	Slack
1	0.0	1.0	0.98235	0.00000	0.017647
2	0.5	0.9	1.13804	0.23804	0.000000
3	1.0	0.7	1.31059	0.61059	0.000000
4	1.5	1.5	1.50000	0.00000	0.000000
5	1.9	2.0	1.66367	0.00000	0.336329
6	2.5	2.4	1.92941	0.00000	0.470588
7	3.0	3.2	2.16941	0.00000	1.030588
8	3.5	2.0	2.42627	0.42627	0.000000
9	4.0	2.7	2.70000	0.00000	0.000000
10	4.5	3.5	2.99059	0.00000	0.509412
11	5.0	1.0	3.29804	2.29804	0.000000
12	5.5	4.0	3.62235	0.00000	0.377647
13	6.0	3.6	3.96353	0.36353	0.000000
14	6.6	2.7	4.39520	1.69520	0.000000
15	7.0	5.7	4.69647	0.00000	1.003529
16	7.6	4.6	5.16861	0.56861	0.000000
17	8.5	6.0	5.92235	0.00000	0.077647
18	9.0	6.8	6.36471	0.00000	0.435294
19	10.0	7.3	7.30000	0.00000	0.000000

Figure 11.4 shows the output from the linear programming solver for the second part of Problem (3).

Figure 11.4 Output from Linear Programming Solver for Second Part of Problem (3)

Problem Summary	
Objective Sense	Minimization
Objective Function	Objective2
Objective Type	Linear
Number of Variables	42
Bounded Above	0
Bounded Below	38
Bounded Below and Above	0
Free	4
Fixed	0
Number of Constraints	38
Linear LE (<=)	0
Linear EQ (=)	19
Linear GE (>=)	19
Linear Range	0
Constraint Coefficients	150

Performance Information	
Execution Mode	Single-Machine
Number of Threads	1

Figure 11.4 *continued*

Solution Summary	
Solver	LP
Algorithm	Dual Simplex
Objective Function	Objective2
Solution Status	Optimal
Objective Value	1.475
Primal Infeasibility	7.105427E-15
Dual Infeasibility	0
Bound Infeasibility	0
Iterations	27
Presolve Time	0.00
Solution Time	0.00

sum_abs_dev	max_abs_dev
16.758	1.475

[1]	Beta
0	2.475
1	-0.625
2	0.125

[1]	x	y	Estimate	Surplus	Slack
1	0.0	1.0	2.4750	1.47500	0.00000
2	0.5	0.9	2.1938	1.29375	0.00000
3	1.0	0.7	1.9750	1.27500	0.00000
4	1.5	1.5	1.8188	0.31875	0.00000
5	1.9	2.0	1.7388	0.00000	0.26125
6	2.5	2.4	1.6938	0.00000	0.70625
7	3.0	3.2	1.7250	0.00000	1.47500
8	3.5	2.0	1.8188	0.00000	0.18125
9	4.0	2.7	1.9750	0.00000	0.72500
10	4.5	3.5	2.1938	0.00000	1.30625
11	5.0	1.0	2.4750	1.47500	0.00000
12	5.5	4.0	2.8188	0.00000	1.18125
13	6.0	3.6	3.2250	0.00000	0.37500
14	6.6	2.7	3.7950	1.09500	0.00000
15	7.0	5.7	4.2250	0.00000	1.47500
16	7.6	4.6	4.9450	0.34500	0.00000
17	8.5	6.0	6.1938	0.19375	0.00000
18	9.0	6.8	6.9750	0.17500	0.00000
19	10.0	7.3	8.7250	1.42500	0.00000

You can find a higher-order polynomial fit simply by increasing the value of the *order* parameter. The dimensions of the Beta and Estimate variables are automatically updated when *order* changes.

The following PROC SGPLOT statements use the output data sets that are created by PROC OPTMODEL to

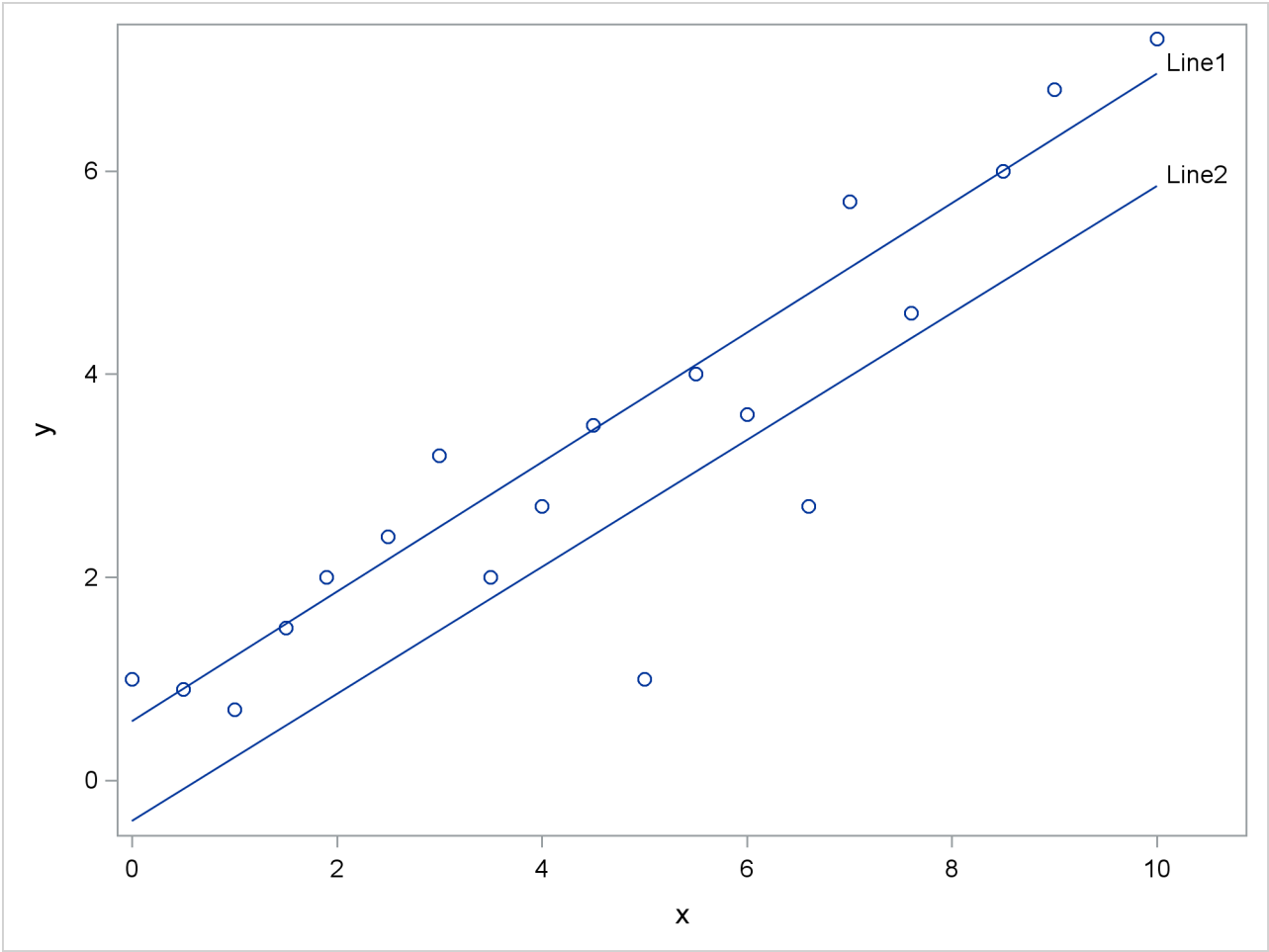
display the results from Problems (1) and (2) in one plot:

```
data plot1;
   merge sol_data1(rename=(Estimate=Line1)) sol_data2(rename=(Estimate=Line2));
run;

proc sgplot data=plot1;
   scatter x=x y=y;
   series x=x y=Line1 / curvelabel;
   series x=x y=Line2 / curvelabel;
run;
```

Figure 11.5 shows the regression lines for Problems (1) and (2), as on page 319 of Williams (1999).

Figure 11.5 Regression Lines for Problems (1) and (2)



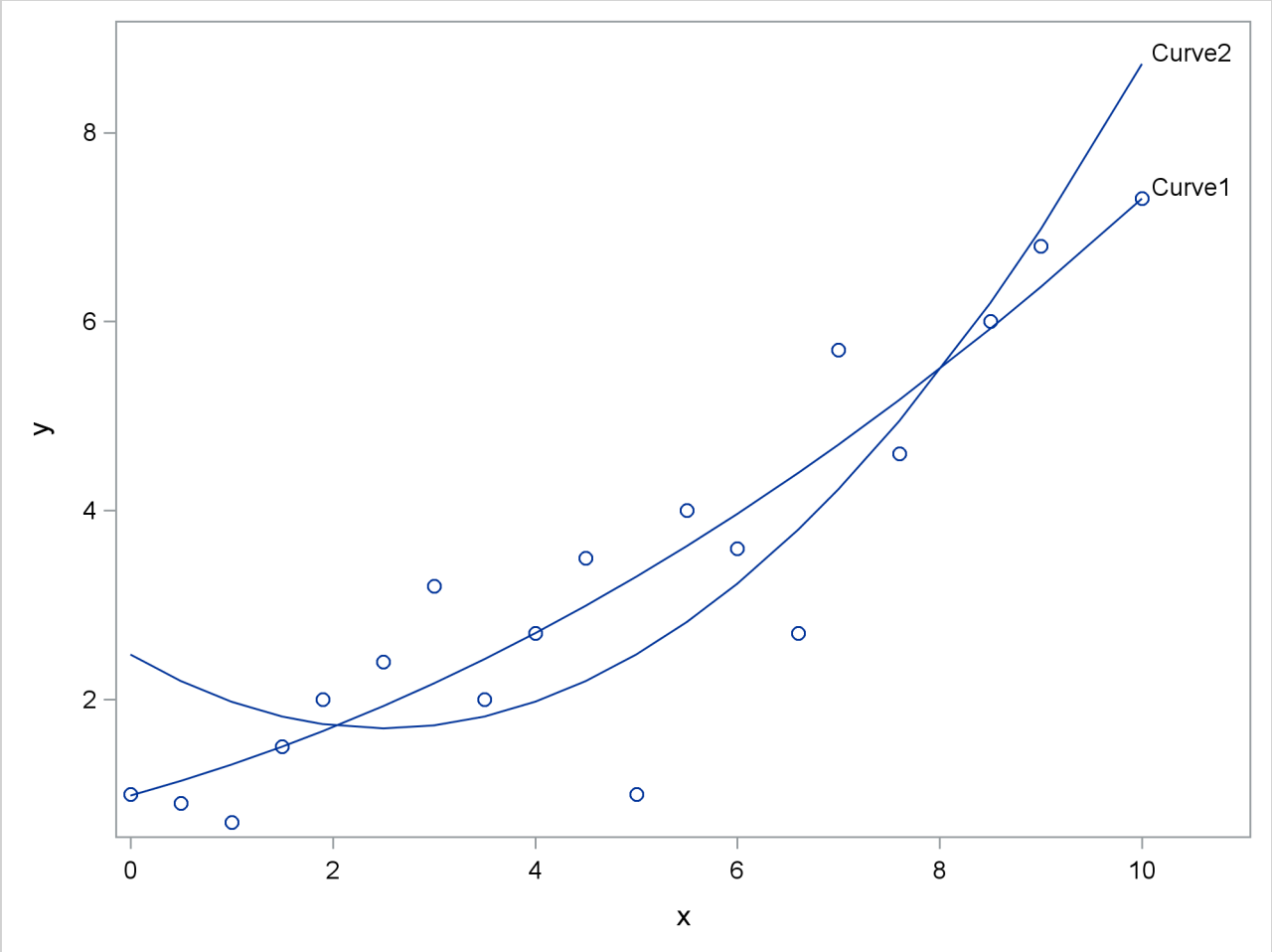
The following PROC SGPLOT statements use the output data sets that are created by PROC OPTMODEL to display the results from both parts of Problem (3) in one plot:

```
data plot2;
   merge sol_data3(rename=(Estimate=Curve1))
      sol_data4(rename=(Estimate=Curve2));
run;

proc sgplot data=plot2;
   scatter x=x y=y;
   series x=x y=Curve1 / curvelabel;
   series x=x y=Curve2 / curvelabel;
run;
```

Figure 11.6 shows the regression curves for Problem (3), as on page 319 of Williams (1999).

Figure 11.6 Regression Curves for Problem (3)



Features Demonstrated

The following features are demonstrated in this example:

- problem types: linear programming (polynomial regression with L_1 and L_∞ norms)
- numeric index set
- IMPVAR statement
- PROBLEM and USE PROBLEM statements
- multiple objectives
- using a variable suffix (such as `.sol`) in the declaration of a numeric parameter
- ABS function
- MAX aggregation operator
- SGPLOT procedure

Chapter 12
Logical Design: Constructing an Electronic System with a Minimum Number of Components

Contents

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Problem Statement

Logical circuits have a given number of inputs and one output.[1] Impulses may be applied to the inputs of a given logical circuit and it will respond either by giving an output (signal 1) or by giving no output (signal 0). The input impulses are of the same kind as the outputs, i.e. 1 (positive input) or 0 (no input).

In this example a logical circuit is to be built up of NOR gates. A NOR gate is a device with two inputs and one output. It has the property that there is positive output (signal 1) if and only if *neither* input is positive, i.e. both inputs have value 0. By connecting such gates together with outputs from one gate possibly being inputs into another gate it is possible to construct a circuit to perform any desired logical function. For example the circuit illustrated in Figure 12.1 will respond to the inputs A and B in the way indicated by the truth table in Table 12.1.

The problem here is to construct a circuit using the *minimum number* of NOR gates which will perform the logical function specified by the truth table in Table 12.2. This problem, together with further references to it, is discussed in Williams (1974).

[1] Reproduced with permission of John Wiley & Sons Ltd. (Williams 1999, pp. 243–244).

Figure 12.1

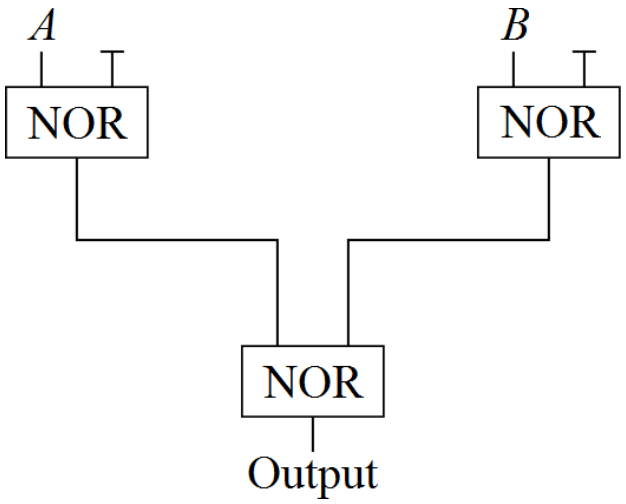


Table 12.1

Inputs		Output
A	*B*	
0	0	0
0	1	0
1	0	0
1	1	1

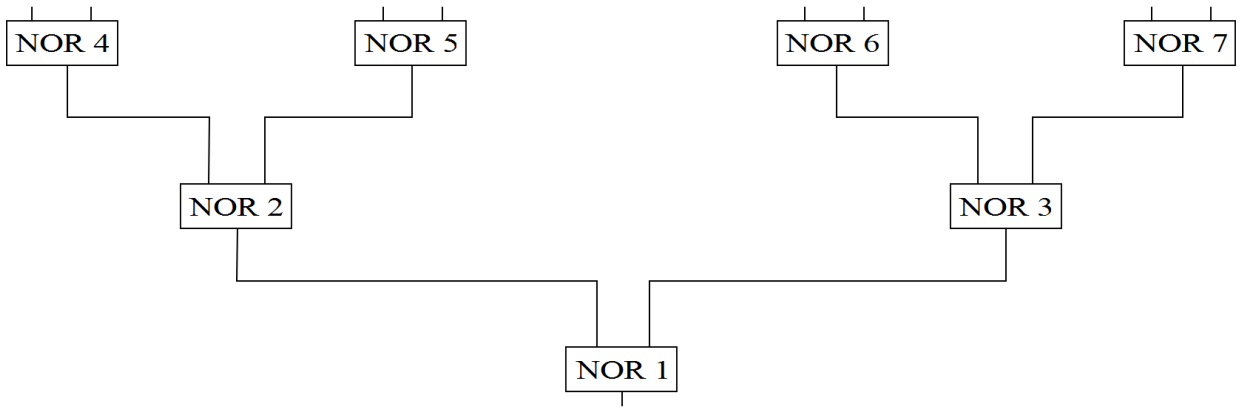
Table 12.2

Inputs		Output
A	*B*	
0	0	0
0	1	1
1	0	1
1	1	0

'Fan-in' and 'fan-out' are not permitted. That is, more than one output from a NOR gate cannot lead into one input, nor can one output lead into more than one input.

It may be assumed throughout that the optimal design is a 'subnet' of the 'maximal' net shown in Figure 12.2.

Figure 12.2



Mathematical Programming Formulation

Index Sets and Their Members

The following index sets and their members are used in this example:

- (i, j), (pred, gate) $\in$ ARCS
- pred, gate $\in$ GATES
- row $\in$ ROWS

Parameters

Table 12.3 shows the parameters that are used in this example.

Table 12.3 Parameters

Parameter Name	Interpretation
inputA[row]	Value of input *A* in each row of truth table
inputB[row]	Value of input *B* in each row of truth table
target_output[row]	Value of target output in each row of truth table

Variables

Table 12.4 shows the variables that are used in this example.

Table 12.4 Variables

Variable Name	Interpretation
UseGate[gate]	1 if gate is used; 0 otherwise
AssignAGate[gate]	1 if input *A* is assigned to gate; 0 otherwise
AssignBGate[gate]	1 if input *B* is assigned to gate; 0 otherwise
Output[gate,row]	Output value for each gate and each row of truth table

Objective

The objective is to minimize the following function:

$$\text{NumGatesUsed} = \sum_{\text{gate} \in \text{GATES}} \text{UseGate[gate]}$$

Constraints

The following constraints are used in this example:

- bounds on variables
- for gate $\in$ GATES,

  $$\text{AssignAGate[gate]} \leq \text{UseGate[gate]}$$

- for gate $\in$ GATES,

  $$\text{AssignBGate[gate]} \leq \text{UseGate[gate]}$$

- for gate $\in$ GATES,

  $$\sum_{(\text{pred},\text{gate}) \in \text{ARCS}} \text{UseGate[pred]} + \text{AssignAGate[gate]} + \text{AssignBGate[gate]} \leq 2$$

- for row $\in$ ROWS,

  $$\text{Output[1,row]} = \textit{target_output[row]}$$

- for gate $\in$ GATES and row $\in$ ROWS,

  $$\text{Output[gate,row]} \leq \text{UseGate[gate]}$$

- for gate $\in$ GATES and row $\in$ ROWS,

$$inputA[row] \cdot \text{AssignAGate[gate]} \leq 1 - \text{Output[gate,row]}$$

- for gate $\in$ GATES and row $\in$ ROWS,

$$inputB[row] \cdot \text{AssignBGate[gate]} \leq 1 - \text{Output[gate,row]}$$

- for (pred, gate) $\in$ ARCS and row $\in$ ROWS,

$$\text{Output[pred,row]} \leq 1 - \text{Output[gate,row]}$$

- for gate $\in$ GATES and row $\in$ ROWS,

$$\begin{aligned} & inputA[row] \cdot \text{AssignAGate[gate]} + inputB[row] \cdot \text{AssignBGate[gate]} \\ & + \sum_{(\text{pred,gate}) \in \text{ARCS}} \text{Output[pred,row]} \\ & \geq \text{UseGate[gate]} - \text{Output[gate,row]} \end{aligned}$$

Input Data

The following data sets contain the input data that are used in this example:

```
data arc_data;
   input i j;
   datalines;
4 2
5 2
6 3
7 3
2 1
3 1
;

/* truth table (for XOR) */
data truth_data;
   input A B output;
   datalines;
0 0 0
0 1 1
1 0 1
1 1 0
;
```

PROC OPTMODEL Statements and Output

The following SET and READ statements declare the ARCS index set and then populate it by reading the arc_data data set:

```
proc optmodel;
   set <num,num> ARCS;
   read data arc_data into ARCS=[i j];
```

Each arc corresponds to a connection from one NOR gate to another in Figure 12.2, and the following SET statement declares and populates the index set GATES by taking a union over ARCS, as in previous examples:

```
set GATES = union {<i,j> in ARCS} {i,j};

var UseGate {GATES} binary;
min NumGatesUsed = sum {gate in GATES} UseGate[gate];

var AssignAGate {GATES} binary;
var AssignBGate {GATES} binary;
```

The following CON statements declare the constraints that if input *A* or *B* is assigned to a gate, that gate must be used:

```
con AssignAGate_def {gate in GATES}:
   AssignAGate[gate] <= UseGate[gate];
con AssignBGate_def {gate in GATES}:
   AssignBGate[gate] <= UseGate[gate];
```

The following CON statement declares the constraint that each gate has at most two inputs:

```
con At_most_two_inputs {gate in GATES}:
   sum {<pred,(gate)> in ARCS} UseGate[pred]
 + AssignAGate[gate] + AssignBGate[gate]
<= 2;
```

The following statements declare the ROWS index set and several parameters and read the truth_data data set that contains the target output:

```
set ROWS;
num inputA {ROWS};
num inputB {ROWS};
num target_output {ROWS};
read data truth_data into ROWS=[_N_] inputA=A inputB=B target_output=output;
```

The following VAR and FIX statements declare the Output variable and fix the output of Gate 1 to the desired values specified in *target_output*:

```
var Output {GATES, ROWS} binary;
for {row in ROWS} fix Output[1,row] = target_output[row];
```

The following CON statement declares the constraint that if any row has a positive Output, that gate must be used.

```
   con Output_link {gate in GATES, row in ROWS}:
      Output[gate,row] <= UseGate[gate];
```

The following CON statements declare the constraints that define each NOR gate:

```
   /* if inputA[row] = 1 and AssignAGate[gate] = 1, then
      Output[gate,row] = 0 */
   con NOR_def1 {gate in GATES, row in ROWS}:
      inputA[row] * AssignAGate[gate] <= 1 - Output[gate,row];

   /* if inputB[row] = 1 and AssignBGate[gate] = 1, then
      Output[gate,row] = 0 */
   con NOR_def2 {gate in GATES, row in ROWS}:
      inputB[row] * AssignBGate[gate] <= 1 - Output[gate,row];

   /* if Output[pred,row] = 1, then Output[gate,row] = 0 */
   con NOR_def3 {<pred,gate> in ARCS, row in ROWS}:
      Output[pred,row] <= 1 - Output[gate,row];

   /* if UseGate[gate] = 1 and Output[gate,row] = 0, then
      (inputA[row] = 1 and AssignAGate[gate] = 1)
      or (inputB[row] = 1 and AssignBGate[gate] = 1)
      or sum {<pred,(gate)> in ARCS} Output[pred,row] >= 1 */
   con NOR_def4 {gate in GATES, row in ROWS}:
      inputA[row] * AssignAGate[gate]
    + inputB[row] * AssignBGate[gate]
    + sum {<pred,(gate)> in ARCS} Output[pred,row]
   >= UseGate[gate] - Output[gate,row];

   solve;
   print UseGate AssignAGate AssignBGate;
   print Output;
   create data sol_data1 from [gate] UseGate AssignAGate AssignBGate;
   create data sol_data2 from [gate row] Output;
quit;
```

Figure 12.3 shows the output from the mixed integer linear programming solver. In this case, five gates are used in the optimal solution.

Figure 12.3 Output from Mixed Integer Linear Programming Solver

The OPTMODEL Procedure

Problem Summary	
Objective Sense	Minimization
Objective Function	NumGatesUsed
Objective Type	Linear
Number of Variables	49
Bounded Above	0
Bounded Below	0
Bounded Below and Above	45
Free	0
Fixed	4
Binary	49
Integer	0
Number of Constraints	157
Linear LE (<=)	129
Linear EQ (=)	0
Linear GE (>=)	28
Linear Range	0
Constraint Coefficients	344

Performance Information	
Execution Mode	Single-Machine
Number of Threads	4

Solution Summary	
Solver	MILP
Algorithm	Branch and Cut
Objective Function	NumGatesUsed
Solution Status	Optimal
Objective Value	5
Relative Gap	0
Absolute Gap	0
Primal Infeasibility	0
Bound Infeasibility	0
Integer Infeasibility	0
Best Bound	5
Nodes	1
Iterations	3
Presolve Time	0.03
Solution Time	0.04

Figure 12.3 *continued*

[1]	UseGate	AssignAGate	AssignBGate
1	1	0	0
2	1	1	1
3	1	0	0
4	0	0	0
5	0	0	0
6	1	1	0
7	1	0	1

	Output			
	1	**2**	**3**	**4**
1	0	1	1	0
2	1	0	0	0
3	0	0	0	1
4	0	0	0	0
5	0	0	0	0
6	1	1	0	0
7	1	0	1	0

Features Demonstrated

The following features are demonstrated in this example:

- problem type: mixed integer linear programming
- numeric and string index sets
- set of tuples
- set operator UNION
- implicit slice
- FIX statement
- modeling if-then constraints by using binary variables

Chapter 13
Market Sharing: Assigning Retailers to Company Divisions

Contents

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Problem Statement

A large company has two divisions D1 and D2.[1] The company supplies retailers with oil and spirit.

It is desired to allocate each retailer to either division D1 or division D2. This division will be the retailer's supplier. As far as possible this division must be made so that D1 controls 40% of the market and D2 the remaining 60%. The retailers are listed below as M1 to M23. Each retailer has an estimated market for oil and spirit. Retailers M1 to M8 are in region 1; retailers M9 to M18 are in region 2; retailers M19 to M23 are in region 3. Certain retailers are considered to have good growth prospects and categorized as group A and the others are in group B. Each retailer has a certain number of delivery points as given below. It is desired to make the 40/60 split between D1 and D2 in each of the following respects:

(1) Total number of delivery points

(2) Control of spirit market

(3) Control of oil market in region 1

(4) Control of oil market in region 2

(5) Control of oil market in region 3

(6) Number of retailers in group A

(7) Number of retailers in group B

[1] Reproduced with permission of John Wiley & Sons Ltd. (Williams 1999, pp. 244–245).

Table 13.1

	Retailer	Oil market (10^6 gallons)	Delivery points	Spirit market (10^6 gallons)	Growth category
	M1	9	11	34	A
	M2	13	47	411	A
	M3	14	44	82	A
Region 1	M4	17	25	157	B
	M5	18	10	5	A
	M6	19	26	183	A
	M7	23	26	14	B
	M8	21	54	215	B
	M9	9	18	102	B
	M10	11	51	21	A
	M11	17	20	54	B
	M12	18	105	0	B
Region 2	M13	18	7	6	B
	M14	17	16	96	B
	M15	22	34	118	A
	M16	24	100	112	B
	M17	36	50	535	B
	M18	43	21	8	B
	M19	6	11	53	B
	M20	15	19	28	A
Region 3	M21	15	14	69	B
	M22	25	10	65	B
	M23	39	11	27	B

There is a certain flexibility in that any share may vary by $\pm 5\%$. That is, the share can vary between the limits 35/65 and 45/55.

The primary aim is to find a feasible solution. If, however, there is some choice, then possible objectives are (i) to minimize the sum of the percentage deviations from the 40/60 split and (ii) to minimize the maximum such deviation.

Build a model to see if the problem has a feasible solution and if so find the optimal solutions.

The numerical data are given in Table 13.1.

Mathematical Programming Formulation

Williams (1999) uses the fact that the company has only two divisions to reduce the number of decision variables and constraints. The formulation shown here generalizes to any number of divisions.

Index Sets and Their Members

The following index sets and their members are used in this example:

- retailer ∈ RETAILERS
- r, reg ∈ REGIONS
- g, group ∈ GROUPS
- retailer ∈ RETAILERS_region[region]
- retailer ∈ RETAILERS_group[group]
- division ∈ DIVISIONS
- category ∈ CATEGORIES

Parameters

Table 13.2 shows the parameters that are used in this example.

Table 13.2 Parameters

Parameter Name	Interpretation
region[retailer]	Region per retailer
oil[retailer]	Estimated market for oil per retailer
delivery[retailer]	Number of delivery points per retailer
spirit[retailer]	Estimated market for spirit per retailer
growth[retailer]	Growth category per retailer
target[division]	Market share target per division
tolerance	Amount by which market share can differ from target
sum_abs_dev	Sum of absolute deviations between market share and target
max_abs_dev	Maximum of absolute deviations between market share and target

Variables

Table 13.3 shows the variables that are used in this example.

Table 13.3 Variables

Variable Name	Interpretation
Assign[retailer,division]	1 if retailer is assigned to division; 0 otherwise
MarketShare[category,division]	Market share for category and division
Surplus[category,division]	$\max\{\text{MarketShare[category,division]} - \textit{target[division]}, 0\}$
Slack[category,division]	$\max\{\textit{target[division]} - \text{MarketShare[category,division]}, 0\}$
MinMax	$\max_{\substack{\text{category}\in\text{CATEGORIES},\\ \text{division}\in\text{DIVISIONS}}} \lvert\text{MarketShare[category,division]} - \textit{target[division]}\rvert$

Objectives

The first objective is to minimize the following (nonlinear, nondifferentiable) L_1 norm function:

$$\text{Objective1} = \sum_{\substack{\text{category}\in\text{CATEGORIES},\\ \text{division}\in\text{DIVISIONS}}} \lvert\text{MarketShare[category,division]} - \textit{target[division]}\rvert$$

The second objective is to minimize the following (nonlinear, nondifferentiable) L_∞ norm function:

$$\text{Objective2} = \max_{\substack{\text{category}\in\text{CATEGORIES},\\ \text{division}\in\text{DIVISIONS}}} \lvert\text{MarketShare[category,division]} - \textit{target[division]}\rvert$$

Constraints

The following constraints are used in this example:

- bounds on variables
- for retailer $\in$ RETAILERS,

$$\sum_{\text{division}\in\text{DIVISIONS}} \text{Assign[retailer,division]} = 1$$

- for division $\in$ DIVISIONS,

$$\text{MarketShare['delivery',division]} = \frac{\sum_{\text{retailer}\in\text{RETAILERS}} \textit{delivery[retailer]} \cdot \text{Assign[retailer,division]}}{\sum_{\text{retailer}\in\text{RETAILERS}} \textit{delivery[retailer]}}$$

- for division ∈ DIVISIONS,

$$\text{MarketShare['spirit',division]} = \frac{\sum_{\text{retailer}\in\text{RETAILERS}} \mathit{spirit[retailer]} \cdot \text{Assign[retailer,division]}}{\sum_{\text{retailer}\in\text{RETAILERS}} \mathit{spirit[retailer]}}$$

- for reg ∈ REGIONS and division ∈ DIVISIONS,

$$\text{MarketShare['oil'||reg,division]} = \frac{\sum_{\text{retailer}\in\text{RETAILERS_region[reg]}} \mathit{oil[retailer]} \cdot \text{Assign[retailer,division]}}{\sum_{\text{retailer}\in\text{RETAILERS_region[reg]}} \mathit{oil[retailer]}}$$

- for group ∈ GROUPS and division ∈ DIVISIONS,

$$\text{MarketShare['growth'||group,division]} = \frac{\sum_{\text{retailer}\in\text{RETAILERS_group[group]}} \text{Assign[retailer,division]}}{|\text{RETAILERS_group[group]}|}$$

- for category ∈ CATEGORIES and division ∈ DIVISIONS,

$$\text{MarketShare[category,division]} - \text{Surplus[category,division]} + \text{Slack[category,division]} = \mathit{target[division]}$$

- for category ∈ CATEGORIES and division ∈ DIVISIONS,

$$\text{MinMax} \geq \text{Surplus[category,division]} + \text{Slack[category,division]}$$

Input Data

The following data sets and macro variables contain the input data that are used in this example:

```
data retailer_data;
   input region oil delivery spirit growth $;
   datalines;
1  9  11  34 A
1 13  47 411 A
1 14  44  82 A
1 17  25 157 B
1 18  10   5 A
1 19  26 183 A
1 23  26  14 B
1 21  54 215 B
2  9  18 102 B
2 11  51  21 A
2 17  20  54 B
2 18 105   0 B
2 18   7   6 B
2 17  16  96 B
2 22  34 118 A
2 24 100 112 B
2 36  50 535 B
2 43  21   8 B
```

```
3  6  11  53 B
3 15  19  28 A
3 15  14  69 B
3 25  10  65 B
3 39  11  27 B
;

data division_data;
   input target;
   datalines;
0.40
0.60
;

%let tolerance = 0.05;
```

PROC OPTMODEL Statements and Output

The following PROC OPTMODEL statements declare an index set and parameters and then read the input data:

```
proc optmodel;
   set RETAILERS;
   num region {RETAILERS};
   num oil {RETAILERS};
   num delivery {RETAILERS};
   num spirit {RETAILERS};
   str growth {RETAILERS};
   read data retailer_data into RETAILERS=[_N_]
      region oil delivery spirit growth;
```

The following statements declare parameters and index sets, which are initialized to be empty and then populated within a FOR loop. Note that both RETAILERS_region and RETAILERS_group are sets that are indexed by other sets:

```
   set REGIONS init {};
   set RETAILERS_region {REGIONS} init {};
   num r;
   set <str> GROUPS init {};
   set RETAILERS_group {GROUPS} init {};
   str g;
   for {retailer in RETAILERS} do;
      r = region[retailer];
      REGIONS = REGIONS union {r};
      RETAILERS_region[r] = RETAILERS_region[r] union {retailer};
      g = growth[retailer];
      GROUPS = GROUPS union {g};
      RETAILERS_group[g] = RETAILERS_group[g] union {retailer};
   end;

   set DIVISIONS;
```

```
num target {DIVISIONS};
read data division_data into DIVISIONS=[_N_] target;

num tolerance = &tolerance;
```

The following declarations are straightforward:

```
var Assign {RETAILERS, DIVISIONS} binary;

con Assign_con {retailer in RETAILERS}:
   sum {division in DIVISIONS} Assign[retailer,division] = 1;

set CATEGORIES = {'delivery','spirit'}
   union (setof {reg in REGIONS} 'oil'||reg)
   union (setof {group in GROUPS} 'growth'||group);
var MarketShare {CATEGORIES, DIVISIONS};
var Surplus     {CATEGORIES, DIVISIONS} >= 0 <= tolerance;
var Slack       {CATEGORIES, DIVISIONS} >= 0 <= tolerance;

min Objective1 =
   sum {category in CATEGORIES, division in DIVISIONS}
      (Surplus[category,division] + Slack[category,division]);

con Delivery_con {division in DIVISIONS}:
   MarketShare['delivery',division]
 = (sum {retailer in RETAILERS} delivery[retailer] *
      Assign[retailer,division])
      / (sum {retailer in RETAILERS} delivery[retailer]);

con Spirit_con {division in DIVISIONS}:
   MarketShare['spirit',division]
 = (sum {retailer in RETAILERS} spirit[retailer] *
      Assign[retailer,division])
      / (sum {retailer in RETAILERS} spirit[retailer]);

con Oil_con {reg in REGIONS, division in DIVISIONS}:
   MarketShare['oil'||reg,division]
 = (sum {retailer in RETAILERS_region[reg]}
      oil[retailer] * Assign[retailer,division])
      / (sum {retailer in RETAILERS_region[reg]} oil[retailer]);

con Growth_con {group in GROUPS, division in DIVISIONS}:
   MarketShare['growth'||group,division]
 = (sum {retailer in RETAILERS_group[group]} Assign[retailer,division])
      / card(RETAILERS_group[group]);

con Abs_dev_con {category in CATEGORIES, division in DIVISIONS}:
   MarketShare[category,division]
 - Surplus[category,division] + Slack[category,division]
 = target[division];
```

The following NUM statements use the ABS function, the `.sol` variable suffix, and the MAX aggregation operator to compute the two norms from the optimal solution:

```
num sum_abs_dev =
   sum {category in CATEGORIES, division in DIVISIONS}
      abs(MarketShare[category,division].sol - target[division]);
num max_abs_dev =
   max {category in CATEGORIES, division in DIVISIONS}
      abs(MarketShare[category,division].sol - target[division]);
```

The MILP solver is called twice, and each SOLVE statement includes the OBJ option to specify which objective to optimize. The first PRINT statement after each SOLVE statement reports the values of both objectives even though only one objective is optimized at a time:

```
solve obj Objective1;
print sum_abs_dev max_abs_dev;
print Assign;
print MarketShare Surplus Slack;
```

Figure 13.1 shows the output that results from the first SOLVE statement. The optimal objective value disagrees with the value 0.0453 presented in Williams (1999), even after you account for the fact that the formulation here counts each deviation twice (once per division).

Figure 13.1 Output from First SOLVE Statement, Minimizing Sum of Deviations

The OPTMODEL Procedure

Problem Summary	
Objective Sense	Minimization
Objective Function	Objective1
Objective Type	Linear
Number of Variables	88
Bounded Above	0
Bounded Below	0
Bounded Below and Above	74
Free	14
Fixed	0
Binary	46
Integer	0
Number of Constraints	51
Linear LE (<=)	0
Linear EQ (=)	51
Linear GE (>=)	0
Linear Range	0
Constraint Coefficients	284

Performance Information	
Execution Mode	Single-Machine
Number of Threads	4

Figure 13.1 *continued*

Solution Summary	
Solver	MILP
Algorithm	Branch and Cut
Objective Function	Objective1
Solution Status	Optimal
Objective Value	0.1059523594
Relative Gap	0
Absolute Gap	0
Primal Infeasibility	1.665335E-16
Bound Infeasibility	0
Integer Infeasibility	0
Best Bound	0.1059523594
Nodes	123
Iterations	1434
Presolve Time	0.02
Solution Time	0.08

sum_abs_dev	max_abs_dev
0.10595	0.025

	Assign	
	1	**2**
1	1	0
2	1	0
3	1	0
4	1	0
5	0	1
6	0	1
7	0	1
8	0	1
9	0	1
10	0	1
11	1	0
12	0	1
13	0	1
14	0	1
15	0	1
16	1	0
17	0	1
18	1	0
19	0	1
20	0	1
21	1	0
22	1	0
23	0	1

Figure 13.1 *continued*

[1]	[2]	MarketShare	Surplus	Slack
delivery	1	0.40000	0.0000000	0.0000000
delivery	2	0.60000	0.0000000	0.0000000
growthA	1	0.37500	0.0000000	0.0250000
growthA	2	0.62500	0.0250000	0.0000000
growthB	1	0.40000	0.0000000	0.0000000
growthB	2	0.60000	0.0000000	0.0000000
oil1	1	0.39552	0.0000000	0.0044776
oil1	2	0.60448	0.0044776	0.0000000
oil2	1	0.39070	0.0000000	0.0093023
oil2	2	0.60930	0.0093023	0.0000000
oil3	1	0.40000	0.0000000	0.0000000
oil3	2	0.60000	0.0000000	0.0000000
spirit	1	0.41420	0.0141962	0.0000000
spirit	2	0.58580	0.0000000	0.0141962

The following statements declare the additional variable, objective, and constraints for problem (ii), with the tighter linearization of the L_∞ norm as in Chapter 11:

```
var MinMax >= 0 init max_abs_dev;
min Objective2 = MinMax;
con MinMax_con {category in CATEGORIES, division in DIVISIONS}:
   MinMax >= Surplus[category,division] + Slack[category,division];
```

The following statements call the MILP solver to solve problem (ii). The PRIMALIN option is used to initialize the MILP solver with the optimal solution that was obtained for problem (i):

```
solve obj Objective2 with MILP / primalin;
print sum_abs_dev max_abs_dev;
print Assign;
print MarketShare Surplus Slack;
quit;
```

Figure 13.2 shows the output that results from the second SOLVE statement.

Figure 13.2 Output from Second SOLVE Statement, Minimizing Maximum Deviation

Problem Summary	
Objective Sense	Minimization
Objective Function	Objective2
Objective Type	Linear
Number of Variables	89
Bounded Above	0
Bounded Below	1
Bounded Below and Above	74
Free	14
Fixed	0
Binary	46
Integer	0
Number of Constraints	65
Linear LE (<=)	0
Linear EQ (=)	51
Linear GE (>=)	14
Linear Range	0
Constraint Coefficients	326

Performance Information	
Execution Mode	Single-Machine
Number of Threads	4

Solution Summary	
Solver	MILP
Algorithm	Branch and Cut
Objective Function	Objective2
Solution Status	Optimal
Objective Value	0.025
Relative Gap	0
Absolute Gap	0
Primal Infeasibility	1.665335E-16
Bound Infeasibility	0
Integer Infeasibility	0
Best Bound	0.025
Nodes	1
Iterations	21
Presolve Time	0.00
Solution Time	0.00

Figure 13.2 *continued*

sum_abs_dev	max_abs_dev
0.10595	0.025

	Assign	
	1	2
1	1	0
2	1	0
3	1	0
4	1	0
5	0	1
6	0	1
7	0	1
8	0	1
9	0	1
10	0	1
11	1	0
12	0	1
13	0	1
14	0	1
15	0	1
16	1	0
17	0	1
18	1	0
19	0	1
20	0	1
21	1	0
22	1	0
23	0	1

[1]	[2]	MarketShare	Surplus	Slack
delivery	**1**	0.40000	0.0000000	0.0000000
delivery	**2**	0.60000	0.0000000	0.0000000
growthA	**1**	0.37500	0.0000000	0.0250000
growthA	**2**	0.62500	0.0250000	0.0000000
growthB	**1**	0.40000	0.0000000	0.0000000
growthB	**2**	0.60000	0.0000000	0.0000000
oil1	**1**	0.39552	0.0000000	0.0044776
oil1	**2**	0.60448	0.0044776	0.0000000
oil2	**1**	0.39070	0.0000000	0.0093023
oil2	**2**	0.60930	0.0093023	0.0000000
oil3	**1**	0.40000	0.0000000	0.0000000
oil3	**2**	0.60000	0.0000000	0.0000000
spirit	**1**	0.41420	0.0141962	0.0000000
spirit	**2**	0.58580	0.0000000	0.0141962

Features Demonstrated

The following features are demonstrated in this example:

- problem type: mixed integer linear programming (L_1 and L_∞ norms)
- numeric and string index sets
- multiple input data sets
- set operators UNION and SETOF
- string concatenation
- sets indexed by other sets
- multiple objectives and the OBJ option
- using a variable suffix (such as `.sol`) in the declaration of a numeric parameter
- CARD function
- ABS function
- MAX aggregation operator
- MILP solver option PRIMALIN

Chapter 14
Opencast Mining: How Much to Excavate

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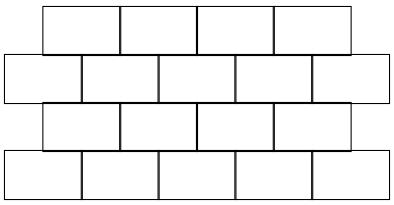
Problem Statement

A company has obtained permission to opencast mine within a square plot 200 ft × 200 ft.[1] The angle of slip of the soil is such that it is not possible for the sides of the excavation to be steeper than 45°. The company has obtained estimates for the value of the ore in various places at various depths. Bearing in mind the restrictions imposed by the angle of slip, the company decides to consider the problem as one of the extracting of rectangular blocks. Each block has horizontal dimensions 50 ft × 50 ft and a vertical dimension of 25 ft. If the blocks are chosen to lie above one another, as illustrated in vertical section in Figure 14.1, then it is only possible to excavate blocks forming an upturned pyramid. (In a three-dimensional representation Figure 14.1 would show four blocks lying above each lower block.)

Figure 14.1



If the estimates of ore value are applied to give values (in percentage of pure metal) for each block in the maximum pyramid which can be extracted then the following values are obtained:

Level 1 (surface)

1.5	1.5	1.5	0.75
1.5	2.0	1.5	0.75
1.0	1.0	0.75	0.5
0.75	0.75	0.5	0.25

Level 2 (25 ft depth)

4.0	4.0	2.0
3.0	3.0	1.0
2.0	2.0	0.5

Level 3 (50 ft depth)

12.0	6.0
5.0	4.0

Level 4 (75 ft depth)

6.0

[1] Reproduced with permission of John Wiley & Sons Ltd. (Williams 1999, pp. 245–247).

The cost of extraction increases with depth. At successive levels the cost of extracting a block is:

Level 1	£3000
Level 2	£6000
Level 3	£8000
Level 4	£10,000

The revenue obtained from a '100% value block' would be £200,000. For each block here the revenue is proportional to ore value.

Build a model to help decide the best blocks to extract. The objective is to maximize revenue − cost.

Mathematical Programming Formulation

Index Sets and Their Members

The following index sets and their members are used in this example:

- block, i, j ∈ BLOCKS
- level ∈ LEVELS

Parameters

Table 14.1 shows the parameters that are used in this example.

Table 14.1 Parameters

Parameter Name	Interpretation
level[block]	Level of block
row[block]	Row of block
column[block]	Column of block
revenue[block]	Revenue obtained from extracting block
cost[level]	Cost of extracting block per level
full_value	Revenue obtained from a 100% value block
profit[block]	Profit obtained from extracting block

Variables

Table 14.2 shows the variables that are used in this example.

Table 14.2 Variables

Variable Name	Interpretation
Extract[block]	1 if block is extracted; 0 otherwise

Objective

The objective is to maximize the following function:

$$\text{TotalProfit} = \sum_{\text{block} \in \text{BLOCKS}} \textit{profit[block]} \cdot \text{Extract[block]}$$

Constraints

The following constraints are used in this example:

- bounds on variables
- for $i \in$ BLOCKS and $j \in$ BLOCKS such that $\textit{level[j]} = \textit{level[i]} - 1$ and $\textit{row[j]} \in \{\textit{row[i]}, \textit{row[i]} + 1\}$ and $\textit{column[j]} \in \{\textit{column[i]}, \textit{column[i]} + 1\}$,

  $$\text{Extract[i]} \leq \text{Extract[j]}$$

Input Data

The following data sets and macro variable contain the input data that are used in this example:

```
data block_data;
   input level row column percent;
   datalines;
1 1 1  1.5
1 1 2  1.5
1 1 3  1.5
1 1 4  0.75
1 2 1  1.5
1 2 2  2.0
1 2 3  1.5
1 2 4  0.75
1 3 1  1.0
```

```
1 3 2  1.0
1 3 3  0.75
1 3 4  0.5
1 4 1  0.75
1 4 2  0.75
1 4 3  0.5
1 4 4  0.25
2 1 1  4.0
2 1 2  4.0
2 1 3  2.0
2 2 1  3.0
2 2 2  3.0
2 2 3  1.0
2 3 1  2.0
2 3 2  2.0
2 3 3  0.5
3 1 1 12.0
3 1 2  6.0
3 2 1  5.0
3 2 2  4.0
4 1 1  6.0
;

data level_data;
   input cost;
   datalines;
3000
6000
8000
10000
;

%let full_value = 200000;
```

PROC OPTMODEL Statements and Output

The following PROC OPTMODEL statements declare index sets and parameters and then read data sets to populate them:

```
proc optmodel;
   set BLOCKS;
   num level {BLOCKS};
   num row {BLOCKS};
   num column {BLOCKS};
   num revenue {BLOCKS};
   read data block_data into BLOCKS=[_N_] level row column revenue=percent;
   for {block in BLOCKS} revenue[block] = &full_value * revenue[block] / 100;

   set LEVELS;
   num cost {LEVELS};
   read data level_data into LEVELS=[_N_] cost;
```

The following statements declare binary decision variables and the objective:

```
   var Extract {BLOCKS} binary;
   num profit {block in BLOCKS} = revenue[block] - cost[level[block]];
   max TotalProfit = sum {block in BLOCKS} profit[block] * Extract[block];
```

The following CON statement uses the colon operator (:) to enforce the rule that if you extract block *i*, then you must also extract each block *j* that lies above it:

```
   con Precedence_con {i in BLOCKS, j in BLOCKS:
      level[j] = level[i] - 1
      and row[j] in {row[i],row[i]+1}
      and column[j] in {column[i],column[i]+1}
      }:
      Extract[i] <= Extract[j];
```

The following SOLVE statement calls the MILP solver. However, since the constraint matrix is totally unimodular, the optimal solution of the LP relaxation is automatically integral, and hence no branching is required.

```
   solve;
   print Extract profit;
```

The following CREATE DATA statement uses the colon operator (:) to output only the blocks that are used in the optimal solution:

```
   create data sol_data from
      [block]={block in BLOCKS: Extract[block].sol > 0.5}
      level row column revenue cost[level[block]] profit;
quit;
```

Figure 14.2 shows the output from the mixed integer linear programming solver.

Figure 14.2 Output from Mixed Integer Linear Programming Solver

The OPTMODEL Procedure

Problem Summary	
Objective Sense	Maximization
Objective Function	TotalProfit
Objective Type	Linear
Number of Variables	30
Bounded Above	0
Bounded Below	0
Bounded Below and Above	30
Free	0
Fixed	0
Binary	30
Integer	0
Number of Constraints	56
Linear LE (<=)	56
Linear EQ (=)	0
Linear GE (>=)	0
Linear Range	0
Constraint Coefficients	112

Performance Information	
Execution Mode	Single-Machine
Number of Threads	4

Solution Summary	
Solver	MILP
Algorithm	Branch and Cut
Objective Function	TotalProfit
Solution Status	Optimal
Objective Value	17500
Relative Gap	0
Absolute Gap	0
Primal Infeasibility	0
Bound Infeasibility	0
Integer Infeasibility	0
Best Bound	17500
Nodes	0
Iterations	0
Presolve Time	0.02
Solution Time	0.02

Figure 14.2 *continued*

[1]	Extract	profit
1	1	0
2	1	0
3	1	0
4	0	-1500
5	1	0
6	1	1000
7	1	0
8	0	-1500
9	1	-1000
10	1	-1000
11	1	-1500
12	0	-2000
13	0	-1500
14	0	-1500
15	0	-2000
16	0	-2500
17	1	2000
18	1	2000
19	0	-2000
20	1	0
21	1	0
22	0	-4000
23	0	-2000
24	0	-2000
25	0	-5000
26	1	16000
27	0	4000
28	0	2000
29	0	0
30	0	2000

Features Demonstrated

The following features are demonstrated in this example:

- problem type: mixed integer linear programming (totally unimodular)
- reading multiple input data sets
- IN expression
- logical operator AND
- using a colon (:) to select members of a set
- modeling if-then constraints by using binary variables
- creating an output data set from a subset of an index set

Chapter 15
Tariff Rates (Power Generation): How to Determine Tariff Rates for the Sale of Electricity

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Problem Statement

A number of power stations are committed to meeting the following electricity load demands over a day:[1]

12	[a.m.]	to	6	a.m.	15,000 megawatts
6	a.m.	to	9	a.m.	30,000 megawatts
9	a.m.	to	3	p.m.	25,000 megawatts
3	p.m.	to	6	p.m.	40,000 megawatts
6	p.m.	to	12	[a.m.]	27,000 megawatts

There are three types of generating unit available: 12 of type 1, 10 of type 2, and five of type 3. Each generator has to work between a minimum and a maximum level. There is an hourly cost of running each generator at minimum level. In addition there is an extra hourly cost for each megawatt at which a unit is operated above minimum level. To start up a generator also involves a cost. All this information is given in Table 15.1 (with costs in £).

In addition to meeting the estimated load demands there must be sufficient generators working at any time to make it possible to meet an increase in load of up to 15%. This increase would have to be accomplished by adjusting the output of generators already operating within their permitted limits.

Which generators should be working in which periods of the day to minimize total cost?

What is the marginal cost of production of electricity in each period of the day; i.e. what tariffs should be charged?

[1] Reproduced with permission of John Wiley & Sons Ltd. (Williams 1999, p. 247).

What would be the saving of lowering the 15% reserve output guarantee; i.e. what does this security of supply guarantee cost?

Table 15.1

	Minimum level	Maximum level	Cost per hour at minimum	Cost per hour per megawatt above minimum	[Start-up] cost
Type 1	850 MW	2000 MW	1000	2	2000
Type 2	1250 MW	1750 MW	2600	1.30	1000
Type 3	1500 MW	4000 MW	3000	3	500

Mathematical Programming Formulation

Index Sets and Their Members

The following index sets and their members are used in this example:

- period ∈ PERIODS
- type ∈ TYPES

Parameters

Table 15.2 shows the parameters that are used in this example.

Table 15.2 Parameters

Parameter Name	Interpretation
length[period]	Length per period (in hours)
demand[period]	Demand per period (in megawatts)
num_avail[type]	Number of units available per generator
min_level[type]	Minimum level (in megawatts) per generator
max_level[type]	Maximum level (in megawatts) per generator
unit_cost[type]	Hourly cost (in pounds) of running each generator at minimum level
excess_cost[type]	Extra hourly cost (in pounds) per megawatt at which a unit is operated above minimum level
startup_cost[type]	Cost (in pounds) of starting up each generator
reserve	Additional fraction of estimated load required to be met by generators already in operation

Variables

Table 15.3 shows the variables that are used in this example.

Table 15.3 Variables

Variable Name	Interpretation
NumWorking[period,type]	Number of units per type of generator operating per period
Excess[period,type]	Number of megawatt-hours generated above minimum level
NumStartup[period,type]	Number of units per type of generator starting up per period
Output[period,type]	Number of megawatt-hours generated

Objective

The objective is to minimize the following function:

$$\begin{aligned}\text{TotalCost} = \sum_{\substack{\text{period}\in\text{PERIODS},\\ \text{type}\in\text{TYPES}}} & (\mathit{unit_cost[type]} \cdot \mathit{length[period]} \cdot \text{NumWorking[period,type]} \\ & + \mathit{excess_cost[type]} \cdot \mathit{length[period]} \cdot \text{Excess[period,type]} \\ & + \mathit{startup_cost[type]} \cdot \text{NumStartup[period,type]})\end{aligned}$$

Constraints

The following constraints are used in this example:

- bounds on variables
- for period $\in$ PERIODS and type $\in$ TYPES,

  $$\text{Output[period,type]} = \mathit{min_level[type]} \cdot \text{NumWorking[period,type]} + \text{Excess[period,type]}$$

- for period $\in$ PERIODS,

  $$\sum_{\text{type}\in\text{TYPES}} \text{Output[period,type]} \geq \mathit{demand[period]}$$

- for period $\in$ PERIODS,

  $$\sum_{\text{type}\in\text{TYPES}} \mathit{max_level[type]} \cdot \text{NumWorking[period,type]} \geq (1 + \mathit{reserve}) \cdot \mathit{demand[period]}$$

- for period $\in$ PERIODS and type $\in$ TYPES,

  $$\text{Excess[period,type]} \leq (\mathit{max_level[type]} - \mathit{min_level[type]}) \cdot \text{NumWorking[period,type]}$$

- for period $\in$ PERIODS and type $\in$ TYPES,

$$\begin{aligned}\text{NumStartup[period,type]} \geq\ & \text{NumWorking[period,type]} \\ & - (\text{if period} - 1 \in \text{PERIODS, then NumWorking[period} - 1, \text{type]}; \\ & \quad \text{else NumWorking[}|\text{PERIODS}|, \text{type]})\end{aligned}$$

Input Data

The following data sets and macro variable contain the input data that are used in this example:

```
data period_data;
   input length demand;
   datalines;
6 15000
3 30000
6 25000
3 40000
6 27000
;

data type_data;
   input num_avail min_level max_level unit_cost excess_cost startup_cost;
   datalines;
12  850 2000 1000 2    2000
10 1250 1750 2600 1.30 1000
 5 1500 4000 3000 3     500
;

%let reserve = 0.15;
```

PROC OPTMODEL Statements and Output

The first several PROC OPTMODEL statements declare index sets and parameters and then read the input data:

```
proc optmodel;
   set PERIODS;
   num length {PERIODS};
   num demand {PERIODS};
   read data period_data into PERIODS=[_N_] length demand;

   set TYPES;
   num num_avail {TYPES};
   num min_level {TYPES};
   num max_level {TYPES};
   num unit_cost {TYPES};
```

```
num excess_cost {TYPES};
num startup_cost {TYPES};
read data type_data into TYPES=[_N_]
   num_avail min_level max_level unit_cost excess_cost startup_cost;
```

The following VAR statements declare decision variables, with bounds that depend on type:

```
var NumWorking {PERIODS, type in TYPES} >= 0 <= num_avail[type] integer;
var Excess {PERIODS, TYPES} >= 0;
var NumStartup {PERIODS, type in TYPES} >= 0 <= num_avail[type] integer;
```

The following IMPVAR statement declares Output as an implicit variable:

```
impvar Output {period in PERIODS, type in TYPES} =
   min_level[type] * NumWorking[period,type] + Excess[period,type];
```

The following MIN statement declares the objective:

```
min TotalCost =
   sum {period in PERIODS, type in TYPES} (
      unit_cost[type] * length[period] * NumWorking[period,type]
      + excess_cost[type] * length[period] * Excess[period,type]
      + startup_cost[type] * NumStartup[period,type]);
```

The following CON statements declare the constraints, with an IF-THEN/ELSE expression in the last constraint to account for a boundary condition in the first period:

```
con Demand_con {period in PERIODS}:
   sum {type in TYPES} Output[period,type]
>= demand[period];

con Reserve_con {period in PERIODS}:
   sum {type in TYPES} max_level[type] * NumWorking[period,type]
>= (1 + &reserve) * demand[period];

con Excess_ub {period in PERIODS, type in TYPES}:
   Excess[period,type]
<= (max_level[type] - min_level[type]) * NumWorking[period,type];

con Startup_con {period in PERIODS, type in TYPES}:
   NumStartup[period,type]
>= NumWorking[period,type]
 - (if period - 1 in PERIODS then NumWorking[period-1,type]
    else NumWorking[card(PERIODS),type]);
```

The following statements call the mixed integer linear programming solver, print the optimal solution, and create a data set that contains the optimal solution, with one observation per period-type pair:

```
solve;
print NumWorking NumStartup Excess Output;
create data sol_data from [period type] NumWorking NumStartup Excess Output;
```

Figure 15.1 shows the output from the mixed integer linear programming solver.

Figure 15.1 Output from Mixed Integer Linear Programming Solver

The OPTMODEL Procedure

Problem Summary	
Objective Sense	Minimization
Objective Function	TotalCost
Objective Type	Linear
Number of Variables	45
Bounded Above	0
Bounded Below	15
Bounded Below and Above	30
Free	0
Fixed	0
Binary	0
Integer	30
Number of Constraints	40
Linear LE (<=)	15
Linear EQ (=)	0
Linear GE (>=)	25
Linear Range	0
Constraint Coefficients	120

Performance Information	
Execution Mode	Single-Machine
Number of Threads	4

Solution Summary	
Solver	MILP
Algorithm	Branch and Cut
Objective Function	TotalCost
Solution Status	Optimal
Objective Value	988540
Relative Gap	0
Absolute Gap	0
Primal Infeasibility	3.637979E-12
Bound Infeasibility	0
Integer Infeasibility	5.551115E-15
Best Bound	988540
Nodes	1
Iterations	33
Presolve Time	0.01
Solution Time	0.02

Figure 15.1 *continued*

[1]	[2]	NumWorking	NumStartup	Excess	Output
1	1	12	0	0	10200
1	2	3	0	1050	4800
1	3	0	0	0	0
2	1	12	0	5800	16000
2	2	8	5	4000	14000
2	3	0	0	0	0
3	1	12	0	800	11000
3	2	8	0	4000	14000
3	3	0	0	0	0
4	1	12	0	11050	21250
4	2	9	1	4500	15750
4	3	2	2	0	3000
5	1	12	0	1050	11250
5	2	9	0	4500	15750
5	3	0	0	0	0

The following statements fix the integer variables to their optimal values, call the linear programming solver, and use the `.dual` constraint suffix to compute marginal costs. The RELAXINT option in the SOLVE statement relaxes the integer variables to be continuous.

```
fix NumWorking;
fix NumStartup;
solve with LP relaxint;
print NumWorking NumStartup Excess Output;
print {period in PERIODS} (demand_con[period].dual / length[period]);
```

Figure 15.2 shows the output from the linear programming solver when the integer variables are fixed to their optimal values. As expected, the same optimal solution is obtained; the point of calling the LP solver is to obtain the dual variables used to compute the marginal costs, as shown in Figure 15.3.

Figure 15.2 Output from Linear Programming Solver, Integer Variables Fixed

Problem Summary	
Objective Sense	Minimization
Objective Function	TotalCost
Objective Type	Linear
Number of Variables	45
Bounded Above	0
Bounded Below	15
Bounded Below and Above	0
Free	0
Fixed	30
Binary	16
Integer	7
Number of Constraints	40
Linear LE (<=)	15
Linear EQ (=)	0
Linear GE (>=)	25
Linear Range	0
Constraint Coefficients	120

Performance Information	
Execution Mode	Single-Machine
Number of Threads	1

Solution Summary	
Solver	LP
Algorithm	Dual Simplex
Objective Function	TotalCost
Solution Status	Optimal
Objective Value	988540
Primal Infeasibility	2.220446E-16
Dual Infeasibility	0
Bound Infeasibility	0
Iterations	6
Presolve Time	0.00
Solution Time	0.00

Figure 15.2 *continued*

[1]	[2]	NumWorking	NumStartup	Excess	Output
1	**1**	12	0	0	10200
1	**2**	3	0	1050	4800
1	**3**	0	0	0	0
2	**1**	12	0	5800	16000
2	**2**	8	5	4000	14000
2	**3**	0	0	0	0
3	**1**	12	0	800	11000
3	**2**	8	0	4000	14000
3	**3**	0	0	0	0
4	**1**	12	0	11050	21250
4	**2**	9	1	4500	15750
4	**3**	2	2	0	3000
5	**1**	12	0	1050	11250
5	**2**	9	0	4500	15750
5	**3**	0	0	0	0

Figure 15.3 Marginal Costs for Demand Constraints

[1]	
1	1.3
2	2.0
3	2.0
4	2.0
5	2.0

The following statements unfix the integer variables, call the linear programming solver, and use the **.dual** constraint suffix to compute marginal costs:

```
   unfix NumWorking;
   unfix NumStartup;
   solve with LP relaxint;
   print NumWorking NumStartup Excess Output;
   print {period in PERIODS} (demand_con[period].dual / length[period]);
   print {period in PERIODS} (reserve_con[period].dual / length[period]);
quit;
```

Figure 15.4 shows the output from the linear programming solver when the integer variables are unfixed and relaxed to be continuous. As expected, some relaxed integer variables now take fractional values, and the total cost is slightly less than before. Figure 15.5 and Figure 15.6 show the resulting marginal costs.

Figure 15.4 Output from Linear Programming Solver, Integer Variables Unfixed and Relaxed

Problem Summary	
Objective Sense	Minimization
Objective Function	TotalCost
Objective Type	Linear
Number of Variables	45
Bounded Above	0
Bounded Below	15
Bounded Below and Above	30
Free	0
Fixed	0
Binary	0
Integer	30
Number of Constraints	40
Linear LE (<=)	15
Linear EQ (=)	0
Linear GE (>=)	25
Linear Range	0
Constraint Coefficients	120

Performance Information	
Execution Mode	Single-Machine
Number of Threads	1

Solution Summary	
Solver	LP
Algorithm	Dual Simplex
Objective Function	TotalCost
Solution Status	Optimal
Objective Value	985164.28571
Primal Infeasibility	7.275958E-12
Dual Infeasibility	0
Bound Infeasibility	0
Iterations	25
Presolve Time	0.00
Solution Time	0.00

Figure 15.4 *continued*

[1]	[2]	NumWorking	NumStartup	Excess	Output
1	1	12.0000	0.0000	0.0	10200
1	2	2.7429	0.0000	1371.4	4800
1	3	0.0000	0.0000	0.0	0
2	1	12.0000	0.0000	5000.0	15200
2	2	8.4571	5.7143	4228.6	14800
2	3	0.0000	0.0000	0.0	0
3	1	12.0000	0.0000	0.0	10200
3	2	8.4571	0.0000	4228.6	14800
3	3	0.0000	0.0000	0.0	0
4	1	12.0000	0.0000	11050.0	21250
4	2	9.6000	1.1429	4800.0	16800
4	3	1.3000	1.3000	0.0	1950
5	1	12.0000	0.0000	0.0	10200
5	2	9.6000	0.0000	4800.0	16800
5	3	0.0000	0.0000	0.0	0

Figure 15.5 Marginal Costs for Demand Constraints

[1]	
1	1.7619
2	2.0000
3	1.7857
4	2.0000
5	1.8601

Figure 15.6 Marginal Costs for Reserve Constraints

[1]	
1	0.000000
2	0.000000
3	0.000000
4	0.041667
5	0.000000

Features Demonstrated

The following features are demonstrated in this example:

- problem types: mixed integer linear programming, linear programming
- multiple input data sets
- IMPVAR statement
- modeling startup costs
- IF-THEN/ELSE expression
- CARD function
- FIX and UNFIX statements
- LP solver option RELAXINT
- `.dual` constraint suffix

Chapter 16
Hydro Power: How to Generate and Combine Hydro and Thermal Electricity Generation

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Problem Statement

This is an extension of the Tariff Rates (Power Generation) problem of Section 12.15 [Chapter 15].[1] In addition to the thermal generators a reservoir powers two hydro generators: one of type A and one of type B. When a hydro generator is running, it operates at a fixed level and the depth of the reservoir decreases. The costs associated with each hydro generator are a fixed start-up cost and a running cost per hour. The characteristics of each type of generator are shown in Table 16.1.

Table 16.1

	Operating level	Cost per hour	Reservoir depth reduction per hour	Start-up cost
Hydro A	900 MW	£90	0.31 metres	£1500
Hydro B	1400 MW	£150	0.47 metres	£1200

For environmental reasons, the reservoir must be maintained at a depth of between 15 and 20 metres. Also, at midnight each night, the reservoir must be 16 metres deep. Thermal generators can be used to pump water into the reservoir. To increase the level of the reservoir by 1 metre requires 3000 MWh of electricity. You may assume that rainfall does not affect the reservoir level.

At any time it must be possible to meet an increase in demand for electricity of up to 15%. This can be achieved by any combination of the following: switching on a hydro generator (even if this would cause the reservoir depth to fall below 15 metres); using the output of a thermal generator which is used for pumping water into the reservoir; and increasing the operating level of a thermal generator to its maximum. Thermal generators cannot be switched on instantaneously to meet increased demand (although hydro generators can be).

Which generators should be working in which periods of the day, and how should the reservoir be maintained to minimize the total cost?

[1] Reproduced with permission of John Wiley & Sons Ltd. (Williams 1999, pp. 247–248).

Mathematical Programming Formulation

This formulation builds on the formulation used in Chapter 15. This section includes only the new elements of the formulation.

Index Sets and Their Members

The following additional index set and its members are used in this example:

- hydro ∈ HYDROS

Parameters

Table 16.2 shows the additional parameters that are used in this example.

Table 16.2 Parameters

Parameter Name	Interpretation
hydro_level[hydro]	Operating level (in MW) per hydro generator
hydro_unit_cost[hydro]	Hourly cost (in pounds) of operating each hydro generator
hydro_depth_rate[hydro]	Reservoir depth reduction (in meters) per hour per hydro generator
hydro_startup_cost[hydro]	Cost of starting up each hydro generator
min_depth	Lower bound on reservoir depth (in meters)
max_depth	Upper bound on reservoir depth (in meters)
midnight_depth	Required depth (in meters) of reservoir at midnight each night
meters_per_mwh	Reservoir level increase (in meters) per megawatt-hour of electricity

Variables

Table 16.3 shows the additional variables that are used in this example.

Table 16.3 Variables

Variable Name	Interpretation
HydroNumWorking[period,hydro]	Number of units per type of hydro generator operating per period
HydroNumStartup[period,hydro]	Number of units per type of hydro generator starting up per period
Depth[period]	Reservoir depth (in meters) per period
Pump[period]	Number of megawatts used by thermal generators to pump per period
HydroOutput[period,hydro]	Number of megawatt-hours generated by hydro generators

Objective

The objective is to minimize the following function:

$$\begin{aligned}\text{TotalCost} = \sum_{\substack{\text{period}\in\text{PERIODS},\\ \text{type}\in\text{TYPES}}} & (\mathit{unit_cost[type]} \cdot \mathit{length[period]} \cdot \text{NumWorking[period,type]} \\ & + \mathit{excess_cost[type]} \cdot \mathit{length[period]} \cdot \text{Excess[period,type]} \\ & + \mathit{startup_cost[type]} \cdot \text{NumStartup[period,type]}) \\ + \sum_{\substack{\text{period}\in\text{PERIODS},\\ \text{hydro}\in\text{HYDROS}}} & (\mathit{hydro_unit_cost[hydro]} \cdot \mathit{length[period]} \cdot \text{HydroNumWorking[period,hydro]} \\ & + \mathit{hydro_startup_cost[hydro]} \cdot \text{HydroNumStartup[period,hydro]})\end{aligned}$$

Constraints

The following modified and additional constraints are used in this example:

- for period $\in$ PERIODS and hydro $\in$ HYDROS,

$$\text{HydroOutput[period,hydro]} = \mathit{hydro_level[hydro]} \cdot \text{HydroNumWorking[period,hydro]}$$

- for period $\in$ PERIODS,

$$\sum_{\text{type}\in\text{TYPES}} \text{Output[period,type]} + \sum_{\text{hydro}\in\text{HYDROS}} \text{HydroOutput[period,hydro]} - \text{Pump[period]} \geq \mathit{demand[period]}$$

- for period $\in$ PERIODS,

$$\sum_{\text{type}\in\text{TYPES}} \mathit{max_level[type]} \cdot \text{NumWorking[period,type]} + \sum_{\text{hydro}\in\text{HYDROS}} \mathit{hydro_level[hydro]} \cdot \mathit{HydroNumWorking[period,hydro].ub} \geq (1 + \mathit{reserve}) \cdot \mathit{demand[period]}$$

- for period $\in$ PERIODS and hydro $\in$ HYDROS,

$$\begin{aligned}&\text{HydroNumStartup[period,hydro]} \\ &\geq \text{HydroNumWorking[period,hydro]} \\ &- (\text{if period} - 1 \in \text{PERIODS, then HydroNumWorking[period} - 1\text{, hydro]}; \\ &\quad \text{else HydroNumWorking[|PERIODS|, hydro]})\end{aligned}$$

- for period $\in$ PERIODS,

$$\begin{aligned}&(\text{if period} + 1 \in \text{PERIODS, then Depth[period} + 1\text{]; else Depth[1]})\\ &= \text{Depth[period]} + \mathit{meters_per_mwh} \cdot \mathit{length[period]} \cdot \text{Pump[period]}\\ &- \sum_{\text{hydro} \in \text{HYDROS}} \mathit{hydro_depth_rate[hydro]} \cdot \mathit{length[period]} \cdot \text{HydroNumWorking[period,hydro]}\end{aligned}$$

Input Data

The following data sets and macro variables contain the additional input data that are used in this example:

```
data hydro_data;
   input hydro $ level unit_cost depth_rate startup_cost;
   datalines;
A  900   90 0.31 1500
B 1400 150 0.47 1200
;

%let min_depth = 15;
%let max_depth = 20;
%let midnight_depth = 16;
%let meters_per_mwh = 1/3000;
```

PROC OPTMODEL Statements and Output

For completeness, all statements are shown. Statements that are new or changed from Chapter 15 are indicated.

```
proc optmodel;
   set PERIODS;
   num length {PERIODS};
   num demand {PERIODS};
   read data period_data into PERIODS=[_N_] length demand;

   set TYPES;
   num num_avail {TYPES};
   num min_level {TYPES};
   num max_level {TYPES};
   num unit_cost {TYPES};
   num excess_cost {TYPES};
   num startup_cost {TYPES};
   read data type_data into TYPES=[_N_]
      num_avail min_level max_level unit_cost excess_cost startup_cost;

   var NumWorking {PERIODS, type in TYPES} >= 0 <= num_avail[type] integer;
   var Excess {PERIODS, TYPES} >= 0;
   var NumStartup {PERIODS, type in TYPES} >= 0 <= num_avail[type] integer;
```

```
impvar Output {period in PERIODS, type in TYPES} =
   min_level[type] * NumWorking[period,type] + Excess[period,type];
```

The following statements declare the additional index set and parameters and then read the additional input data:

```
set <str> HYDROS;
num hydro_level {HYDROS};
num hydro_unit_cost {HYDROS};
num hydro_depth_rate {HYDROS};
num hydro_startup_cost {HYDROS};
read data hydro_data into HYDROS=[hydro]
   hydro_level=level hydro_unit_cost=unit_cost hydro_depth_rate=depth_rate
   hydro_startup_cost=startup_cost;
```

The following statements declare additional variables and fix the value of Depth[1]:

```
var HydroNumWorking {PERIODS, HYDROS} binary;
var HydroNumStartup {PERIODS, HYDROS} binary;
var Depth {PERIODS} >= &min_depth <= &max_depth;
fix Depth[1] = &midnight_depth;
var Pump {PERIODS} >= 0;
```

The following IMPVAR statement declares HydroOutput as a new implicit variable:

```
impvar HydroOutput {period in PERIODS, hydro in HYDROS} =
   hydro_level[hydro] * HydroNumWorking[period,hydro];
```

The following MIN statement is a modification of the objective declaration from Chapter 15:

```
min TotalCost =
   sum {period in PERIODS, type in TYPES} (
      unit_cost[type] * length[period] * NumWorking[period,type]
      + excess_cost[type] * length[period] * Excess[period,type]
      + startup_cost[type] * NumStartup[period,type])
   + sum {period in PERIODS, hydro in HYDROS} (
      hydro_unit_cost[hydro] * length[period] *
         HydroNumWorking[period,hydro]
      + hydro_startup_cost[hydro] * HydroNumStartup[period,hydro]);
```

The following two CON statements are modified from Chapter 15:

```
con Demand_con {period in PERIODS}:
   sum {type in TYPES} Output[period,type]
 + sum {hydro in HYDROS} HydroOutput[period,hydro]
 - Pump[period]
>= demand[period];

con Reserve_con {period in PERIODS}:
   sum {type in TYPES} max_level[type] * NumWorking[period,type]
 + sum {hydro in HYDROS} hydro_level[hydro] *
      HydroNumWorking[period,hydro].ub
>= (1 + &reserve) * demand[period];

con Excess_ub {period in PERIODS, type in TYPES}:
   Excess[period,type]
<= (max_level[type] - min_level[type]) * NumWorking[period,type];
```

```
con Startup_con {period in PERIODS, type in TYPES}:
   NumStartup[period,type]
>= NumWorking[period,type]
 - (if period - 1 in PERIODS then NumWorking[period-1,type]
    else NumWorking[card(PERIODS),type]);
```

The following two CON statements declare the final two additional constraints:

```
con Hydro_startup_con {period in PERIODS, hydro in HYDROS}:
   HydroNumStartup[period,hydro]
>= HydroNumWorking[period,hydro]
 - (if period - 1 in PERIODS then HydroNumWorking[period-1,hydro]
    else HydroNumWorking[card(PERIODS),hydro]);

con Depth_con {period in PERIODS}:
   (if period + 1 in PERIODS then Depth[period+1] else Depth[1])
 = Depth[period]
 + &meters_per_mwh * length[period] * Pump[period]
 - sum {hydro in HYDROS} hydro_depth_rate[hydro] * length[period] *
      HydroNumWorking[period,hydro];
```

The following statements call the mixed integer linear programming solver, print the optimal solution, and create several data sets that contain various parts of the optimal solution, with variables grouped according to their index sets:

```
   solve;
   print NumWorking NumStartup Excess Output;
   print HydroNumWorking HydroNumStartup HydroOutput;
   print Pump Depth;
   create data sol_data1 from [period type]
      NumWorking NumStartup Excess Output;
   create data sol_data2 from [period hydro]
      HydroNumWorking HydroNumStartup HydroOutput;
   create data sol_data3 from [period] Pump Depth;
quit;
```

Figure 16.1 shows the output from the mixed integer linear programming solver.

Figure 16.1 Output from Mixed Integer Linear Programming Solver

The OPTMODEL Procedure

Problem Summary	
Objective Sense	Minimization
Objective Function	TotalCost
Objective Type	Linear
Number of Variables	75
Bounded Above	0
Bounded Below	20
Bounded Below and Above	54
Free	0
Fixed	1
Binary	20
Integer	30
Number of Constraints	55
Linear LE (<=)	15
Linear EQ (=)	5
Linear GE (>=)	35
Linear Range	0
Constraint Coefficients	190

Performance Information	
Execution Mode	Single-Machine
Number of Threads	4

Solution Summary	
Solver	MILP
Algorithm	Branch and Cut
Objective Function	TotalCost
Solution Status	Optimal
Objective Value	986630
Relative Gap	0
Absolute Gap	0
Primal Infeasibility	1.554312E-15
Bound Infeasibility	0
Integer Infeasibility	0
Best Bound	986630
Nodes	24
Iterations	623
Presolve Time	0.01
Solution Time	0.05

Figure 16.1 *continued*

[1]	[2]	NumWorking	NumStartup	Excess	Output
1	**1**	12	0	365	10565
1	**2**	3	0	1500	5250
1	**3**	0	0	0	0
2	**1**	12	0	4050	14250
2	**2**	9	6	4500	15750
2	**3**	0	0	0	0
3	**1**	12	0	0	10200
3	**2**	9	0	4500	15750
3	**3**	0	0	0	0
4	**1**	12	0	11150	21350
4	**2**	9	0	4500	15750
4	**3**	1	1	0	1500
5	**1**	12	0	0	10200
5	**2**	9	0	4500	15750
5	**3**	0	0	0	0

[1]	[2]	HydroNumWorking	HydroNumStartup	HydroOutput
1	**A**	0	0	0
1	**B**	0	0	0
2	**A**	0	0	0
2	**B**	0	0	0
3	**A**	0	0	0
3	**B**	0	0	0
4	**A**	0	0	0
4	**B**	1	1	1400
5	**A**	0	0	0
5	**B**	1	0	1400

[1]	Pump	Depth
1	815	16.00
2	0	17.63
3	950	17.63
4	0	19.53
5	350	18.12

Features Demonstrated

The following features are demonstrated in this example:

- problem type: mixed integer linear programming
- multiple input and output data sets
- FIX statement
- IMPVAR statement
- `.ub` variable suffix
- modeling startup costs
- IF-THEN/ELSE expression
- CARD function

Chapter 17
Three-Dimensional Noughts and Crosses: A Combinatorial Problem

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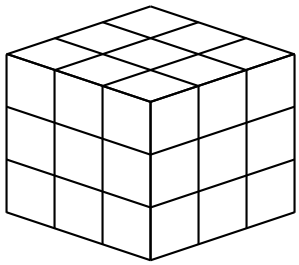
Problem Statement

Twenty-seven cells are arranged $3 \times 3 \times 3$ in a three-dimensional array as shown in Figure 17.1.[1]

Figure 17.1



Three cells are regarded as lying in the same line if they are on the same horizontal or vertical line or the same diagonal. Diagonals exist on each horizontal and vertical section and connecting opposite vertices of the cube. (There are 49 lines altogether.)

Given 13 white balls (noughts) and 14 black balls (crosses), arrange them, one to a cell, so as to minimize the number of lines with balls all of one colour.

[1] Reproduced with permission of John Wiley & Sons Ltd. (Williams 1999, pp. 248–249).

Mathematical Programming Formulation

The formulation shown here generalizes the problem to an $n \times n \times n$ array and an arbitrary number of colors.

Index Sets and Their Members

The following index sets and their members are used in this example:

- $(i, j, k) \in$ CELLS
- color $\in$ COLORS
- line $\in$ LINES
- $(i, j, k) \in$ CELLS_line[line]

Parameters

Table 17.1 shows the parameters that are used in this example.

Table 17.1 Parameters

Parameter Name	Interpretation
n	Number of cells per line
num_lines	Number of lines
num_balls[color]	Number of balls of each color
assigned_color[i,j,k]	Color assigned to cell (i, j, k)

Variables

Table 17.2 shows the variables that are used in this example.

Table 17.2 Variables

Variable Name	Interpretation
IsColor[i,j,k,color]	1 if cell (i, j, k) is color; 0 otherwise
IsMonochromatic[line]	1 if line is monochromatic; 0 otherwise

Objective

The objective is to minimize the following function:

$$\text{NumMonochromaticLines} = \sum_{\text{line} \in \text{LINES}} \text{IsMonochromatic[line]}$$

Constraints

The following constraints are used in this example:

- bounds on variables
- for $(i, j, k) \in$ CELLS,

$$\sum_{\text{color} \in \text{COLORS}} \text{IsColor[i,j,k,color]} = 1$$

- for color $\in$ COLORS,

$$\sum_{(i,j,k) \in \text{CELLS}} \text{IsColor[i,j,k,color]} = \textit{num_balls[color]}$$

- for line $\in$ LINES and color $\in$ COLORS,

$$\sum_{(i,j,k) \in \text{CELLS_line[line]}} \text{IsColor[i,j,k,color]} - |\text{CELLS_line}[\text{line}]| + 1 \leq \text{IsMonochromatic[line]}$$

Input Data

The following data set and macro variable contain the input data that are used in this example:

```
data color_data;
   input num_balls;
   datalines;
13
14
;

%let n = 3;
```

PROC OPTMODEL Statements and Output

The first several PROC OPTMODEL statements declare parameters and index sets and then read the input data:

```
proc optmodel;
   num n = &n;
   set CELLS = 1..n cross 1..n cross 1..n;

   set COLORS;
   num num_balls {COLORS};
   read data color_data into COLORS=[_N_] num_balls;
```

The following statements declare the IsColor variables and the first two families of constraints:

```
   var IsColor {CELLS, COLORS} binary;
   con IsColor_con {<i,j,k> in CELLS}:
      sum {color in COLORS} IsColor[i,j,k,color] = 1;
   con Num_balls_con {color in COLORS}:
      sum {<i,j,k> in CELLS} IsColor[i,j,k,color] = num_balls[color];
```

The following statements declare the IsMonochromatic variables and the objective:

```
   num num_lines init 0;
   set LINES = 1..num_lines;
   var IsMonochromatic {LINES} binary;

   min NumMonochromaticLines = sum {line in LINES} IsMonochromatic[line];
```

The following SET statement declares (but does not populate) an index set that will contain the cells for each line:

```
   set <num,num,num> CELLS_line {LINES};
```

The following FOR loops use the SETOF operator to populate CELLS_line for each row:

```
   for {i in 1..n, j in 1..n} do;
      num_lines = num_lines + 1;
      CELLS_line[num_lines] = setof {k in 1..n} <i,j,k>;
   end;
   for {i in 1..n, k in 1..n} do;
      num_lines = num_lines + 1;
      CELLS_line[num_lines] = setof {j in 1..n} <i,j,k>;
   end;
   for {j in 1..n, k in 1..n} do;
      num_lines = num_lines + 1;
      CELLS_line[num_lines] = setof {i in 1..n} <i,j,k>;
   end;
```

The following FOR loops use the SETOF operator to populate CELLS_line for each face diagonal:

```
for {i in 1..n} do;
   num_lines = num_lines + 1;
   CELLS_line[num_lines] = setof {j in 1..n} <i,j,j>;
   num_lines = num_lines + 1;
   CELLS_line[num_lines] = setof {j in 1..n} <i,j,n+1-j>;
end;
for {j in 1..n} do;
   num_lines = num_lines + 1;
   CELLS_line[num_lines] = setof {i in 1..n} <i,j,i>;
   num_lines = num_lines + 1;
   CELLS_line[num_lines] = setof {i in 1..n} <i,j,n+1-i>;
end;
for {k in 1..n} do;
   num_lines = num_lines + 1;
   CELLS_line[num_lines] = setof {i in 1..n} <i,i,k>;
   num_lines = num_lines + 1;
   CELLS_line[num_lines] = setof {i in 1..n} <i,n+1-i,k>;
end;
```

The following statements use the SETOF operator to populate CELLS_line for each cube diagonal:

```
num_lines = num_lines + 1;
CELLS_line[num_lines] = setof {t in 1..n} <t,t,t>;
num_lines = num_lines + 1;
CELLS_line[num_lines] = setof {t in 1..n} <t,t,n+1-t>;
num_lines = num_lines + 1;
CELLS_line[num_lines] = setof {t in 1..n} <t,n+1-t,t>;
num_lines = num_lines + 1;
CELLS_line[num_lines] = setof {t in 1..n} <t,n+1-t,n+1-t>;
```

The following PUT statements demonstrate that *num_lines* is $((n+2)^3 - n^3)/2$, which is 49 when $n = 3$.

```
put num_lines=;
put (((n+2)^3 - n^3) / 2)=;
```

The following CON statement enforces the rule that if all cells in a line are the same color, then IsMonochromatic[line] $= 1$:

```
con Link_con {line in LINES, color in COLORS}:
   sum {<i,j,k> in CELLS_line[line]} IsColor[i,j,k,color]
 - card(CELLS_line[line]) + 1
<= IsMonochromatic[line];
```

The following statements call the mixed integer linear programming solver and use the **.sol** variable suffix to populate *assigned_color[i,j,k]*:

```
solve;
num assigned_color {CELLS};
for {<i,j,k> in CELLS} do;
   for {color in COLORS: IsColor[i,j,k,color].sol > 0.5} do;
      assigned_color[i,j,k] = color;
      leave;
   end;
end;
```

The following statements print the values of *assigned_color[i,j,k]* for each *i* and display in the log which cells make up each monochromatic line:

```
   for {i in 1..n}
      print {j in 1..n, k in 1..n} assigned_color[i,j,k];
   for {line in LINES: IsMonochromatic[line].sol > 0.5}
      put CELLS_line[line]=;
```

By default, the PUT statement writes to the log. You can use the following FILE statement to redirect log output to the listing:

```
   file print;
   for {line in LINES: IsMonochromatic[line].sol > 0.5}
      put CELLS_line[line]=;
quit;
```

Figure 17.2 shows the output from the mixed integer linear programming solver.

Figure 17.2 Output from Mixed Integer Linear Programming Solver

The OPTMODEL Procedure

Problem Summary	
Objective Sense	Minimization
Objective Function	NumMonochromaticLines
Objective Type	Linear
Number of Variables	103
Bounded Above	0
Bounded Below	0
Bounded Below and Above	103
Free	0
Fixed	0
Binary	103
Integer	0
Number of Constraints	127
Linear LE (<=)	98
Linear EQ (=)	29
Linear GE (>=)	0
Linear Range	0
Constraint Coefficients	500

Performance Information	
Execution Mode	Single-Machine
Number of Threads	4

Figure 17.2 *continued*

Solution Summary	
Solver	MILP
Algorithm	Branch and Cut
Objective Function	NumMonochromaticLines
Solution Status	Optimal
Objective Value	4
Relative Gap	0
Absolute Gap	0
Primal Infeasibility	0
Bound Infeasibility	0
Integer Infeasibility	0
Best Bound	4
Nodes	239
Iterations	5025
Presolve Time	0.01
Solution Time	0.09

assigned_color			
	1	**2**	**3**
1	2	1	2
2	1	1	2
3	2	2	1

assigned_color			
	1	**2**	**3**
1	1	2	2
2	2	2	1
3	1	1	2

assigned_color			
	1	**2**	**3**
1	2	1	1
2	1	2	2
3	1	2	1

Figure 17.3 shows the output from the PUT statement.

Figure 17.3 Output from PUT Statement

```
The OPTMODEL Procedure

CELLS_line[35]={<1,1,3>,<2,1,2>,<3,1,1>}
CELLS_line[39]={<1,3,3>,<2,3,2>,<3,3,1>}
CELLS_line[41]={<1,3,1>,<2,2,1>,<3,1,1>}
CELLS_line[45]={<1,3,3>,<2,2,3>,<3,1,3>}
```

Features Demonstrated

The following features are demonstrated in this example:

- problem type: mixed integer linear programming
- sets of tuples
- set operators CROSS and SETOF
- sets indexed by other sets
- modeling if-then constraints by using binary variables
- CARD function
- PUT statement
- `.sol` variable suffix
- FILE statement

Chapter 18
Optimizing a Constraint: Reconstructing an Integer Programming Constraint More Simply

Contents

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Problem Statement

In an integer programming problem the following constraint occurs:[1]

$$9x_1 + 13x_2 - 14x_3 + 17x_4 + 13x_5 - 19x_6 + 23x_7 + 21x_8 \leq 37.$$

All the variables occurring in this constraint are 0-1 variables, i.e. they can only take the value of 0 or 1.

Find the 'simplest' version of this constraint. The objective is to find another constraint involving these variables which is logically equivalent to the original constraint but which has the smallest possible absolute value of the right-hand side (with all coefficients of similar signs to the original coefficients).

If the objective were to find an equivalent constraint where the sum of the absolute values of the coefficients (apart from the right-hand side coefficient) were a minimum what would be the result?

Mathematical Programming Formulation

The formulation shown here differs from Williams (1999) and does not require transforming the original constraint into a standard form with positive coefficients. The new constraint to be found is

$$\sum_{j \in \text{VARS}} \text{Alpha[j]} \cdot x_j \leq \text{Alpha[0]}$$

[1] Reproduced with permission of John Wiley & Sons Ltd. (Williams 1999, p. 249).

Index Sets and Their Members

The following index sets and their members are used in this example:

- $j \in$ VARS
- $j \in$ VARS0 $=$ VARS $\cup \{0\}$
- point $\in$ FEAS_POINTS
- point $\in$ INFEAS_POINTS

Parameters

Table 18.1 shows the parameters that are used in this example.

Table 18.1 Parameters

Parameter Name	Interpretation
n	Number of decision variables in constraint
a[j]	Original constraint coefficient of variable *j*, with *a[0]* equal to original right-hand side
x_feas[point,j]	Value of variable *j* in feasible point
x_infeas[point,j]	Value of variable *j* in infeasible point

Variables

Table 18.2 shows the variables that are used in this example.

Table 18.2 Variables

Variable Name	Interpretation
Scale[j]	Nonnegative scalar multiplier for coefficient of variable *j*
Alpha[j]	Coefficient of variable *j* in new constraint

Objectives

The first objective is to minimize the absolute value of the new right-hand side:

$$\text{Objective1} = |a[0]| \cdot \text{Scale}[0]$$

The second objective is to minimize the sum of absolute values of the new left-hand side coefficients:

$$\text{Objective2} = \sum_{j \in \text{VARS}} |a[j]| \cdot \text{Scale[j]}$$

Constraints

The following constraints are used in this example:

- bounds on variables
- for $j \in$ VARS0,

  $$\text{Alpha[j]} = a[j] \cdot \text{Scale[j]}$$

- for point $\in$ FEAS_POINTS,

  $$\sum_{j \in \text{VARS}} \text{Alpha[j]} \cdot x_feas[point,j] \leq \text{Alpha[0]}$$

- for point $\in$ INFEAS_POINTS,

  $$\sum_{j \in \text{VARS}} \text{Alpha[j]} \cdot x_infeas[point,j] \geq \text{Alpha[0]} + 1$$

Input Data

The following data set and macro variables contain the input data that are used in this example:

```
data a_data;
   input a @@;
   datalines;
9 13 -14 17 13 -19 23 21
;

%let b = 37;

/* populate macro variable n as number of decision variables */
%let dsid = %sysfunc(open(a_data));
%let n = %sysfunc(attrn(&dsid, NOBS));
%let rc = %sysfunc(close(&dsid));
```

PROC OPTMODEL Statements and Output

The following PROC TRANSPOSE statements and DATA step create an input data set for the CLP procedure in SAS/OR software:

```
proc transpose data=a_data out=trans(drop=_name_) prefix=x;
run;

data condata_feas(drop=j);
   length _type_ $8;
   array x[&n];
   set trans;
   _type_ = 'le';
   _rhs_ = &b;
   output;
   do j = 1 to &n;
      x[j] = 1;
   end;
   _type_ = 'binary';
   _rhs_ = .;
   output;
run;
```

The following PROC CLP statements find all feasible solutions and save them in the out_feas data set:

```
proc clp data=condata_feas out=out_feas allsolns usecondatavars=1;
run;
```

The following DATA step creates another input data set for PROC CLP, for the purpose of finding all binary solutions that violate the original constraint:

```
data condata_infeas;
   set condata_feas;
   if _N_ = 1 then do;
      _type_ = 'ge';
      _rhs_ = _rhs_ + 1;
   end;
run;
```

The following PROC CLP statements find all such solutions and save them in the out_infeas data set:

```
proc clp data=condata_infeas out=out_infeas allsolns usecondatavars=1;
run;
```

The first several PROC OPTMODEL statements declare sets and parameters and read the original constraint data:

```
proc optmodel;
   set VARS;
   set VARS0 = VARS union {0};
   num a {VARS0};
   read data a_data into VARS=[_N_] a;
   a[0] = &b;
```

The following statements declare the FEAS_POINTS set and the *x_feas* parameter and populate them by reading the out_feas data set:

```
   set FEAS_POINTS;
   num x_feas {FEAS_POINTS, VARS};
   read data out_feas into FEAS_POINTS=[_N_]
      {j in VARS} <x_feas[_N_,j]=col('x'||j)>;
```

The following statements declare the INFEAS_POINTS set and the *x_infeas* parameter and populate them by reading the out_infeas data set:

```
   set INFEAS_POINTS;
   num x_infeas {INFEAS_POINTS, VARS};
   read data out_infeas into INFEAS_POINTS=[_N_]
      {j in VARS} <x_infeas[_N_,j]=col('x'||j)>;
```

The following statements declare the variables and constraints:

```
   var Scale {VARS0} >= 0;
   impvar Alpha {j in VARS0} = a[j] * Scale[j];

   con Feas_con {point in FEAS_POINTS}:
      sum {j in VARS} Alpha[j] * x_feas[point,j] <= Alpha[0];
   con Infeas_con {point in INFEAS_POINTS}:
      sum {j in VARS} Alpha[j] * x_infeas[point,j] >= Alpha[0] + 1;
```

The following statements solve the problem by using the first objective and then print the solution:

```
   min Objective1 = abs(a[0]) * Scale[0];
   solve;
   print a Scale Alpha;
```

Figure 18.1 shows the output from the linear programming solver for the first objective.

Figure 18.1 Output from Linear Programming Solver, First Objective

The OPTMODEL Procedure

Problem Summary	
Objective Sense	Minimization
Objective Function	Objective1
Objective Type	Linear
Number of Variables	9
Bounded Above	0
Bounded Below	9
Bounded Below and Above	0
Free	0
Fixed	0
Number of Constraints	256
Linear LE (<=)	152
Linear EQ (=)	0
Linear GE (>=)	104
Linear Range	0
Constraint Coefficients	1280

Performance Information	
Execution Mode	Single-Machine
Number of Threads	1

Solution Summary	
Solver	LP
Algorithm	Dual Simplex
Objective Function	Objective1
Solution Status	Optimal
Objective Value	25
Primal Infeasibility	7.105427E-15
Dual Infeasibility	0
Bound Infeasibility	0
Iterations	20
Presolve Time	0.02
Solution Time	0.02

Figure 18.1 *continued*

[1]	a	Scale	Alpha
0	37	0.67568	25
1	9	0.66667	6
2	13	0.69231	9
3	-14	0.71429	-10
4	17	0.70588	12
5	13	0.69231	9
6	-19	0.68421	-13
7	23	0.69565	16
8	21	0.66667	14

The following statements solve the problem by using the second objective and then print the solution:

```
   min Objective2 = sum {j in VARS} abs(a[j]) * Scale[j];
   solve;
   print a Scale Alpha;
quit;
```

Figure 18.2 shows the output from the linear programming solver for the second objective.

Figure 18.2 Output from Linear Programming Solver, Second Objective

Problem Summary	
Objective Sense	Minimization
Objective Function	Objective2
Objective Type	Linear
Number of Variables	9
Bounded Above	0
Bounded Below	9
Bounded Below and Above	0
Free	0
Fixed	0
Number of Constraints	256
Linear LE (<=)	152
Linear EQ (=)	0
Linear GE (>=)	104
Linear Range	0
Constraint Coefficients	1280

Performance Information	
Execution Mode	Single-Machine
Number of Threads	1

Figure 18.2 *continued*

Solution Summary	
Solver	LP
Algorithm	Dual Simplex
Objective Function	Objective2
Solution Status	Optimal
Objective Value	89
Primal Infeasibility	3.552714E-15
Dual Infeasibility	0
Bound Infeasibility	0
Iterations	19
Presolve Time	0.00
Solution Time	0.00

[1]	a	Scale	Alpha
0	37	0.67568	25
1	9	0.66667	6
2	13	0.69231	9
3	-14	0.71429	-10
4	17	0.70588	12
5	13	0.69231	9
6	-19	0.68421	-13
7	23	0.69565	16
8	21	0.66667	14

For this test instance, it turns out that the optimal solutions for the two different objectives agree.

In SAS/OR 13.2, you can also access the CLP solver from within PROC OPTMODEL by using the SOLVE WITH CLP statement. The first several PROC OPTMODEL statements are the same as before:

```
proc optmodel;
   set VARS;
   set VARS0 = VARS union {0};
   num a {VARS0};
   read data a_data into VARS=[_N_] a;
   a[0] = &b;
```

The following statements declare variables and a constraint for the CLP problem:

```
   var X {VARS} binary;
   con CLP_con:
      sum {j in VARS} a[j] * X[j] <= a[0];
```

The following statements call the CLP solver and use the FINDALLSOLNS option to find all solutions, and then they use the predeclared numeric parameter _NSOL_ and the `.sol` variable suffix to retrieve the resulting solutions:

```
solve with CLP / findallsolns;
set FEAS_POINTS;
FEAS_POINTS = 1.._NSOL_;
num x_feas {FEAS_POINTS, VARS};
for {s in FEAS_POINTS, j in VARS} x_feas[s,j]=X[j].sol[s];
```

The following statements modify the right-hand side of the constraint by changing the `.lb` constraint suffix and then use the CONSTANT function to effectively remove the previously declared upper bound by replacing it with the largest machine-representable number:

```
CLP_con.lb = CLP_con.ub + 1;
CLP_con.ub = constant('BIG');
```

The following statements call the CLP solver again and retrieve the resulting solutions:

```
solve with CLP / findallsolns;
set INFEAS_POINTS;
INFEAS_POINTS = 1.._NSOL_;
num x_infeas {INFEAS_POINTS, VARS};
for {s in INFEAS_POINTS, j in VARS} x_infeas[s,j]=X[j].sol[s];
```

The remaining statements are the same as before, except that now the PROBLEM and USE PROBLEM statements are used to switch the focus from the CLP problem to the LP problem:

```
var Scale {VARS0} >= 0;
impvar Alpha {j in VARS0} = a[j] * Scale[j];

con Feas_con {point in FEAS_POINTS}:
   sum {j in VARS} Alpha[j] * x_feas[point,j] <= Alpha[0];
con Infeas_con {point in INFEAS_POINTS}:
   sum {j in VARS} Alpha[j] * x_infeas[point,j] >= Alpha[0] + 1;

min Objective1 = abs(a[0]) * Scale[0];

problem LP_problem include Scale Feas_con Infeas_con Objective1;
use problem LP_problem;

solve;
print a Scale Alpha;

min Objective2 = sum {j in VARS} abs(a[j]) * Scale[j];
solve;
print a Scale Alpha;
quit;
```

Features Demonstrated

The following features are demonstrated in this example:

- problem type: linear programming
- CLP procedure
- set operator UNION
- IMPVAR statement
- ABS function
- reading multiple data sets
- multiple objectives
- SOLVE WITH CLP
- CLP solver option FINDALLSOLNS
- multiple solutions
- predeclared numeric parameter _NSOL_
- using the **.lb** and **.ub** constraint suffixes to modify the right-hand side of a constraint
- CONSTANT function
- PROBLEM and USE PROBLEM statements

Chapter 19
Distribution 1: Which Factories and Depots to Supply Which Customers

Contents

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Problem Statement

A company has two factories, one at Liverpool and one at Brighton.[1] In addition it has four depots with storage facilities at Newcastle, Birmingham, London and Exeter. The company sells its product to six customers C1, C2, . . . , C6. Customers can be supplied either from a depot or from the factory direct (see Figure 19.1).

[1] Reproduced with permission of John Wiley & Sons Ltd. (Williams 1999, pp. 249–251).

Figure 19.1


Newcastle
C1
Liverpool
C2
Birmingham
C3
London
C4
Brighton
C5
Exeter
C6
Factories
Depots
Customers

Table 19.1

Supplied to	Supplier[a]					
	Liverpool factory	**Brighton factory**	**Newcastle depot**	**Birmingham depot**	**London depot**	**Exeter depot**
Depots						
Newcastle	0.5	—				
Birmingham	0.5	0.3				
London	1.0	0.5				
Exeter	0.2	0.2				
Customers						
C1	1.0	2.0	—	1.0	—	—
C2	—	—	1.5	0.5	1.5	—
C3	1.5	—	0.5	0.5	2.0	0.2
C4	2.0	—	1.5	1.0	—	1.5
C5	—	—	—	0.5	0.5	0.5
C6	1.0	—	1.0	—	1.5	1.5

[a] A dash indicates the impossibility of certain suppliers for certain depots or customers.

The distribution costs (which are borne by the company) are known; they are given in Table 19.1 (in £ per ton delivered).

Certain customers have expressed preferences for being supplied from factories or depots which they are used to. The preferred suppliers are

C1	Liverpool (factory)
C2	Newcastle (depot)
C3	No preferences
C4	No preferences
C5	Birmingham (depot)
C6	Exeter or London (depots)

Each factory has a monthly capacity given below which cannot be exceeded:

Liverpool	150,000 tons
Brighton	200,000 tons

Each depot has a maximum monthly throughput given below which cannot be exceeded:

Newcastle	70,000 tons
Birmingham	50,000 tons
London	100,000 tons
Exeter	40,000 tons

Each customer has a monthly requirement given below which must be met:

C1	50,000 tons
C2	10,000 tons
C3	40,000 tons
C4	35,000 tons
C5	60,000 tons
C6	20,000 tons

The company would like to determine:

(1) What distribution pattern would minimize overall cost?

(2) What the effect of increasing factory and depot capacities would be on distribution costs?

(3) What the effects of small changes in costs, capacities and requirements would be on the distribution pattern?

(4) Would it be possible to meet all customers preferences regarding suppliers and if so what would the extra cost of doing this be?

Mathematical Programming Formulation

The problem is represented as a network flow problem with side constraints. Each node corresponds to a factory, depot, or customer, and each arc represents a shipment of the product from one place to another.

Index Sets and Their Members

The following index sets and their members are used in this example:

- $(i, j) \in$ ARCS
- i, factory $\in$ FACTORIES
- i, depot $\in$ DEPOTS
- i, customer $\in$ CUSTOMERS
- $i \in$ NODES
- $(i, j) \in$ PREFERRED_ARCS
- $j \in$ CUSTOMERS_WITH_PREFERENCES

Parameters

Table 19.2 shows the parameters that are used in this example.

Table 19.2 Parameters

Parameter Name	Interpretation
cost[i,j]	Distribution costs (in £ per ton delivered)
capacity[factory]	Monthly capacity per factory (in tons)
throughput[depot]	Maximum monthly throughput per depot (in tons)
demand[customer]	Monthly demand per customer (in tons)
supply[i]	Supply at node i (in tons)

Variables

Table 19.3 shows the variables that are used in this example.

Table 19.3 Variables

Variable Name	Interpretation
Flow[i,j]	Flow across arc (i, j)

Objectives

One objective is to minimize the following function:

$$\text{TotalCost} = \sum_{(i,j)\in\text{ARCS}} cost[i,j] \cdot \text{Flow[i,j]}$$

Another objective is to minimize the following function:

$$\text{NonpreferredFlow} = \sum_{\substack{(i,j)\in\text{ARCS}\setminus\text{PREFERRED_ARCS:}\\ j\in\text{CUSTOMERS_WITH_PREFERENCES}}} \text{Flow[i,j]}$$

Constraints

The following constraints are used in this example:

- bounds on variables
- for $i \in$ NODES,

$$\sum_{(i,j)\in\text{ARCS}} \text{Flow[i,j]} - \sum_{(j,i)\in\text{ARCS}} \text{Flow[j,i]} \leq supply[i]$$

- for $i \in$ DEPOTS,

$$\sum_{(i,j)\in \text{ARCS}} \text{Flow[i,j]} \leq \textit{throughput[i]}$$

- NonpreferredFlow $\leq$ *NonpreferredFlow.sol*

Input Data

The following data sets contain the input data that are used in this example:

```
data arc_data;
   input j $11. i $11. cost;
   datalines;
Newcastle  Liverpool  0.5
Birmingham Liverpool  0.5
Birmingham Brighton   0.3
London     Liverpool  1.0
London     Brighton   0.5
Exeter     Liverpool  0.2
Exeter     Brighton   0.2
C1         Liverpool  1.0
C1         Brighton   2.0
C1         Birmingham 1.0
C2         Newcastle  1.5
C2         Birmingham 0.5
C2         London     1.5
C3         Liverpool  1.5
C3         Newcastle  0.5
C3         Birmingham 0.5
C3         London     2.0
C3         Exeter     0.2
C4         Liverpool  2.0
C4         Newcastle  1.5
C4         Birmingham 1.0
C4         Exeter     1.5
C5         Birmingham 0.5
C5         London     0.5
C5         Exeter     0.5
C6         Liverpool  1.0
C6         Newcastle  1.0
C6         London     1.5
C6         Exeter     1.5
;

data customer_data;
   input customer $ demand;
   datalines;
C1 50000
C2 10000
```

```
C3 40000
C4 35000
C5 60000
C6 20000
;

data factory_data;
   input factory $10. capacity;
   datalines;
Liverpool 150000
Brighton  200000
;

data depot_data;
   input depot $11. throughput;
   datalines;
Newcastle   70000
Birmingham  50000
London     100000
Exeter      40000
;

data preferred_arc_data;
   input j $ i $11.;
   datalines;
C1 Liverpool
C2 Newcastle
C5 Birmingham
C6 Exeter
C6 London
;
```

PROC OPTMODEL Statements and Output

The first several PROC OPTMODEL statements declare index sets and parameters and then read the input data:

```
proc optmodel;
   set <str,str> ARCS;
   num cost {ARCS};
   read data arc_data into ARCS=[i j] cost;

   set <str> FACTORIES;
   num capacity {FACTORIES};
   read data factory_data into FACTORIES=[factory] capacity;

   set <str> DEPOTS;
   num throughput {DEPOTS};
   read data depot_data into DEPOTS=[depot] throughput;

   set <str> CUSTOMERS;
   num demand {CUSTOMERS};
   read data customer_data into CUSTOMERS=[customer] demand;
```

The following statements use the UNION set operator to declare the NODES index set, declare the *supply* parameter with an initial value of 0, and populate *supply* for both factories and customers:

```
set NODES = FACTORIES union DEPOTS union CUSTOMERS;
num supply {NODES} init 0;
for {i in FACTORIES} supply[i] = capacity[i];
for {i in CUSTOMERS} supply[i] = -demand[i];
```

The following statements declare the variables, constraints, and TotalCost objective:

```
var Flow {ARCS} >= 0;

con Flow_balance_con {i in NODES}:
   sum {<(i),j> in ARCS} Flow[i,j] - sum {<j,(i)> in ARCS} Flow[j,i]
<= supply[i];

con Depot_con {i in DEPOTS}:
   sum {<(i),j> in ARCS} Flow[i,j] <= throughput[i];

min TotalCost = sum {<i,j> in ARCS} cost[i,j] * Flow[i,j];
```

The following statements call the default linear programming algorithm (which is the dual simplex algorithm), print the positive variables in the resulting optimal solution, and print the left-hand side (`.body`), right-hand side (`.ub`), and dual value (`.dual`) of each constraint:

```
put 'Minimizing TotalCost...';
solve;
print {<i,j> in ARCS: Flow[i,j].sol > 0} Flow;
print Flow_balance_con.body Flow_balance_con.ub Flow_balance_con.dual;
print Depot_con.body Depot_con.ub Depot_con.dual;
```

Figure 19.2 shows the output when you use the (default) dual simplex algorithm.

Figure 19.2 Output from Dual Simplex Algorithm, Minimizing TotalCost

The OPTMODEL Procedure

Problem Summary	
Objective Sense	Minimization
Objective Function	TotalCost
Objective Type	Linear
Number of Variables	29
Bounded Above	0
Bounded Below	29
Bounded Below and Above	0
Free	0
Fixed	0
Number of Constraints	16
Linear LE (<=)	16
Linear EQ (=)	0
Linear GE (>=)	0
Linear Range	0
Constraint Coefficients	75

Performance Information	
Execution Mode	Single-Machine
Number of Threads	1

Solution Summary	
Solver	LP
Algorithm	Dual Simplex
Objective Function	TotalCost
Solution Status	Optimal
Objective Value	198500
Primal Infeasibility	0
Dual Infeasibility	0
Bound Infeasibility	0
Iterations	14
Presolve Time	0.00
Solution Time	0.00

Figure 19.2 *continued*

[1]	[2]	Flow
Birmingham	**C2**	10000
Birmingham	**C4**	35000
Birmingham	**C5**	5000
Brighton	**Birmingham**	50000
Brighton	**London**	55000
Exeter	**C3**	40000
Liverpool	**C1**	50000
Liverpool	**C6**	20000
Liverpool	**Exeter**	40000
London	**C5**	55000

[1]	Flow_balance_con.BODY	Flow_balance_con.UB	Flow_balance_con.DUAL
Birmingham	0	0	-0.3
Brighton	105000	200000	0.0
C1	-50000	-50000	-1.0
C2	-10000	-10000	-1.0
C3	-40000	-40000	-1.0
C4	-35000	-35000	-1.5
C5	-60000	-60000	-1.0
C6	-20000	-20000	-1.0
Exeter	0	0	-0.2
Liverpool	110000	150000	0.0
London	0	0	-0.5
Newcastle	0	0	-0.5

[1]	Depot_con.BODY	Depot_con.UB	Depot_con.DUAL
Birmingham	50000	50000	-0.2
Exeter	40000	40000	-0.6
London	55000	100000	0.0
Newcastle	0	70000	0.0

The following statements call the network simplex linear programming algorithm and print the same solution information as before:

```
put 'Minimizing TotalCost by using network simplex...';
solve with LP / algorithm=ns;
print {<i,j> in ARCS: Flow[i,j].sol > 0} Flow;
print Flow_balance_con.body Flow_balance_con.ub Flow_balance_con.dual;
print Depot_con.body Depot_con.ub Depot_con.dual;
```

Figure 19.3 shows the output when you use the ALGORITHM=NS option to invoke the network simplex algorithm.

Figure 19.3 Output from Network Simplex Algorithm, Minimizing TotalCost

Problem Summary	
Objective Sense	Minimization
Objective Function	TotalCost
Objective Type	Linear
Number of Variables	29
Bounded Above	0
Bounded Below	29
Bounded Below and Above	0
Free	0
Fixed	0
Number of Constraints	16
Linear LE (<=)	16
Linear EQ (=)	0
Linear GE (>=)	0
Linear Range	0
Constraint Coefficients	75

Performance Information	
Execution Mode	Single-Machine
Number of Threads	1

Solution Summary	
Solver	LP
Algorithm	Network Simplex
Objective Function	TotalCost
Solution Status	Optimal
Objective Value	198500
Primal Infeasibility	0
Dual Infeasibility	0
Bound Infeasibility	0
Iterations	15
Iterations2	5
Presolve Time	0.00
Solution Time	0.00

Figure 19.3 *continued*

[1]	[2]	Flow
Birmingham	**C2**	10000
Birmingham	**C4**	35000
Birmingham	**C5**	5000
Brighton	**Birmingham**	50000
Brighton	**London**	55000
Exeter	**C3**	40000
Liverpool	**C1**	50000
Liverpool	**C6**	20000
Liverpool	**Exeter**	40000
London	**C5**	55000

[1]	Flow_balance_con.BODY	Flow_balance_con.UB	Flow_balance_con.DUAL
Birmingham	0	0	-0.3
Brighton	105000	200000	0.0
C1	-50000	-50000	-1.0
C2	-10000	-10000	-1.0
C3	-40000	-40000	-1.0
C4	-35000	-35000	-1.5
C5	-60000	-60000	-1.0
C6	-20000	-20000	-1.0
Exeter	0	0	-0.2
Liverpool	110000	150000	0.0
London	0	0	-0.5
Newcastle	0	0	-0.5

[1]	Depot_con.BODY	Depot_con.UB	Depot_con.DUAL
Birmingham	50000	50000	-0.2
Exeter	40000	40000	-0.6
London	55000	100000	0.0
Newcastle	0	70000	0.0

The following statements call the linear programming solver to minimize NonpreferredFlow and print both objectives, the positive variables in the resulting optimal solution, and the flow along nonpreferred arcs:

```
set <str,str> PREFERRED_ARCS;
read data preferred_arc_data into PREFERRED_ARCS=[i j];
set CUSTOMERS_WITH_PREFERENCES = setof {<i,j> in PREFERRED_ARCS} j;
min NonpreferredFlow =
   sum {<i,j> in ARCS diff PREFERRED_ARCS: j in CUSTOMERS_WITH_PREFERENCES}
      Flow[i,j];

put 'Minimizing NonpreferredFlow...';
solve;
print TotalCost NonpreferredFlow;
print {<i,j> in ARCS: Flow[i,j].sol > 0} Flow;
print
   {<i,j> in ARCS diff PREFERRED_ARCS: j in CUSTOMERS_WITH_PREFERENCES}
      Flow;
```

Figure 19.4 shows the output from the linear programming solver.

Figure 19.4 Output from Linear Programming Solver, Minimizing NonpreferredFlow

Problem Summary	
Objective Sense	Minimization
Objective Function	NonpreferredFlow
Objective Type	Linear
Number of Variables	29
Bounded Above	0
Bounded Below	29
Bounded Below and Above	0
Free	0
Fixed	0
Number of Constraints	16
Linear LE (<=)	16
Linear EQ (=)	0
Linear GE (>=)	0
Linear Range	0
Constraint Coefficients	75

Performance Information	
Execution Mode	Single-Machine
Number of Threads	1

Solution Summary	
Solver	LP
Algorithm	Dual Simplex
Objective Function	NonpreferredFlow
Solution Status	Optimal
Objective Value	10000
Primal Infeasibility	0
Dual Infeasibility	0
Bound Infeasibility	0
Iterations	17
Presolve Time	0.00
Solution Time	0.00

TotalCost	NonpreferredFlow
356000	10000

Figure 19.4 *continued*

[1]	[2]	Flow
Birmingham	C5	50000
Brighton	Birmingham	50000
Brighton	Exeter	40000
Brighton	London	30000
Exeter	C3	40000
Liverpool	C1	50000
Liverpool	C4	35000
Liverpool	Newcastle	65000
London	C5	10000
London	C6	20000
Newcastle	C2	65000

[1]	[2]	Flow
Birmingham	C1	0
Birmingham	C2	0
Brighton	C1	0
Exeter	C5	0
Liverpool	C6	0
London	C2	0
London	C5	10000
Newcastle	C6	0

The following CON statement declares a constraint that limits NonpreferredFlow to the minimum value found in the previous solve:

```
con Objective_cut:
   NonpreferredFlow <= NonpreferredFlow.sol;
```

The following statements call the linear programming solver to minimize TotalCost with a constrained NonpreferredFlow and print the same solution information as before:

```
put 'Minimizing TotalCost with constrained NonpreferredFlow...';
solve obj TotalCost;
print TotalCost NonpreferredFlow;
print {<i,j> in ARCS: Flow[i,j].sol > 0} Flow;
print
   {<i,j> in ARCS diff PREFERRED_ARCS: j in CUSTOMERS_WITH_PREFERENCES}
      Flow;
quit;
```

Figure 19.5 shows the output from the linear programming solver. As expected, NonpreferredFlow remains at its minimum value and TotalCost is less than its value from Figure 19.4.

Figure 19.5 Output from Linear Programming Solver, Minimizing TotalCost with Constrained NonpreferredFlow

Problem Summary	
Objective Sense	Minimization
Objective Function	TotalCost
Objective Type	Linear
Number of Variables	29
Bounded Above	0
Bounded Below	29
Bounded Below and Above	0
Free	0
Fixed	0
Number of Constraints	17
Linear LE (<=)	17
Linear EQ (=)	0
Linear GE (>=)	0
Linear Range	0
Constraint Coefficients	83

Performance Information	
Execution Mode	Single-Machine
Number of Threads	1

Solution Summary	
Solver	LP
Algorithm	Dual Simplex
Objective Function	TotalCost
Solution Status	Optimal
Objective Value	246000
Primal Infeasibility	0
Dual Infeasibility	2.220446E-16
Bound Infeasibility	0
Iterations	18
Presolve Time	0.00
Solution Time	0.00

TotalCost	NonpreferredFlow
246000	10000

Figure 19.5 *continued*

[1]	[2]	Flow
Birmingham	C5	50000
Brighton	Birmingham	50000
Brighton	London	30000
Exeter	C3	40000
Liverpool	C1	50000
Liverpool	C4	35000
Liverpool	Exeter	40000
Liverpool	Newcastle	10000
London	C5	10000
London	C6	20000
Newcastle	C2	10000

[1]	[2]	Flow
Birmingham	C1	0
Birmingham	C2	0
Brighton	C1	0
Exeter	C5	0
Liverpool	C6	0
London	C2	0
London	C5	10000
Newcastle	C6	0

Features Demonstrated

The following features are demonstrated in this example:

- problem type: linear programming (network flow with side constraints)
- sets of tuples
- reading multiple data sets
- set operators UNION, SETOF, and DIFF
- INIT option
- implicit slice
- using a colon (:) to select members of a set
- `.body` constraint suffix
- `.ub` constraint suffix
- `.dual` constraint suffix
- multiple objectives and the OBJ option
- ALGORITHM= option

Chapter 20
Depot Location (Distribution 2): Where Should New Depots Be Built

Contents

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Problem Statement

In the distribution problem there is a possibility of opening new depots at Bristol and Northampton as well as of enlarging the Birmingham depot.[1]

It is not considered desirable to have more than four depots and if necessary Newcastle or Exeter (or both) can be closed down.

The monthly costs (in interest charges) of the possible new depots and expansion at Birmingham are given in Table 20.1 together with the potential monthly throughputs.

The monthly savings of closing down the Newcastle and Exeter depots are given in Table 20.2.

Table 20.1

	Cost (£1000)	Throughput (1000 tons)
Bristol	12	30
Northampton	4	25
Birmingham (expansion)	3	20

Table 20.2

	Saving (£1000)
Newcastle	10
Exeter	5

[1] Reproduced with permission of John Wiley & Sons Ltd. (Williams 1999, pp. 251–252).

Table 20.3

Supplied to	Supplier: Liverpool factory	Supplier: Brighton factory	Supplier: Bristol depot	Supplier: Northampton depot
New depots				
Bristol	0.6	0.4		
Northampton	0.4	0.3		
Customers				
C1			1.2	—
C2			0.6	0.4
C3	As given for		0.5	—
C4	distribution problem		—	0.5
C5			0.3	0.6
C6			0.8	0.9

The distribution costs involving the new depots are given in Table 20.3 (in £ per ton delivered).

Which new depots should be built? Should Birmingham be expanded? Should Exeter or Newcastle be closed down? What would be the best resultant distribution pattern to minimize overall costs?

Mathematical Programming Formulation

This formulation builds on the formulation used in Chapter 19. This section includes only the new elements of the formulation.

Index Sets and Their Members

The following additional index set and its members are used in this example:

- $i, \text{depot} \in \text{EXPAND_DEPOTS}$

Parameters

Table 20.4 shows the additional parameters that are used in this example.

Table 20.4 Parameters

Parameter Name	Interpretation
open_cost[depot]	Monthly costs for opening each depot (in £)
close_savings[depot]	Monthly savings for closing each depot (in £)
expand_throughput[depot]	Potential monthly throughput per depot (in tons)
expand_cost[depot]	Monthly costs for expanding each depot (in £)
max_num_depots	Maximum number of depots allowed to be open

Variables

Table 20.5 shows the additional variables that are used in this example.

Table 20.5 Variables

Variable Name	Interpretation
IsOpen[depot]	1 if depot is open; 0 otherwise
Expand[depot]	1 if depot is expanded; 0 otherwise
FixedCost	Fixed cost that depends only on the binary variables
VariableCost	Variable cost that depends only on the continuous variables

Objective

The objective is to minimize the following function:

$$\text{TotalCost} = \text{FixedCost} + \text{VariableCost}$$

where

$$\text{FixedCost} = \sum_{\text{depot} \in \text{DEPOTS}} (\textit{open_cost[depot]} \cdot \text{IsOpen[depot]} - \textit{close_savings[depot]} \cdot (1 - \text{IsOpen[depot]})) + \sum_{\text{depot} \in \text{EXPAND_DEPOTS}} \textit{expand_cost[depot]} \cdot \text{Expand[depot]}$$

and

$$\text{VariableCost} = \sum_{(i,j) \in \text{ARCS}} \textit{cost[i,j]} \cdot \text{Flow[i,j]}$$

Constraints

The following additional and modified constraints are used in this example:

- bounds on variables
- $\sum_{i \in \mathrm{DEPOTS}} \mathsf{IsOpen[i]} \leq max_num_depots$
- for $i \in \mathrm{DEPOTS}$,

  $\sum_{(i,j) \in \mathrm{ARCS}} \mathsf{Flow[i,j]} \leq throughput[i] \cdot \mathsf{IsOpen[i]} + (\text{if } i \in \mathrm{EXPAND_DEPOTS}, \text{ then } expand_throughput[i] \cdot \mathsf{Expand[i]})$

- for $i \in \mathrm{EXPAND_DEPOTS}$,

  $\mathsf{Expand[i]} \leq \mathsf{IsOpen[i]}$

Input Data

The following additional and modified data sets and macro variable contain the input data that are used in this example:

```
data arc_data;
   input j $12. i $12. cost;
   datalines;
Newcastle   Liverpool   0.5
Birmingham  Liverpool   0.5
Birmingham  Brighton    0.3
London      Liverpool   1.0
London      Brighton    0.5
Exeter      Liverpool   0.2
Exeter      Brighton    0.2
C1          Liverpool   1.0
C1          Brighton    2.0
C1          Birmingham  1.0
C2          Newcastle   1.5
C2          Birmingham  0.5
C2          London      1.5
C3          Liverpool   1.5
C3          Newcastle   0.5
C3          Birmingham  0.5
C3          London      2.0
C3          Exeter      0.2
C4          Liverpool   2.0
C4          Newcastle   1.5
C4          Birmingham  1.0
C4          Exeter      1.5
```

```
C5          Birmingham  0.5
C5          London      0.5
C5          Exeter      0.5
C6          Liverpool   1.0
C6          Newcastle   1.0
C6          London      1.5
C6          Exeter      1.5
Bristol     Liverpool   0.6
Bristol     Brighton    0.4
Northampton Liverpool   0.4
Northampton Brighton    0.3
C1          Bristol     1.2
C2          Bristol     0.6
C2          Northampton 0.4
C3          Bristol     0.5
C4          Northampton 0.5
C5          Bristol     0.3
C5          Northampton 0.6
C6          Bristol     0.8
C6          Northampton 0.9
;

data depot_data;
   input depot $12. throughput cost savings;
   datalines;
Newcastle    70000     0 10000
Birmingham   50000     0     .
London      100000     0     .
Exeter       40000     0  5000
Bristol      30000 12000     0
Northampton  25000  4000     0
;

data expand_depot_data;
   input depot $12. throughput cost;
   datalines;
Birmingham   20000  3000
;

%let max_num_depots = 4;
```

PROC OPTMODEL Statements and Output

The first several PROC OPTMODEL statements declare index sets and parameters and then read the input data:

```
proc optmodel;
   set <str,str> ARCS;
   num cost {ARCS};
   read data arc_data into ARCS=[i j] cost;

   set <str> FACTORIES;
   num capacity {FACTORIES};
   read data factory_data into FACTORIES=[factory] capacity;

   set <str> DEPOTS;
   num throughput {DEPOTS};
   num open_cost {DEPOTS};
   num close_savings {DEPOTS};
   read data depot_data into DEPOTS=[depot]
      throughput open_cost=cost close_savings=savings;

   set <str> EXPAND_DEPOTS;
   num expand_throughput {EXPAND_DEPOTS};
   num expand_cost {EXPAND_DEPOTS};
   read data expand_depot_data into EXPAND_DEPOTS=[depot]
      expand_throughput=throughput expand_cost=cost;

   set <str> CUSTOMERS;
   num demand {CUSTOMERS};
   read data customer_data into CUSTOMERS=[customer] demand;
```

The following statements are the same as in Chapter 19:

```
   set NODES = FACTORIES union DEPOTS union CUSTOMERS;
   num supply {NODES} init 0;
   for {i in FACTORIES} supply[i] = capacity[i];
   for {i in CUSTOMERS} supply[i] = -demand[i];

   var Flow {ARCS} >= 0;

   con Flow_balance_con {i in NODES}:
      sum {<(i),j> in ARCS} Flow[i,j] - sum {<j,(i)> in ARCS} Flow[j,i]
   <= supply[i];
```

The following statements declare the additional variables and constraints:

```
var IsOpen {DEPOTS} binary;
var Expand {EXPAND_DEPOTS} binary;

con Max_num_depots_con:
   sum {i in DEPOTS} IsOpen[i] <= &max_num_depots;

con Depot_con {i in DEPOTS}:
   sum {<(i),j> in ARCS} Flow[i,j]
<= throughput[i] * IsOpen[i]
+ (if i in EXPAND_DEPOTS then expand_throughput[i] * Expand[i]);

con Expand_con {i in EXPAND_DEPOTS}:
   Expand[i] <= IsOpen[i];
```

The following statements fix the IsOpen variable to 1 for depots that are not eligible to be closed (indicated by a missing value for *close_savings* in the input data):

```
for {i in DEPOTS: close_savings[i] = .} do;
   close_savings[i] = 0;
   fix IsOpen[i] = 1;
end;
```

The following statements declare FixedCost and VariableCost as implicit variables and TotalCost as the objective:

```
impvar FixedCost =
   sum {depot in DEPOTS}
      (open_cost[depot] * IsOpen[depot] -
         close_savings[depot] * (1 - IsOpen[depot]))
 + sum {depot in EXPAND_DEPOTS} expand_cost[depot] * Expand[depot];
impvar VariableCost = sum {<i,j> in ARCS} cost[i,j] * Flow[i,j];
min TotalCost = FixedCost + VariableCost;
```

The following statements call the mixed integer linear programming solver and then print the various parts of the objective, the positive variables in the resulting optimal solution, and the additional decision variables:

```
   solve;
   print FixedCost VariableCost TotalCost;
   print {<i,j> in ARCS: Flow[i,j].sol > 0} Flow;
   print IsOpen Expand;
quit;
```

Figure 20.1 shows the output from the mixed integer linear programming solver.

Figure 20.1 Output from Mixed Integer Linear Programming Solver

The OPTMODEL Procedure

Problem Summary	
Objective Sense	Minimization
Objective Function	TotalCost
Objective Type	Linear
Number of Variables	49
Bounded Above	0
Bounded Below	42
Bounded Below and Above	5
Free	0
Fixed	2
Binary	7
Integer	0
Number of Constraints	22
Linear LE (<=)	22
Linear EQ (=)	0
Linear GE (>=)	0
Linear Range	0
Constraint Coefficients	125

Performance Information	
Execution Mode	Single-Machine
Number of Threads	4

Solution Summary	
Solver	MILP
Algorithm	Branch and Cut
Objective Function	TotalCost
Solution Status	Optimal
Objective Value	174000
Relative Gap	0
Absolute Gap	0
Primal Infeasibility	0
Bound Infeasibility	0
Integer Infeasibility	0
Best Bound	174000
Nodes	1
Iterations	20
Presolve Time	0.01
Solution Time	0.02

Figure 20.1 *continued*

FixedCost	VariableCost	TotalCost
-3000	177000	174000

[1]	[2]	Flow
Birmingham	**C2**	10000
Birmingham	**C4**	10000
Birmingham	**C5**	50000
Brighton	**Birmingham**	70000
Brighton	**London**	10000
Brighton	**Northampton**	25000
Exeter	**C3**	40000
Liverpool	**C1**	50000
Liverpool	**C6**	20000
Liverpool	**Exeter**	40000
London	**C5**	10000
Northampton	**C4**	25000

[1]	IsOpen	Expand
Birmingham	1	1
Bristol	0	
Exeter	1	
London	1	
Newcastle	0	
Northampton	1	

Features Demonstrated

The following features are demonstrated in this example:

- problem type: mixed integer linear programming (fixed costs)
- sets of tuples
- reading multiple data sets
- set operator UNION
- INIT option
- implicit slice
- IF-THEN expression
- modeling if-then constraints by using binary variables
- using a colon (:) to select members of a set
- FIX statement
- IMPVAR statement

Chapter 21
Agricultural Pricing: What Prices to Charge for Dairy Products

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Problem Statement

The government of a country wants to decide what prices should be charged for its dairy products, milk, butter and cheese.[1] All these products arise directly or indirectly from the country's raw milk production. This raw milk is usefully divided into the two components of fat and dry matter. After subtracting the quantities of fat and dry matter which are used for making products for export or consumption on the farms there is a total yearly availability of 600,000 tons of fat and 750,000 tons of dry matter. This is all available for producing milk, butter and two kinds of cheese for domestic consumption.

The percentage compositions of the products are given in Table 21.1.

For the previous year the domestic consumption and prices for the products are given in Table 21.2.

Table 21.1

	Fat	Dry matter	Water
Milk	4	9	87
Butter	80	2	18
Cheese 1	35	30	35
Cheese 2	25	40	35

[1] Reproduced with permission of John Wiley & Sons Ltd. (Williams 1999, pp. 252–253).

Table 21.2

	Milk	Butter	Cheese 1	Cheese 2
Domestic consumption (1000 tons)	4820	320	210	70
Price (£/ton)	297	720	1050	815

Price elasticities of demand, relating consumer demand to the prices of each product, have been calculated on the basis of past statistics. The price elasticity E of a product is defined by

$$E = \frac{\text{percentage decrease in demand}}{\text{percentage increase in price}}.$$

For the two makes of cheese there will be some degree of substitution in consumer demand depending on relative prices. This is measured by cross-elasticity of demand with respect to price. The cross-elasticity E_{AB} from a product A to a product B is defined by

$$E_{\mathrm{AB}} = \frac{\text{percentage increase in demand for A}}{\text{percentage increase in price of B}}.$$

The elasticities and cross-elasticities are given in Table 21.3.

The objective is to determine what prices and resultant demand will maximize total revenue.

It is, however, politically unacceptable to allow a certain price index to rise. As a result of the way this index is calculated this limitation simply demands that the new prices must be such that the total cost of last year's consumption would not be increased. A particularly important additional requirement is to quantify the economic cost of this political limitation.

Table 21.3

Milk	Butter	Cheese 1	Cheese 2	Cheese 1 to Cheese 2	Cheese 2 to Cheese 1
0.4	2.7	1.1	0.4	0.1	0.4

Mathematical Programming Formulation

Index Sets and Their Members

The following index sets and their members are used in this example:

- raw ∈ RAWS
- product, i, j ∈ PRODUCTS

Parameters

Table 21.4 shows the parameters that are used in this example.

Table 21.4 Parameters

Parameter Name	Interpretation
supply[raw]	Supply of each raw material (in tons)
prev_demand[product]	Demand for each product in previous year (in tons)
prev_price[product]	Price of each product in previous year (in £/ton)
percent[product,raw]	Percentage composition for each product and raw
elasticity[i,j]	$\frac{\text{Percentage increase in demand for product } i}{\text{Percentage increase in price of product } j}$

Variables

Table 21.5 shows the variables that are used in this example.

Table 21.5 Variables

Variable Name	Interpretation
Price[product]	Price of each product (in £/ton)
Demand[product]	Demand for each product (in tons)

Objective

The objective is to maximize the following (bilinear) quadratic function:

$$\text{TotalRevenue} = \sum_{\text{product} \in \text{PRODUCTS}} \text{Price[product]} \cdot \text{Demand[product]}$$

Constraints

The following constraints are used in this example:

- bounds on variables
- for $i \in$ PRODUCTS,

$$\frac{\text{Demand[i]} - \mathit{prev_demand[i]}}{\mathit{prev_demand[i]}} = \sum_{j \in \text{PRODUCTS}} \mathit{elasticity[i,j]} \cdot \frac{\text{Price[j]} - \mathit{prev_price[j]}}{\mathit{prev_price[j]}}$$

- for raw $\in$ RAWS such that *supply[raw]* is not missing,

$$\sum_{\text{product}\in\text{PRODUCTS}} \frac{percent[product,raw]}{100} \cdot \text{Demand[product]} \leq supply[raw]$$

- $$\sum_{\text{product}\in\text{PRODUCTS}} prev_demand[product] \cdot \text{Price[product]}$$
$$\leq \sum_{\text{product}\in\text{PRODUCTS}} prev_demand[product] \cdot prev_price[product]$$

Input Data

The following data sets contain the input data that are used in this example:

```
/* missing supply indicates unbounded */
data raw_material_data;
   input raw $10. supply;
   datalines;
Fat        600000
DryMatter 750000
Water      .
;

data product_data;
   input product $ Fat DryMatter Water prev_demand prev_price;
   datalines;
Milk     4  9 87 4820000  297
Butter  80  2 18  320000  720
Cheese1 35 30 35  210000 1050
Cheese2 25 40 35   70000  815
;

data elasticity_data;
   input i $ j $ elasticity;
   datalines;
Milk    Milk    -0.4
Butter  Butter  -2.7
Cheese1 Cheese1 -1.1
Cheese2 Cheese2 -0.4
Cheese1 Cheese2  0.1
Cheese2 Cheese1  0.4
;
```

PROC OPTMODEL Statements and Output

The first several PROC OPTMODEL statements declare index sets and parameters and then read the input data:

```
proc optmodel;
   set <str> RAWS;
   num supply {RAWS};
   read data raw_material_data into RAWS=[raw] supply;

   set <str> PRODUCTS;
   num prev_demand {PRODUCTS};
   num prev_price {PRODUCTS};
   num percent {PRODUCTS, RAWS};
   read data product_data into PRODUCTS=[product]
      {raw in RAWS} <percent[product,raw]=col(raw)> prev_demand prev_price;

   num elasticity {PRODUCTS, PRODUCTS} init 0;
   read data elasticity_data into [i j] elasticity;
```

The following model declaration statements correspond directly to the mathematical programming formulation that is described earlier:

```
   var Price {PRODUCTS} >= 0;

   var Demand {PRODUCTS} >= 0;

   max TotalRevenue
      = sum {product in PRODUCTS} Price[product] * Demand[product];

   con Demand_con {i in PRODUCTS}:
      (Demand[i] - prev_demand[i]) / prev_demand[i]
    = sum {j in PRODUCTS} elasticity[i,j] * (Price[j] - prev_price[j]) /
      prev_price[j];

   con Supply_con {raw in RAWS: supply[raw] ne .}:
      sum {product in PRODUCTS} (percent[product,raw]/100) * Demand[product]
   <= supply[raw];

   con Price_index_con:
      sum {product in PRODUCTS} prev_demand[product] * Price[product]
   <= sum {product in PRODUCTS} prev_demand[product] * prev_price[product];
```

In this example, all variables are real, the objective function is quadratic, and all constraints are linear. So PROC OPTMODEL automatically recognizes that this model is a quadratic programming problem, and the first SOLVE statement calls the quadratic programming solver. In this case, the QP solver detects that this maximization problem has a nonconcave objective, and PROC OPTMODEL instead calls the default nonlinear programming algorithm, which is the interior point algorithm.

```
solve;
print Price Demand;
print Price_index_con.dual;
```

Figure 21.1 shows the output when you use the (default) NLP interior point algorithm.

Figure 21.1 Output from NLP Interior Point Algorithm

The OPTMODEL Procedure

Problem Summary	
Objective Sense	Maximization
Objective Function	TotalRevenue
Objective Type	Quadratic
Number of Variables	8
Bounded Above	0
Bounded Below	8
Bounded Below and Above	0
Free	0
Fixed	0
Number of Constraints	7
Linear LE (<=)	3
Linear EQ (=)	4
Linear GE (>=)	0
Linear Range	0
Constraint Coefficients	22

Performance Information	
Execution Mode	Single-Machine
Number of Threads	4

Solution Summary	
Solver	NLP
Algorithm	Interior Point
Objective Function	TotalRevenue
Solution Status	Optimal
Objective Value	1991585738.7
Optimality Error	9.4885594E-7
Infeasibility	9.4885594E-7
Iterations	30
Presolve Time	0.02
Solution Time	0.34

Figure 21.1 *continued*

[1]	Price	Demand
Butter	667.29	383255
Cheese1	889.89	251695
Cheese2	1066.17	57101
Milk	303.83	4775679

Price_index_con.DUAL
0.6161

To invoke the active set algorithm (which is not the default NLP algorithm), you can use the ALGORITHM= option in the SOLVE WITH NLP statement:

```
    solve with NLP / algorithm=activeset;
    print Price Demand;
    print Price_index_con.dual;
quit;
```

Figure 21.2 shows the output when you use the ALGORITHM=ACTIVESET option to invoke the active set algorithm.

Figure 21.2 Output from NLP Active Set Algorithm

Problem Summary	
Objective Sense	Maximization
Objective Function	TotalRevenue
Objective Type	Quadratic
Number of Variables	8
Bounded Above	0
Bounded Below	8
Bounded Below and Above	0
Free	0
Fixed	0
Number of Constraints	7
Linear LE (<=)	3
Linear EQ (=)	4
Linear GE (>=)	0
Linear Range	0

Performance Information	
Execution Mode	Single-Machine
Number of Threads	4

Figure 21.2 *continued*

Solution Summary	
Solver	NLP
Algorithm	Active Set
Objective Function	TotalRevenue
Solution Status	Best Feasible
Objective Value	1991585738.7
Optimality Error	0.0024623377
Infeasibility	9.4885594E-7
Iterations	66
Presolve Time	0.00
Solution Time	0.07

[1]	Price	Demand
Butter	667.29	383255
Cheese1	889.89	251695
Cheese2	1066.17	57101
Milk	303.83	4775679

Price_index_con.DUAL
.

The optimal solutions for the two algorithms agree. The dual value is missing for the active set algorithm because the solution status is Best Feasible.

You can also replace the Demand and Demand_con declarations with the following IMPVAR and CON statements to perform the substitutions that are described on page 299 of Williams (1999):

```
impvar Demand {i in PRODUCTS} =
   prev_demand[i] * (1 +
      sum {j in PRODUCTS}
         elasticity[i,j] * (Price[j] - prev_price[j]) / prev_price[j]);

con Demand_nonnegative {i in PRODUCTS}:
   Demand[i] >= 0;
```

The resulting formulation is mathematically equivalent but yields a concave objective function, as shown in Williams (1999).

Figure 21.3 shows the output when you use the (default) quadratic programming solver on the reformulated problem.

Figure 21.3 Output from Quadratic Programming Solver, Reformulated Problem

The OPTMODEL Procedure

Problem Summary	
Objective Sense	Maximization
Objective Function	TotalRevenue
Objective Type	Quadratic
Number of Variables	4
Bounded Above	0
Bounded Below	4
Bounded Below and Above	0
Free	0
Fixed	0
Number of Constraints	7
Linear LE (<=)	3
Linear EQ (=)	0
Linear GE (>=)	4
Linear Range	0
Constraint Coefficients	18

Performance Information	
Execution Mode	Single-Machine
Number of Threads	4

Solution Summary	
Solver	QP
Algorithm	Interior Point
Objective Function	TotalRevenue
Solution Status	Optimal
Objective Value	1991585591.9
Primal Infeasibility	0
Dual Infeasibility	6.522632E-14
Bound Infeasibility	0
Duality Gap	1.307146E-12
Complementarity	0
Iterations	10
Presolve Time	0.00
Solution Time	0.01

Figure 21.3 *continued*

[1]	Price	Demand
Butter	667.29	383255
Cheese1	889.89	251695
Cheese2	1066.17	57101
Milk	303.83	4775678

Price_index_con.DUAL
0.6161

Figure 21.4 shows the output when you use the WITH NLP option to invoke the nonlinear programming interior point algorithm on the reformulated problem.

Figure 21.4 Output from NLP Interior Point Algorithm, Reformulated Problem

Problem Summary	
Objective Sense	Maximization
Objective Function	TotalRevenue
Objective Type	Quadratic
Number of Variables	4
Bounded Above	0
Bounded Below	4
Bounded Below and Above	0
Free	0
Fixed	0
Number of Constraints	7
Linear LE (<=)	3
Linear EQ (=)	0
Linear GE (>=)	4
Linear Range	0

Performance Information	
Execution Mode	Single-Machine
Number of Threads	4

Figure 21.4 *continued*

Solution Summary	
Solver	NLP
Algorithm	Interior Point
Objective Function	TotalRevenue
Solution Status	Optimal
Objective Value	1991585591.9
Optimality Error	5E-7
Infeasibility	0
Iterations	7
Presolve Time	0.00
Solution Time	0.01

[1]	Price	Demand
Butter	667.29	383255
Cheese1	889.89	251695
Cheese2	1066.17	57101
Milk	303.83	4775678

Price_index_con.DUAL
0.6161

Figure 21.5 shows the output when you use the ALGORITHM=ACTIVESET option to invoke the active set algorithm on the reformulated problem.

Figure 21.5 Output from NLP Active Set Algorithm, Reformulated Problem

Problem Summary	
Objective Sense	Maximization
Objective Function	TotalRevenue
Objective Type	Quadratic
Number of Variables	4
Bounded Above	0
Bounded Below	4
Bounded Below and Above	0
Free	0
Fixed	0
Number of Constraints	7
Linear LE (<=)	3
Linear EQ (=)	0
Linear GE (>=)	4
Linear Range	0

Figure 21.5 *continued*

Performance Information	
Execution Mode	Single-Machine
Number of Threads	4

Solution Summary	
Solver	NLP
Algorithm	Active Set
Objective Function	TotalRevenue
Solution Status	Optimal
Objective Value	1991585591.9
Optimality Error	4.301741E-13
Infeasibility	5.6461431E-8
Iterations	16
Presolve Time	0.00
Solution Time	0.02

[1]	Price	Demand
Butter	667.29	383255
Cheese1	889.89	251695
Cheese2	1066.17	57101
Milk	303.83	4775678

Price_index_con.DUAL
0.6161

The optimal solutions and dual values for all three algorithms agree with the previous results.

Features Demonstrated

The following features are demonstrated in this example:

- problem type: nonlinear programming (quadratic)
- reading multiple data sets
- INIT option
- reading sparse two-dimensional data
- using a colon (:) to select members of a set
- WITH clause
- ALGORITHM= option
- `.dual` constraint suffix
- IMPVAR statement

Chapter 22
Efficiency Analysis: How to Use Data Envelopment Analysis to Compare Efficiencies of Garages

Contents

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Problem Statement

A car manufacturer wants to evaluate the efficiencies of different garages who have received a franchise to sell its cars.[1] The method to be used is Data Envelopment Analysis (DEA). References to this technique are given in Section 3.2. Each garage has a certain number of measurable 'inputs'. These are taken to be: *Staff*, *Showroom Space*, *Catchment Population* in different economic categories and annual *Enquiries* for different brands of car. Each garage also has a certain number of measurable 'outputs'. These are taken to be: *Number Sold* of different brands of car and annual *Profit*. Table 22.1 gives the inputs and outputs for each of the 28 franchised garages.

A central assumption of DEA (although modified models can be built to alter this assumption) is that constant returns to scale are possible, i.e. doubling a garage's inputs should lead to a doubling of all its outputs. A garage is deemed to be efficient if it is not possible to find a mixture of proportions of other garages whose combined inputs do not exceed those of the garage being considered, but whose outputs are equal to, or exceed, those of the garage. Should this not be possible then the garage is deemed to be inefficient and the comparator garages can be identified.

[1] Reproduced with permission of John Wiley & Sons Ltd. (Williams 1999, pp. 253–255).

Table 22.1

		Inputs						Outputs		
Garage		Staff	Show-room space (100 m^2)	Popn. cat. 1 (1000s)	Popn. cat. 2 (1000s)	Enq. Alpha model (100s)	Enq. Beta model (100s)	Alpha sales (1000s)	Beta sales (1000s)	Profit (millions)
1	Winchester	7	8	10	12	8.5	4	2	0.6	1.5
2	Andover	6	6	20	30	9	4.5	2.3	0.7	1.6
3	Basingstoke	2	3	40	40	2	1.5	0.8	0.25	0.5
4	Poole	14	9	20	25	10	6	2.6	0.86	1.9
5	Woking	10	9	10	10	11	5	2.4	1	2
6	Newbury	24	15	15	13	25	1.9	8	2.6	4.5
7	Portsmouth	6	7	50	40	8.5	3	2.5	0.9	1.6
8	Alresford	8	7.5	5	8	9	4	2.1	0.85	2
9	Salisbury	5	5	10	10	5	2.5	2	0.65	0.9
10	Guildford	8	10	30	35	9.5	4.5	2.05	0.75	1.7
11	Alton	7	8	7	8	3	2	1.9	0.70	0.5
12	Weybridge	5	6.5	9	12	8	4.5	1.8	0.63	1.4
13	Dorchester	6	7.5	10	10	7.5	4	1.5	0.45	1.45
14	Bridport	11	8	8	10	10	6	2.2	0.65	2.2
15	Weymouth	4	5	10	10	7.5	3.5	1.8	0.62	1.6
16	Portland	3	3.5	3	20	2	1.5	0.9	0.35	0.5
17	Chichester	5	5.5	8	10	7	3.5	1.2	0.45	1.3
18	Petersfield	21	12	6	6	15	8	6	0.25	2.9
19	Petworth	6	5.5	2	2	8	5	1.5	0.55	1.55
20	Midhurst	3	3.6	3	3	2.5	1.5	0.8	0.20	0.45
21	Reading	30	29	120	80	35	20	7	2.5	8
22	Southampton	25	16	110	80	27	12	6.5	3.5	5.4
23	Bournemouth	19	10	90	22	25	13	5.5	3.1	4.5
24	Henley	7	6	5	7	8.5	4.5	1.2	0.48	2
25	Maidenhead	12	8	7	10	12	7	4.5	2	2.3
26	Fareham	4	6	1	1	7.5	3.5	1.1	0.48	1.7
27	Romsey	2	2.5	1	1	2.5	1	0.4	0.1	0.55
28	Ringwood	2	3.5	2	2	1.9	1.2	0.3	0.09	0.4

A linear programming model can be built to identify efficient and inefficient garages and their comparators.

Mathematical Programming Formulation

The formulation that is described here applies to each garage k. The PROC OPTMODEL statements call the linear programming solver in a loop, with one solve per garage.

Index Sets and Their Members

The following index sets and their members are used in this example:

- $i \in$ INPUTS
- $i \in$ OUTPUTS
- $j, k \in$ GARAGES
- $j, g_1 \in$ INEFFICIENT_GARAGES
- $j, g_2 \in$ EFFICIENT_GARAGES

Parameters

Table 22.2 shows the parameters that are used in this example.

Table 22.2 Parameters

Parameter Name	Interpretation
garage_name[garage]	Name of garage
input[i,j]	Input coefficient of input *i* for garage *j*
output[i,j]	Output coefficient of output *i* for garage *j*
k	Dummy index for looping over garages
efficiency_number[k]	Efficiency number of garage *k*
weight_sol[k,j]	Weight of garage *j* when maximizing inefficiency of garage *k*

Variables

Table 22.3 shows the variables that are used in this example.

Table 22.3 Variables

Variable Name	Interpretation
Weight[j]	Weight of garage *j*
Inefficiency	Inefficiency of current garage

Objective

The objective is to maximize Inefficiency.

Constraints

The following constraints are used in this example:

- bounds on variables
- for $i \in$ INPUTS,

$$\sum_{j \in \text{GARAGES}} input[i,j] \cdot \text{Weight[j]} \leq input[i,k]$$

- for $i \in$ OUTPUTS,

$$\sum_{j \in \text{GARAGES}} output[i,j] \cdot \text{Weight[j]} \geq output[i,k] \cdot \text{Inefficiency}$$

Input Data

The following data sets contain the input data that are used in this example:

```
data inputs;
   input input $9.;
   datalines;
staff
showroom
pop1
pop2
alpha_enq
beta_enq
;

data outputs;
   input output $11.;
   datalines;
alpha_sales
beta_sales
profit
;

data garage_data;
   input garage_name $12. staff showroom pop1 pop2 alpha_enq beta_enq
      alpha_sales beta_sales profit;
   datalines;
Winchester  7  8   10  12 8.5 4   2    0.6  1.5
Andover     6  6   20  30 9   4.5 2.3  0.7  1.6
Basingstoke 2  3   40  40 2   1.5 0.8  0.25 0.5
Poole       14 9   20  25 10  6   2.6  0.86 1.9
Woking      10 9   10  10 11  5   2.4  1    2
Newbury     24 15  15  13 25  19  8    2.6  4.5
```

```
Portsmouth  6  7   50  40 8.5 3   2.5  0.9  1.6
Alresford   8  7.5 5   8  9   4   2.1  0.85 2
Salisbury   5  5   10  10 5   2.5 2    0.65 0.9
Guildford   8  10  30  35 9.5 4.5 2.05 0.75 1.7
Alton       7  8   7   8  3   2   1.9  0.7  0.5
Weybridge   5  6.5 9   12 8   4.5 1.8  0.63 1.4
Dorchester  6  7.5 10  10 7.5 4   1.5  0.45 1.45
Bridport    11 8   8   10 10  6   2.2  0.65 2.2
Weymouth    4  5   10  10 7.5 3.5 1.8  0.62 1.6
Portland    3  3.5 3   2  2   1.5 0.9  0.35 0.5
Chichester  5  5.5 8   10 7   3.5 1.2  0.45 1.3
Petersfield 21 12  6   8  15  8   6    0.25 2.9
Petworth    6  5.5 2   2  8   5   1.5  0.55 1.55
Midhurst    3  3.6 3   3  2.5 1.5 0.8  0.2  0.45
Reading     30 29  120 80 35  20  7    2.5  8
Southampton 25 16  110 80 27  12  6.5  3.5  5.4
Bournemouth 19 10  90  12 25  13  5.5  3.1  4.5
Henley      7  6   5   7  8.5 4.5 1.2  0.48 2
Maidenhead  12 8   7   10 12  7   4.5  2    2.3
Fareham     4  6   1   1  7.5 3.5 1.1  0.48 1.7
Romsey      2  2.5 1   1  2.5 1   0.4  0.1  0.55
Ringwood    2  3.5 2   2  1.9 1.2 0.3  0.09 0.4
;
```

PROC OPTMODEL Statements and Output

The first several PROC OPTMODEL statements declare index sets and parameters and then read the input data:

```
proc optmodel;
   set <str> INPUTS;
   read data inputs into INPUTS=[input];

   set <str> OUTPUTS;
   read data outputs into OUTPUTS=[output];

   set <num> GARAGES;
   str garage_name {GARAGES};
   num input  {INPUTS, GARAGES};
   num output {OUTPUTS, GARAGES};
   read data garage_data into GARAGES=[_N_] garage_name
      {i in INPUTS}  <input[i,_N_]=col(i)>
      {i in OUTPUTS} <output[i,_N_]=col(i)>;

   num k;
   num efficiency_number {GARAGES};
   num weight_sol {GARAGES, GARAGES};
```

The following statements correspond directly to the mathematical programming formulation that is described earlier:

```
var Weight {GARAGES} >= 0;
var Inefficiency >= 0;

max Objective = Inefficiency;

con Input_con {i in INPUTS}:
   sum {j in GARAGES} input[i,j] * Weight[j] <= input[i,k];

con Output_con {i in OUTPUTS}:
   sum {j in GARAGES} output[i,j] * Weight[j] >= output[i,k] * Inefficiency;
```

The following statements loop over all garages, call the linear programming solver once per garage, and store the results in the parameters *efficiency_number* and *weight_sol*:

```
do k = GARAGES;
   solve;
   efficiency_number[k] = 1 / Inefficiency.sol;
   for {j in GARAGES}
      weight_sol[k,j] = (if Weight[j].sol > 1e-6 then Weight[j].sol else .);
end;
```

Note that the DO loop contains no declaration statements. As the value of *k* changes, the SOLVE statement automatically updates the constraints to use the values of *input[i,k]* and *output[i,k]*. The approach shown here is much more efficient than the following alternatives:

- Calling PROC OPTMODEL once per garage
- Enlarging the set of decision variables by including an additional index, resulting in variables Weight[k,j] and Inefficiency[k]. Within the DO loop, you must then fix most of the variables to 0 and rely on the presolver to remove them, but that approach uses much more memory and computational time.

In SAS/OR 13.1, you can instead solve the same set of problems in parallel by replacing the DO loop with the following COFOR loop:

```
cofor {kk in GARAGES} do;
   k = kk;
   solve;
   efficiency_number[k] = 1 / Inefficiency.sol;
   for {j in GARAGES}
      weight_sol[k,j] = (if Weight[j].sol > 1e-6 then Weight[j].sol else .);
end;
```

After the DO or COFOR loop terminates, the following statements partition the garages into two sets by using a threshold on the resulting efficiency numbers:

```
set EFFICIENT_GARAGES = {j in GARAGES: efficiency_number[j] >= 1};
set INEFFICIENT_GARAGES = GARAGES diff EFFICIENT_GARAGES;
```

The following statements print the efficiency numbers, as shown in Figure 22.1, and write them to the efficiency_data data set:

```
   print garage_name efficiency_number;
   create data efficiency_data from [garage] garage_name efficiency_number;
```

Figure 22.1 *efficiency_number* Parameter

The OPTMODEL Procedure

[1]	garage_name	efficiency_number
1	Winchester	0.84017
2	Andover	0.91738
3	Basingstoke	1.00000
4	Poole	0.86189
5	Woking	0.86732
6	Newbury	1.00000
7	Portsmouth	1.00000
8	Alresford	1.00000
9	Salisbury	1.00000
10	Guildford	0.81417
11	Alton	1.00000
12	Weybridge	0.85435
13	Dorchester	0.83920
14	Bridport	0.97101
15	Weymouth	1.00000
16	Portland	1.00000
17	Chichester	0.82434
18	Petersfield	1.00000
19	Petworth	0.98824
20	Midhurst	0.82928
21	Reading	0.98205
22	Southampton	1.00000
23	Bournemouth	1.00000
24	Henley	1.00000
25	Maidenhead	1.00000
26	Fareham	1.00000
27	Romsey	1.00000
28	Ringwood	0.87587

The following CREATE DATA statements write the inefficient garages and the corresponding multiples of efficient garages to SAS data sets (in both dense and sparse form), as in Table 14.8 in Williams (1999):

```
   create data weight_data_dense from [inefficient_garage]=INEFFICIENT_GARAGES
      garage_name
      efficiency_number
      {efficient_garage in EFFICIENT_GARAGES} <col('w'||efficient_garage)
         =weight_sol[inefficient_garage,efficient_garage]>;
   create data weight_data_sparse from
      [inefficient_garage efficient_garage]=
   {g1 in INEFFICIENT_GARAGES, g2 in EFFICIENT_GARAGES: weight_sol[g1,g2] ne .}
      weight_sol;
quit;
```

The following statements sort the efficiency_data data set by efficiency and print the results, shown in Figure 22.2:

```
proc sort data=efficiency_data;
   by descending efficiency_number;
run;

proc print;
run;
```

Figure 22.2 efficiency_data Data Set

Obs	garage	garage_name	efficiency_number
1	25	Maidenhead	1.00000
2	26	Fareham	1.00000
3	18	Petersfield	1.00000
4	3	Basingstoke	1.00000
5	6	Newbury	1.00000
6	7	Portsmouth	1.00000
7	8	Alresford	1.00000
8	9	Salisbury	1.00000
9	11	Alton	1.00000
10	16	Portland	1.00000
11	23	Bournemouth	1.00000
12	24	Henley	1.00000
13	27	Romsey	1.00000
14	15	Weymouth	1.00000
15	22	Southampton	1.00000
16	19	Petworth	0.98824
17	21	Reading	0.98205
18	14	Bridport	0.97101
19	2	Andover	0.91738
20	28	Ringwood	0.87587
21	5	Woking	0.86732
22	4	Poole	0.86189
23	12	Weybridge	0.85435
24	1	Winchester	0.84017
25	13	Dorchester	0.83920
26	20	Midhurst	0.82928
27	17	Chichester	0.82434
28	10	Guildford	0.81417

The following statements sort the weight_data_dense data set by efficiency and print the results, shown in Figure 22.3:

```
proc sort data=weight_data_dense;
   by descending efficiency_number;
run;

proc print;
run;
```

Figure 22.3 weight_data_dense Data Set

Obs	inefficient_garage	garage_name	efficiency_number	w3	w6	w7	w8	w9
1	15	Weymouth	1.00000	.	.	.	.	.
2	22	Southampton	1.00000	.	.	.	.	.
3	19	Petworth	0.98824	.	0.066345	.	.	.
4	21	Reading	0.98205	1.26862	.	.	.	.
5	14	Bridport	0.97101	0.03278	.	.	.	.
6	2	Andover	0.91738	.	.	.	.	.
7	28	Ringwood	0.87587	0.00771	.	.	.	.
8	5	Woking	0.86732	.	.	.	0.95253	.
9	4	Poole	0.86189	0.32859	.	.	.	.
10	12	Weybridge	0.85435	.	.	.	.	.
11	1	Winchester	0.84017	.	.	0.00528	0.41627	0.40328
12	13	Dorchester	0.83920	0.13436	.	.	0.10448	.
13	20	Midhurst	0.82928	.	.	.	.	0.05957
14	17	Chichester	0.82434	0.05825	.	.	0.09682	.
15	10	Guildford	0.81417	0.42459	.	0.14961	0.62272	.

Obs	w11	w16	w18	w23	w24	w25	w26	w27
1	.	.	.	.	.	.	.	.
2	.	.	.	.	.	.	.	.
3	.	.	0.015212	.	.	0.03409	0.67493	.
4	.	1.19914	.	.	2.86247	0.13753	.	.
5	.	0.46969	.	.	0.78310	0.19489	.	.
6	.	.	.	.	.	0.21429	.	.
7	.	0.31973	.	.	0.14649	.	.	.
8	0.021078	.	.	.	.	0.14838	.	.
9	.	0.75733	.	.	0.43442	0.34463	.	.
10	.	.	.	.	.	0.14524	0.01773	.
11	.	0.09614	.	.	.	.	.	.
12	.	0.75163	.	.	0.03532	.	0.47905	.
13	.	0.47189	0.043482	.	.	0.00894	.	.
14	.	0.16523	.	.	0.23637	.	0.15424	.
15	.	0.16807	.	.	.	.	.	.

The weight_data_sparse data set contains the same information in sparse format, as shown in Figure 22.4:

```
proc print data=weight_data_sparse;
run;
```

Figure 22.4 weight_data_sparse Data Set

Obs	inefficient_garage	efficient_garage	weight_sol
1	4	3	0.32859
2	10	3	0.42459
3	13	3	0.13436
4	14	3	0.03278
5	17	3	0.05825
6	21	3	1.26862
7	28	3	0.00771
8	19	6	0.06635
9	1	7	0.00528
10	10	7	0.14961
11	1	8	0.41627
12	5	8	0.95253
13	10	8	0.62272
14	13	8	0.10448
15	17	8	0.09682
16	1	9	0.40328
17	20	9	0.05957
18	5	11	0.02108
19	1	16	0.09614
20	4	16	0.75733
21	10	16	0.16807
22	13	16	0.75163
23	14	16	0.46969
24	17	16	0.16523
25	20	16	0.47189
26	21	16	1.19914
27	28	16	0.31973
28	19	18	0.01521
29	20	18	0.04348
30	4	24	0.43442
31	13	24	0.03532
32	14	24	0.78310
33	17	24	0.23637
34	21	24	2.86247
35	28	24	0.14649
36	2	25	0.21429
37	4	25	0.34463
38	5	25	0.14838
39	12	25	0.14524
40	14	25	0.19489
41	19	25	0.03409
42	20	25	0.00894
43	21	25	0.13753
44	12	26	0.01773
45	13	26	0.47905
46	17	26	0.15424
47	19	26	0.67493

Features Demonstrated

The following features are demonstrated in this example:

- problem type: linear programming (data envelopment analysis)
- numeric and string index sets
- reading and writing multiple data sets
- reading dense two-dimensional data
- calling a solver in a DO loop
- calling a solver in a COFOR loop
- `.sol` variable suffix
- storing a solution in a numeric parameter
- using a colon (:) to select members of a set
- set operator DIFF
- writing dense two-dimensional data
- writing sparse two-dimensional data

Chapter 23
Milk Collection: How to Route and Assign Milk Collection Lorries to Farms

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Problem Statement

A small milk processing company is committed to collecting milk from 20 farms and taking it back to the depot for processing.[1] The company has one tanker lorry with a capacity for carrying 80,000 litres of milk. Eleven of the farms are small and need a collection only every other day. The other nine farms need a collection every day. The positions of the farms in relation to the depot (numbered 1) are given in Table 23.1 together with their collection requirements.

Find the optimal route for the tanker lorry on each day, bearing in mind that it has to (i) visit all the 'every day' farms, (ii) visit some of the 'every other day' farms, and (iii) work within its capacity. On alternate days it must again visit the 'every day' farms but also visit the 'every other day' farms not visited on the previous day.

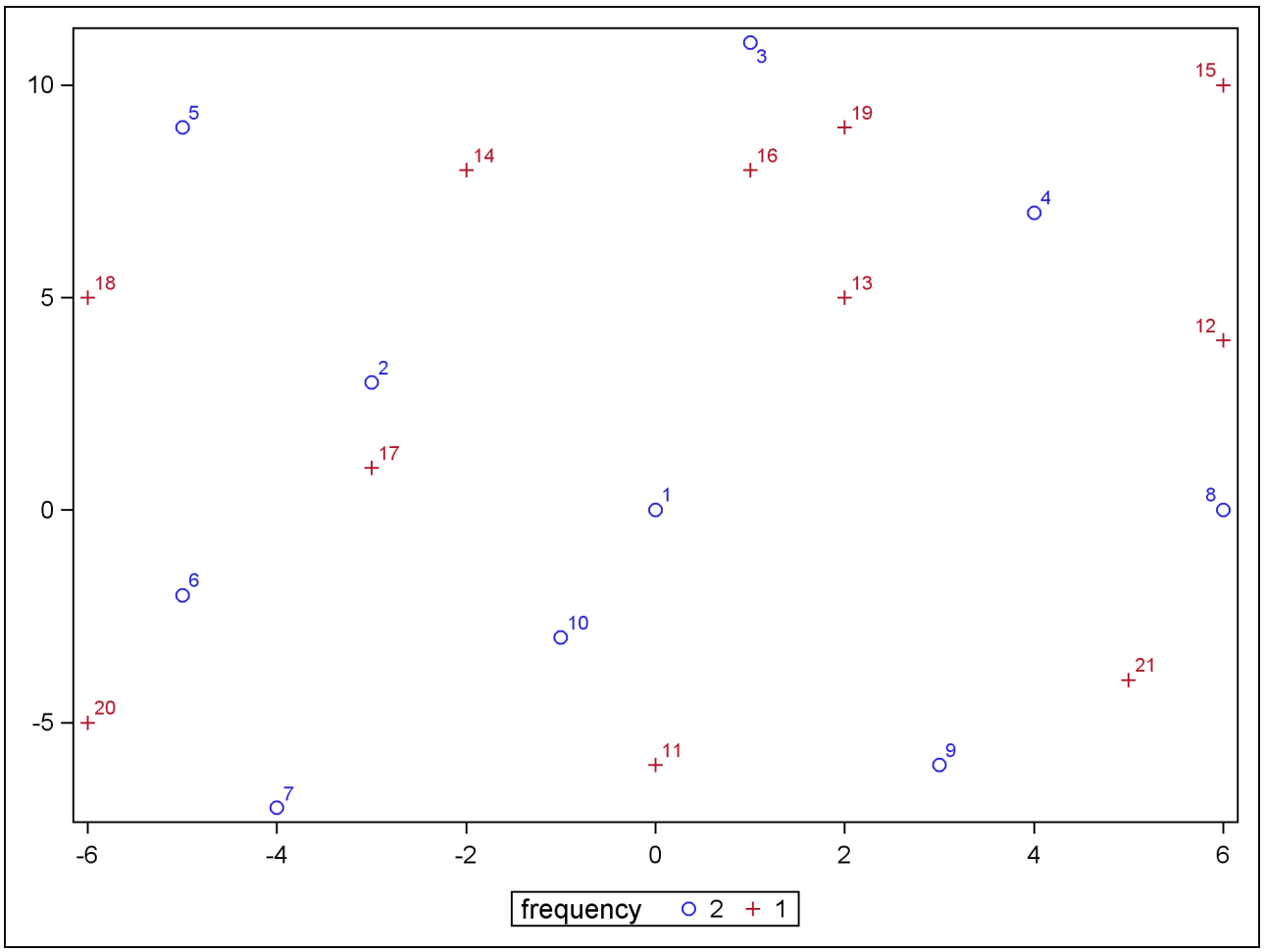
For convenience a map of the area considered is given in Figure 23.1.

[1] Reproduced with permission of John Wiley & Sons Ltd. (Williams 1999, pp. 255–256).

Table 23.1

Farm	Position 10 miles: East	North	Collection frequency	Collection requirement (1000 litres)
1 (Depot)	0	0	—	—
2	−3	3	Every day	5
3	1	11	Every day	4
4	4	7	Every day	3
5	−5	9	Every day	6
6	−5	−2	Every day	7
7	−4	−7	Every day	3
8	6	0	Every day	4
9	3	−6	Every day	6
10	−1	−3	Every day	5
11	0	−6	Every other day	4
12	6	4	Every other day	7
13	2	5	Every other day	3
14	−2	8	Every other day	4
15	6	10	Every other day	5
16	1	8	Every other day	6
17	−3	1	Every other day	8
18	−6	5	Every other day	5
19	2	9	Every other day	7
20	−6	−5	Every other day	6
21	5	−4	Every other day	6

Figure 23.1



Mathematical Programming Formulation

The problem is formulated as a periodic vehicle routing problem with a node for each farm. Williams (1999) uses the fact that the number of days in the planning horizon is two to reduce the number of decision variables and constraints. The formulation shown here generalizes to any number of days and arbitrary frequencies (not just 1 or 2).

Index Sets and Their Members

The following index sets and their members are used in this example:

- $i, j \in \text{NODES}$
- $(i, j) \in \text{EDGES}$

- $d \in$ DAYS
- $i, j \in$ NODES_TEMP
- $(i, j) \in$ EDGES_SOL[iter,d]
- $c_i, c_j \in$ COMPONENT_IDS
- $k \in$ COMPONENT[ci]
- $k \in$ SUBTOUR[s]

Parameters

Table 23.2 shows the parameters that are used in this example.

Table 23.2 Parameters

Parameter Name	Interpretation
east[i]	East coordinate of node *i*
north[i]	North coordinate of node *i*
frequency[i]	Number of times node *i* must be visited (1 for every other day, 2 for every day)
requirement[i]	Collection requirement of node *i* (in 1,000 litres)
distance[i,j]	Distance between nodes *i* and *j* (in miles)
distance_scale	Distance scale
num_days	Number of days in planning horizon
capacity	Capacity of the lorry (in 1,000 litres)
depot	Node number of depot
num_subtours	Number of subtours in current formulation
iter	Iteration number for each round of subtour elimination constraints
num_components[d]	Number of connected components in solution for day *d*
component_id[i]	Connected component containing node *i*
ci,cj	Dummy indices for members of COMPONENT_IDS

Variables

Table 23.3 shows the variables that are used in this example.

Table 23.3 Variables

Variable Name	Interpretation
UseNode[i,d]	1 if node *i* is visited on day *d*; 0 otherwise
UseEdge[i,j,d]	1 if edge (i, j) is traversed on day *d*; 0 otherwise

Objective

The objective is to minimize the following function:

$$\text{TotalDistance} = \sum_{(i,j)\in \text{EDGES}} \sum_{d\in \text{DAYS}} \textit{distance[i,j]} \cdot \text{UseEdge[i,j,d]}$$

Constraints

The following constraints are used in this example:

- bounds on variables
- for $d \in$ DAYS,

$$\sum_{i\in \text{NODES}} \textit{requirement[i]} \cdot \text{UseNode[i,d]} \leq \textit{capacity}$$

- for $i \in$ NODES,

$$\sum_{d\in \text{DAYS}} \text{UseNode[i,d]} = \textit{frequency[i]}$$

- for $k \in$ NODES and $d \in$ DAYS,

$$\sum_{(i,j)\in \text{EDGES}: k\in\{i,j\}} \text{UseEdge[i,j,d]} = 2 \cdot \text{UseNode[k,d]}$$

- for $d \in \text{DAYS} \setminus \{1\}$,

$$\sum_{i\in \text{NODES}} \text{UseNode[i,d]} \leq \sum_{i\in \text{NODES}} \text{UseNode[i,d} - 1\text{]}$$

- for $s \in \{1, \ldots, \textit{num_subtours}\}$ and $k \in$ SUBTOUR[s] and $d \in$ DAYS,

$$\sum_{\substack{i\in \text{NODES}\setminus \text{SUBTOUR[s]},\\ j\in \text{SUBTOUR[s]}:\\ (i,j)\in \text{EDGES}}} \text{UseEdge[i,j,d]} + \sum_{\substack{i\in \text{SUBTOUR[s]},\\ j\in \text{NODES}\setminus \text{SUBTOUR[s]}:\\ (i,j)\in \text{EDGES}}} \text{UseEdge[i,j,d]} \geq 2 \cdot \text{UseNode[k,d]}$$

Input Data

The following data set and macro variables contain the input data that are used in this example:

```
data farm_data;
   farm = _N_;
   input east north frequency requirement;
   datalines;
 0  0 2 0
-3  3 2 5
 1 11 2 4
 4  7 2 3
-5  9 2 6
-5 -2 2 7
-4 -7 2 3
 6  0 2 4
 3 -6 2 6
-1 -3 2 5
 0 -6 1 4
 6  4 1 7
 2  5 1 3
-2  8 1 4
 6 10 1 5
 1  8 1 6
-3  1 1 8
-6  5 1 5
 2  9 1 7
-6 -5 1 6
 5 -4 1 6
;

%let distance_scale = 10;
%let num_days = 2;
%let capacity = 80;
%let depot = 1;
```

PROC OPTMODEL Statements and Output

The first several PROC OPTMODEL statements declare index sets and parameters and then read the input data:

```
proc optmodel;
   set NODES;
   num east {NODES};
   num north {NODES};
   num frequency {NODES};
   num requirement {NODES};
   read data farm_data into NODES=[_N_] east north frequency requirement;
```

```
set EDGES = {i in NODES, j in NODES: i < j};
num distance {<i,j> in EDGES} =
   &distance_scale * sqrt((east[i]-east[j])^2+(north[i]-north[j])^2);

set DAYS = 1..&num_days;
```

The following model declaration statements correspond directly to the mathematical programming formulation that is described earlier:

```
var UseNode {NODES, DAYS} binary;
var UseEdge {EDGES, DAYS} binary;

min TotalDistance
   = sum {<i,j> in EDGES, d in DAYS} distance[i,j] * UseEdge[i,j,d];

con Capacity_con {d in DAYS}:
   sum {i in NODES} requirement[i] * UseNode[i,d] <= &capacity;

con Frequency_con {i in NODES}:
   sum {d in DAYS} UseNode[i,d] = frequency[i];

con Two_match {k in NODES, d in DAYS}:
   sum {<i,j> in EDGES: k in {i,j}} UseEdge[i,j,d] = 2 * UseNode[k,d];
```

The following statements declare optional symmetry-breaking constraints to reduce the number of essentially identical branch-and-bound nodes that are explored by the mixed integer linear programming solver:

```
   /* several alternatives for symmetry-breaking constraints */
*  con Symmetry {d in DAYS diff {1}}:
      sum {<i,j> in EDGES} distance[i,j] * UseEdge[i,j,d]
   <= sum {<i,j> in EDGES} distance[i,j] * UseEdge[i,j,d-1];
*  con Symmetry {d in DAYS diff {1}}:
      sum {i in NODES} requirement[i] * UseNode[i,d]
   <= sum {i in NODES} requirement[i] * UseNode[i,d-1];
   con Symmetry {d in DAYS diff {1}}:
      sum {i in NODES} UseNode[i,d]
   <= sum {i in NODES} UseNode[i,d-1];
```

Williams (1999) breaks symmetry instead by fixing UseNode[11,1] = 1. The symmetry-breaking constraints shown here apply to the general formulation with any number of days and arbitrary frequencies.

In SAS/OR 13.1, the mixed integer linear programming solver automatically detects and exploits symmetry without you having to explicitly declare such symmetry-breaking constraints. You can control the aggressiveness of symmetry detection by using the SYMMETRY= option in the SOLVE WITH MILP statement.

The following statements declare the subtour elimination constraints:

```
num num_subtours init 0;
/* subset of nodes not containing depot node */
set SUBTOUR {1..num_subtours};
/* if node k in SUBTOUR[s] is used on day d, then
   must use at least two edges across partition induced by SUBTOUR[s] */
con Subtour_elimination {s in 1..num_subtours, k in SUBTOUR[s], d in DAYS}:
   sum {i in NODES diff SUBTOUR[s], j in SUBTOUR[s]: <i,j> in EDGES}
```

```
      UseEdge[i,j,d]
 + sum {i in SUBTOUR[s], j in NODES diff SUBTOUR[s]: <i,j> in EDGES}
      UseEdge[i,j,d]
>= 2 * UseNode[k,d];
```

The following statements declare the index sets and parameters that are needed to detect violated subtour elimination constraints:

```
num iter init 0;
num num_iters init 0;
set ITERS = 1..num_iters;
num num_components {DAYS};
set NODES_TEMP;
set <num,num> EDGES_SOL {ITERS, DAYS};
num component_id {NODES_TEMP};
set COMPONENT_IDS;
set COMPONENT {COMPONENT_IDS};
num ci;
num cj;
```

The following DO UNTIL loop implements dynamic generation of subtour elimination constraints ("row generation"):

```
/* loop until each day's support graph is connected */
do until (and {d in DAYS} num_components[d] = 1);
   iter = iter + 1;
   num_iters = iter;
   solve;
   /* find connected components for each day */
   for {d in DAYS} do;
      NODES_TEMP = {i in NODES: UseNode[i,d].sol > 0.5};
      EDGES_SOL[iter,d] = {<i,j> in EDGES: UseEdge[i,j,d].sol > 0.5};
      /* initialize each node to its own component */
      COMPONENT_IDS = NODES_TEMP;
      num_components[d] = card(NODES_TEMP);
      for {i in NODES_TEMP} do;
         component_id[i] = i;
         COMPONENT[i] = {i};
      end;
      /* if i and j are in different components, merge the two components */
      for {<i,j> in EDGES_SOL[iter,d]} do;
         ci = component_id[i];
         cj = component_id[j];
         if ci ne cj then do;
            /* update smaller component */
            if card(COMPONENT[ci]) < card(COMPONENT[cj]) then do;
               for {k in COMPONENT[ci]} component_id[k] = cj;
               COMPONENT[cj] = COMPONENT[cj] union COMPONENT[ci];
               COMPONENT_IDS = COMPONENT_IDS diff {ci};
            end;
            else do;
               for {k in COMPONENT[cj]} component_id[k] = ci;
               COMPONENT[ci] = COMPONENT[ci] union COMPONENT[cj];
               COMPONENT_IDS = COMPONENT_IDS diff {cj};
            end;
```

```
            end;
         end;
         num_components[d] = card(COMPONENT_IDS);
         put num_components[d]=;
         /* create subtour from each component not containing depot node */
         for {k in COMPONENT_IDS: &depot not in COMPONENT[k]} do;
            num_subtours = num_subtours + 1;
            SUBTOUR[num_subtours] = COMPONENT[k];
            put SUBTOUR[num_subtours]=;
         end;
      end;
      print capacity_con.body capacity_con.ub;
      print num_components;
   end;
```

The body of the loop calls the mixed integer linear programming solver, finds the connected components of the support graph of the resulting solution, and adds any subtours found. Note that the DO UNTIL loop contains no declaration statements. As the value of *num_subtours* changes, the SOLVE statement automatically updates the subtour elimination constraints. See Chapter 27 for an alternative approach that instead uses the SOLVE WITH NETWORK statement together with the CONCOMP option to find the connected components.

After the DO UNTIL loop terminates, the following statements output the edges that appear in each iteration of subtour elimination and write the value of *iter* to a SAS macro variable named *num_iters*:

```
   create data sol_data from
      [iter d i j]={it in ITERS, d in DAYS, <i,j> in EDGES_SOL[it,d]}
      x1=east[i] y1=north[i] x2=east[j] y2=north[j];
   call symput('num_iters',put(num_iters,best.));
quit;
```

The following SAS macro calls PROC SGPLOT to plot the solution that results from each iteration:

```
%macro showPlots;
   %do iter = 1 %to &num_iters;
      %do d = 1 %to &num_days;
         /* create annotate data set to draw subtours */
         data sganno(keep=drawspace linethickness function x1 y1 x2 y2);
            retain drawspace "datavalue" linethickness 1;
            set sol_data;
            where iter = &iter and d = &d;
            function = 'line';
         run;

         title1 "iter = &iter, day = &d";
         title2;
         proc sgplot data=farm_data sganno=sganno;
            scatter y=north x=east / group=frequency datalabel=farm;
            xaxis display=(nolabel);
            yaxis display=(nolabel);
         run;
      %end;
   %end;
%mend showPlots;
```

```
%showPlots;
```

Figure 23.2 and Figure 23.3 show the output from the mixed integer linear programming solver for the first iteration, before any subtour elimination constraints have been generated.

Figure 23.2 Output from Mixed Integer Linear Programming Solver, Iteration 1, Day 1

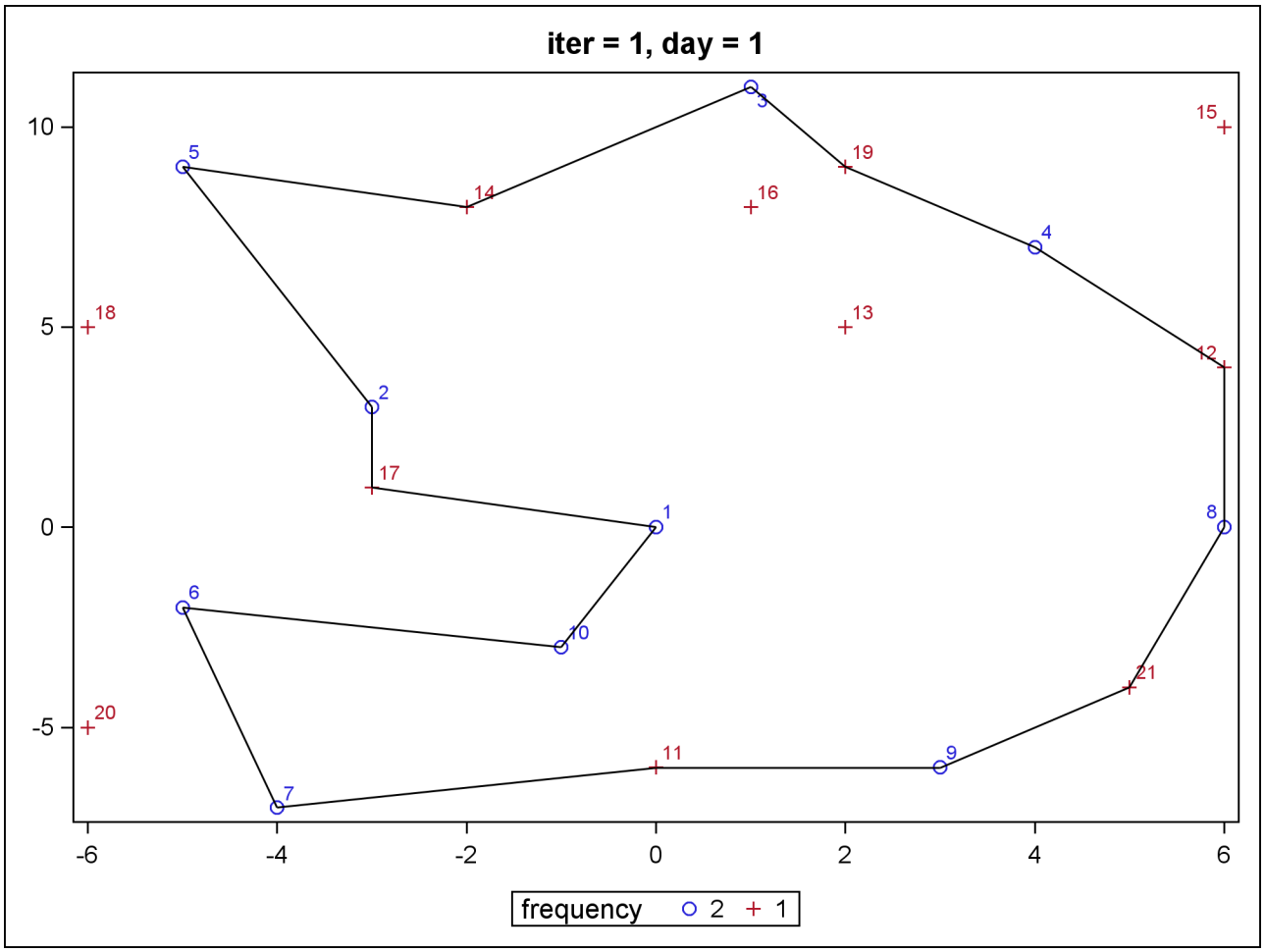


Figure 23.3 Output from Mixed Integer Linear Programming Solver, Iteration 1, Day 2

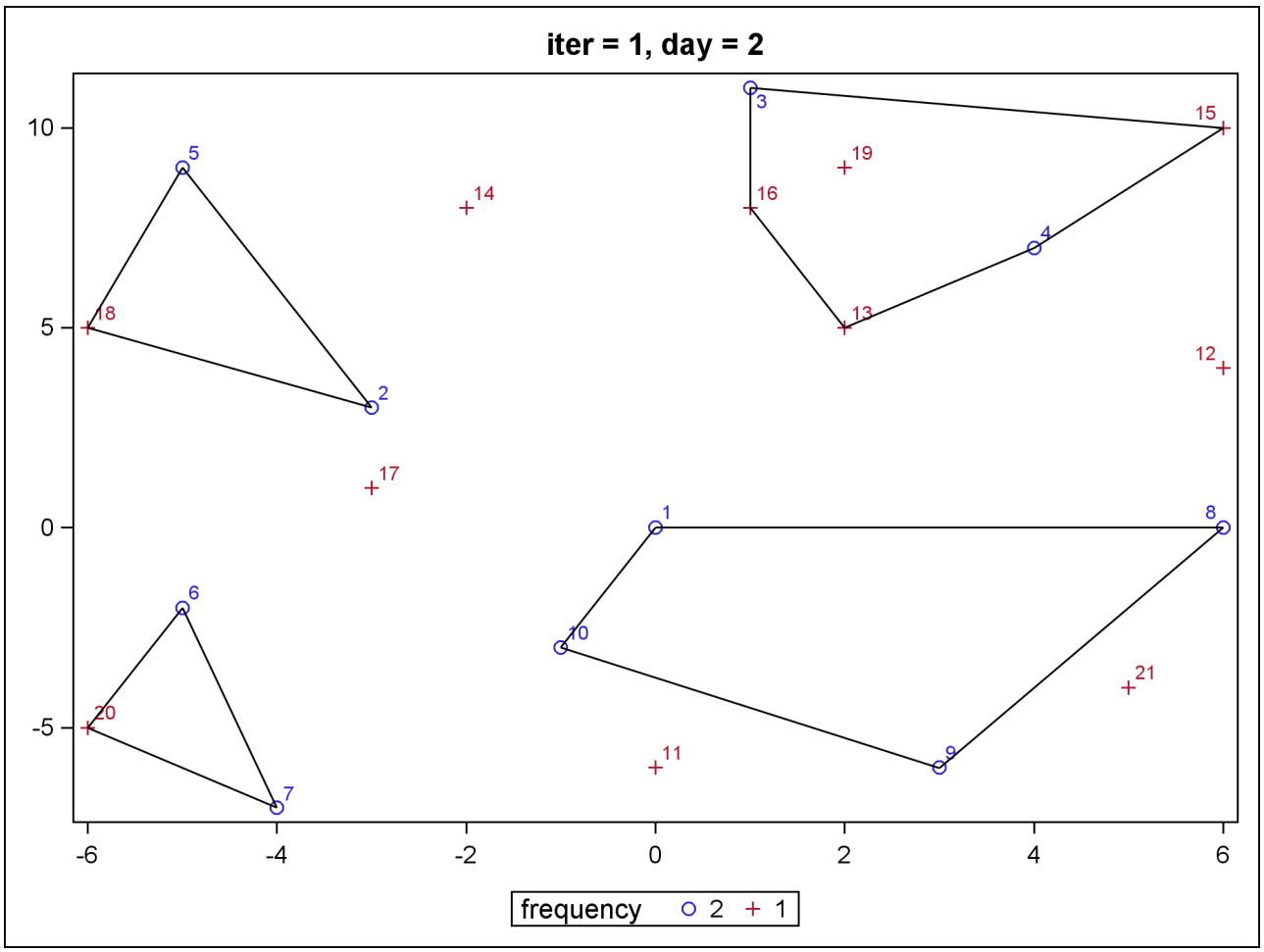


Figure 23.4 and Figure 23.5 show the output from the mixed integer linear programming solver for the second iteration, after the first round of subtour elimination constraints.

Figure 23.4 Output from Mixed Integer Linear Programming Solver, Iteration 2, Day 1

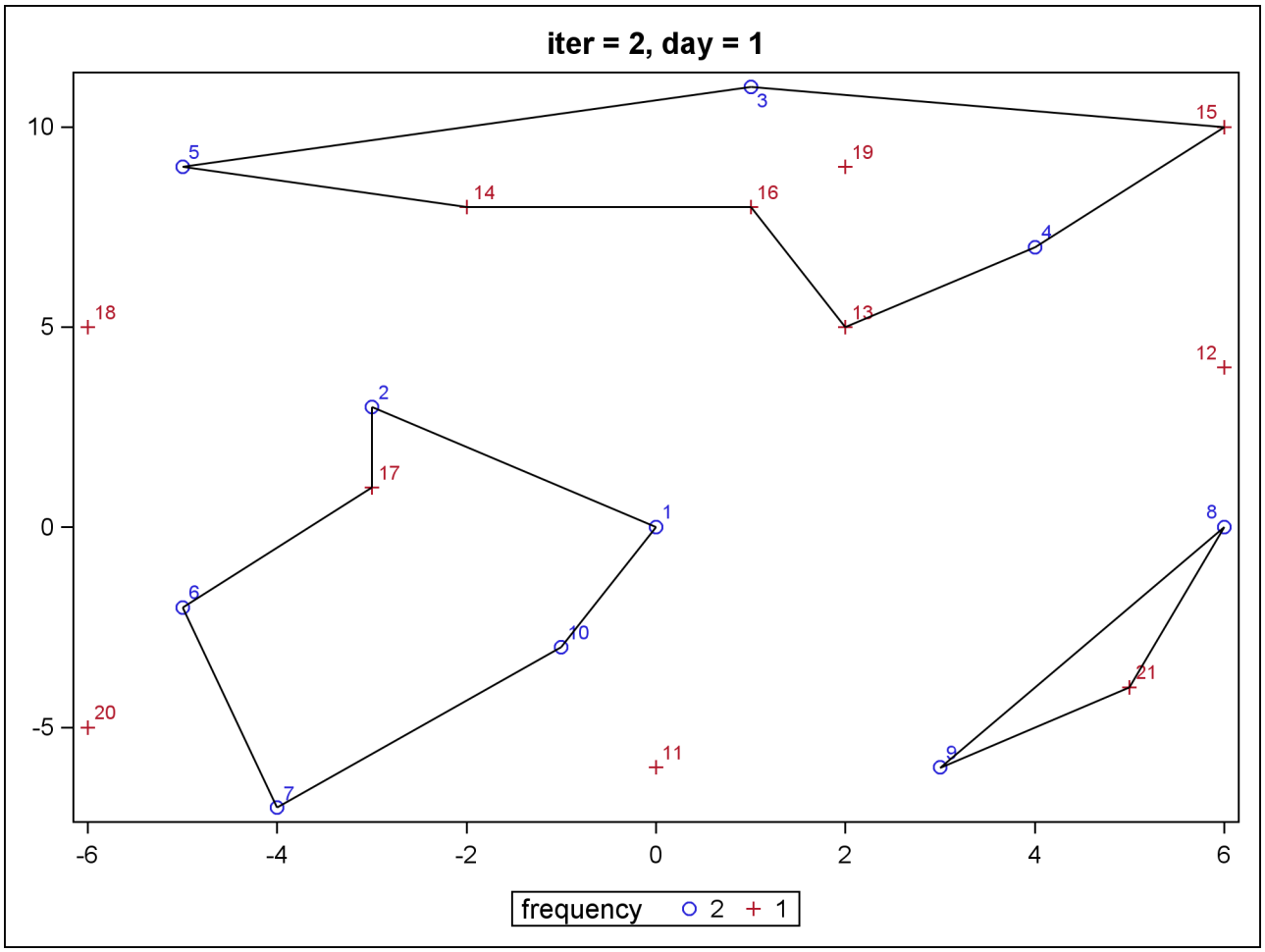


Figure 23.5 Output from Mixed Integer Linear Programming Solver, Iteration 2, Day 2

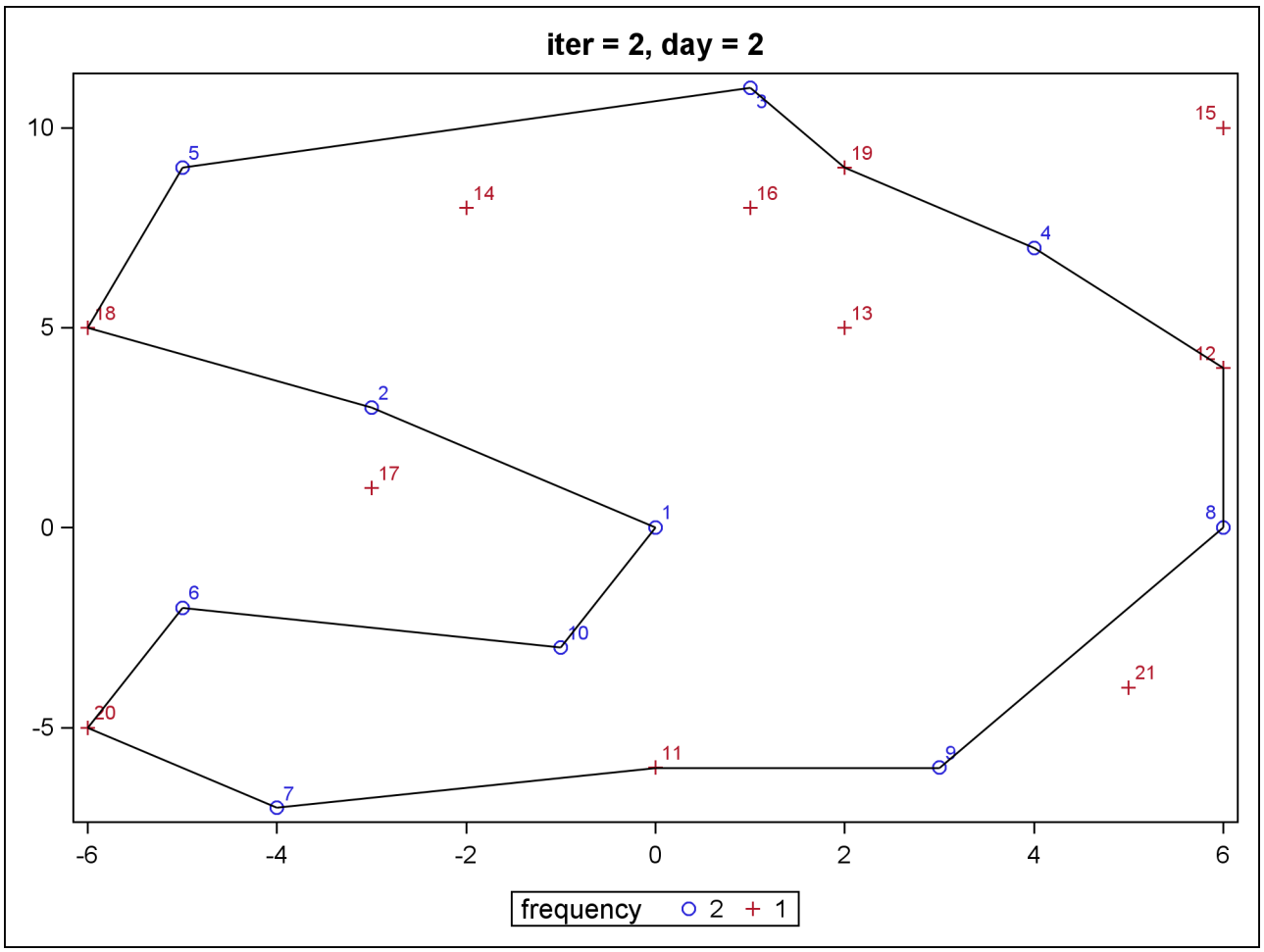


Figure 23.6 and Figure 23.7 show the output from the mixed integer linear programming solver for the third iteration, after the second round of subtour elimination constraints.

Figure 23.6 Output from Mixed Integer Linear Programming Solver, Iteration 3, Day 1

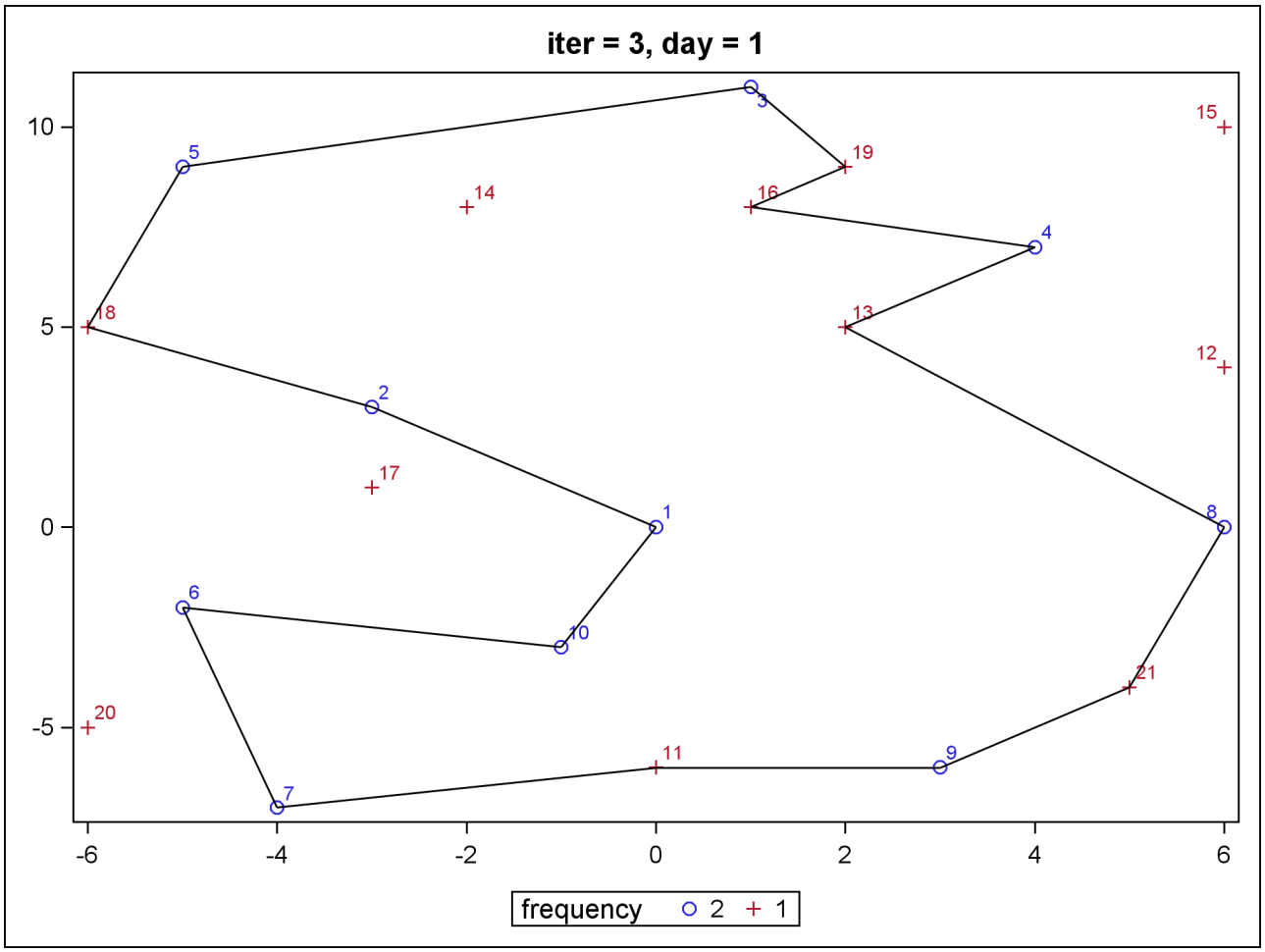
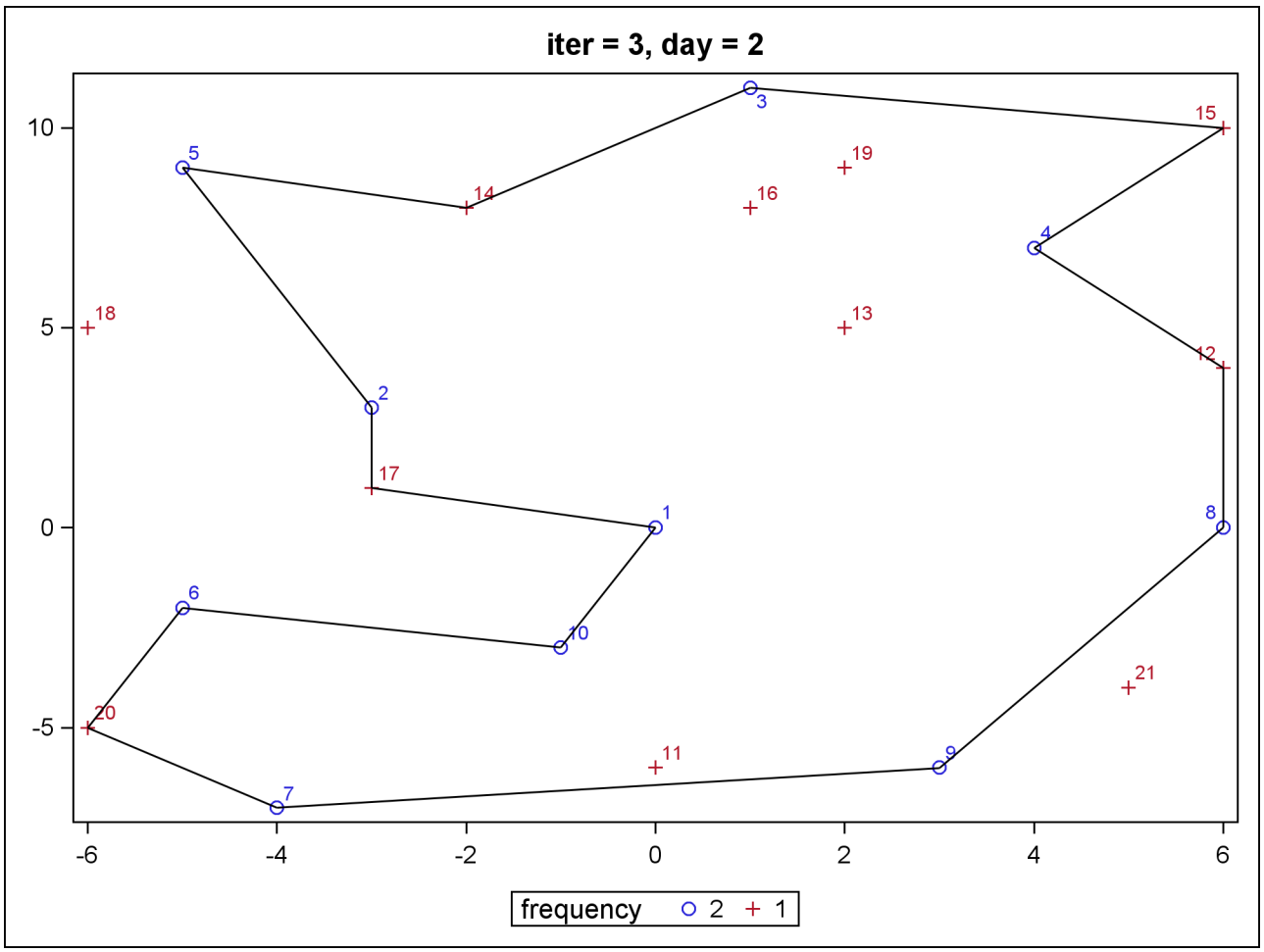


Figure 23.7 Output from Mixed Integer Linear Programming Solver, Iteration 3, Day 2



Since no more subtour elimination constraints are violated, the DO UNTIL loop terminates with an optimal solution. Figure 23.8 shows the final problem and solution summaries from the mixed integer linear programming solver.

Figure 23.8 Final Problem and Solution Summaries from Mixed Integer Linear Programming Solver

Problem Summary	
Objective Sense	Minimization
Objective Function	TotalDistance
Objective Type	Linear
Number of Variables	462
Bounded Above	0
Bounded Below	0
Bounded Below and Above	462
Free	0
Fixed	0
Binary	462
Integer	0
Number of Constraints	108
Linear LE (<=)	3
Linear EQ (=)	63
Linear GE (>=)	42
Linear Range	0
Constraint Coefficients	4192

Performance Information	
Execution Mode	Single-Machine
Number of Threads	4

Solution Summary	
Solver	MILP
Algorithm	Branch and Cut
Objective Function	TotalDistance
Solution Status	Optimal
Objective Value	1230.5623785
Relative Gap	0
Absolute Gap	0
Primal Infeasibility	8.881784E-16
Bound Infeasibility	6.661338E-16
Integer Infeasibility	6.661338E-16
Best Bound	1230.5623785
Nodes	1
Iterations	177
Presolve Time	0.01
Solution Time	0.05

The optimal objective value differs slightly from the one given in Williams (1999), perhaps because of rounding of distances by Williams.

Features Demonstrated

The following features are demonstrated in this example:

- problem type: mixed integer linear programming (vehicle routing)
- sets of tuples
- sets indexed by other sets
- set operators UNION and DIFF
- INIT option
- CARD function
- using a colon (:) to select members of a set
- AND aggregation operator
- symmetry-breaking constraints
- MILP solver option SYMMETRY=
- calling a solver in a DO UNTIL loop
- row generation
- connected components
- `.sol` variable suffix
- CALL SYMPUT
- SGPLOT procedure

Chapter 24
Yield Management: What Quantities of Airline Tickets to Sell at What Prices and What Times

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Problem Statement

An airline is selling tickets for flights to a particular destination.[1] The flight will depart in three weeks' time. It can use up to six planes each costing £50,000 to hire. Each plane has:

37	First Class seats
38	Business Class seats
47	Economy Class seats

Up to 10% of seats in any one category can be transferred to an adjacent category.

It wishes to decide a price for each of these seats. There will be further opportunities to update these prices after one week and two weeks. Once a customer has purchased a ticket there is no cancellation option.

For administrative simplicity three price level options are possible in each class (one of which must be chosen). The same option need not be chosen for each class. These are given in Table 24.1 for the current period (period 1) and two future periods.

[1] Reproduced with permission of John Wiley & Sons Ltd. (Williams 1999, pp. 256–258).

Table 24.1

	Option 1	Option 2	Option 3	
First	£1200	£1000	£950	
Business	£900	£800	£600	Period 1
Economy	£500	£300	£200	
First	£1400	£1300	£1150	
Business	£1100	£900	£750	Period 2
Economy	£700	£400	£350	
First	£1500	£900	£850	
Business	£820	£800	£500	Period 3
Economy	£480	£470	£450	

Demand is uncertain but will be affected by price. Forecasts have been made of these demands according to a probability distribution which divides the demand levels into three scenarios for each period. The probabilities of the three scenarios in each period are:

Scenario 1	0.1
Scenario 2	0.7
Scenario 3	0.2

The forecast demands are shown in Table 24.2.

Decide price levels for the current period, how many seats to sell in each class (depending on demand), the provisional number of planes to book and provisional price levels and seats to sell in future periods in order to maximize expected yield. You should schedule to be able to meet commitments under all possible combinations of scenarios.

With hindsight (i.e. not known until the beginning of the next period) it turned out that demand in each period (depending on the price level you chose) was as shown in Table 24.3.

Use the actual demands that resulted from the prices you set in period 1 to rerun the model at the beginning of period 2 to set price levels for period 2 and provisional price levels for period 3.

Repeat this procedure with a rerun at the beginning of period 3. Give the final operational solution.

Contrast this solution to one obtained at the beginning of period 1 by pricing to maximize yield based on expected demands.

Table 24.2

	Price option 1	Price option 2	Price option 3	
First	10	15	20	Period 1
Business	20	25	35	Scenario 1
Economy	45	55	60	
First	20	25	35	Period 1
Business	40	42	45	Scenario 2
Economy	50	52	63	
First	45	50	60	Period 1
Business	45	46	47	Scenario 3
Economy	55	56	64	
First	20	25	35	Period 2
Business	42	45	46	Scenario 1
Economy	50	52	60	
First	10	40	50	Period 2
Business	50	60	80	Scenario 2
Economy	60	65	90	
First	50	55	80	Period 2
Business	20	30	50	Scenario 3
Economy	10	40	60	
First	30	35	40	Period 3
Business	40	50	55	Scenario 1
Economy	50	60	80	
First	30	40	60	Period 3
Business	10	40	45	Scenario 2
Economy	50	60	70	
First	50	70	80	Period 3
Business	40	45	60	Scenario 3
Economy	60	65	70	

Table 24.3

	Price option 1	Price option 2	Price option 3	
First	25	30	40	
Business	50	40	45	Period 1
Economy	50	53	65	
First	22	45	50	
Business	45	55	75	Period 2
Economy	50	60	80	
First	45	60	75	
Business	20	40	50	Period 3
Economy	55	60	75	

Mathematical Programming Formulation

Index Sets and Their Members

The following index sets and their members are used in this example:

- period $\in$ PERIODS
- class $\in$ CLASSES
- option $\in$ OPTIONS
- scenario, $i \in$ SCENARIOS
- $(i, j) \in$ SCENARIOS2 = SCENARIOS $\times$ SCENARIOS
- $(i, j, k) \in$ SCENARIOS3 = SCENARIOS2 $\times$ SCENARIOS

Parameters

Table 24.4 shows the parameters that are used in this example.

Table 24.4 Parameters

Parameter Name	Interpretation
num_seats[class]	Number of seats per class per plane
price[period,class,option]	Price level (in £)
prob[scenario]	Probability of demand level scenario per period
demand[period,scenario,class,option]	Demand for period, scenario, and class if price level option is chosen
actual_demand[period,class,option]	Actual demand for period and class if price level option is chosen
expected_demand[period,class,option]	Expected demand for period and class if price level option is chosen
num_periods	Number of periods
num_planes	Number of planes available to be used
plane_cost	Cost to hire each plane (in £)
transfer_fraction_ub	Upper bound on fraction of seats that can be transferred to another category
num_options	Number of price level options
actual_price[period,class]	Actual price level for period and class
actual_sales[period,class]	Actual sales for period and class
actual_revenue[period,class]	Actual revenue for period and class
current_period	Current period for optimization

Variables

Table 24.5 shows the variables that are used in this example.

Table 24.5 Variables

Variable Name	Interpretation
P1[class,option]	1 if price option is chosen for class in period 1; 0 otherwise
P2[i,class,option]	1 if price option is chosen for class in period 2 as a result of scenario *i* in period 1; 0 otherwise
P3[i,j,class,option]	1 if price option is chosen for class in period 3 as a result of scenarios *i* and *j* in periods 1 and 2, respectively; 0 otherwise
S1[i,class,option]	Sales for class in period 1 under price option and scenario *i*
S2[i,j,class,option]	Sales for class in period 2 under price option and scenarios *i* and *j* in periods 1 and 2, respectively
S3[i,j,k,class,option]	Sales for class in period 3 under price option and scenarios i, j, k in periods 1, 2, 3, respectively
R1[i,class,option]	Revenue for class in period 1 under price option and scenario *i*
R2[i,j,class,option]	Revenue for class in period 2 under price option and scenarios *i* and *j* in periods 1 and 2, respectively
R3[i,j,k,class,option]	Revenue for class in period 3 under price option and scenarios i, j, k in periods 1, 2, 3, respectively
TransferFrom[i,j,k,class]	Number of seats transferred from class under scenarios i, j, k in periods 1, 2, 3, respectively
TransferTo[i,j,k,class]	Number of seats transferred to class under scenarios i, j, k in periods 1, 2, 3, respectively
NumPlanes	Number of planes to use

Objective

The objective is to maximize the following quadratic function:

$$
\begin{aligned}
&\text{ExpectedYield} \\
&= \left(\text{if } \mathit{current_period} \le 1, \text{then} \sum_{\substack{i \in \text{SCENARIOS}, \\ \text{class} \in \text{CLASSES}, \\ \text{option} \in \text{OPTIONS}}} \mathit{prob[i]} \cdot \text{R1[i,class,option]} \right) \\
&+ \left(\text{if } \mathit{current_period} \le 2, \text{then} \sum_{\substack{(i,j) \in \text{SCENARIOS2}, \\ \text{class} \in \text{CLASSES}, \\ \text{option} \in \text{OPTIONS}}} \mathit{prob[i]} \cdot \mathit{prob[j]} \cdot \text{R2[i,j,class,option]} \right) \\
&+ \left(\text{if } \mathit{current_period} \le 3, \text{then} \sum_{\substack{(i,j,k) \in \text{SCENARIOS3}, \\ \text{class} \in \text{CLASSES}, \\ \text{option} \in \text{OPTIONS}}} \mathit{prob[i]} \cdot \mathit{prob[j]} \cdot \mathit{prob[k]} \cdot \text{R3[i,j,k,class,option]} \right) \\
&+ \sum_{\substack{\text{period} \in 1 \ldots \mathit{current_period}-1, \\ \text{class} \in \text{CLASSES}}} \mathit{actual_revenue[period,class]} \\
&- \mathit{plane_cost} \cdot \text{NumPlanes}
\end{aligned}
$$

where

$$
\begin{aligned}
\text{R1[i,class,option]} &= \mathit{price[1,class,option]} \cdot \text{P1[class,option]} \cdot \text{S1[i,class,option]} \\
\text{R2[i,j,class,option]} &= \mathit{price[2,class,option]} \cdot \text{P2[class,option]} \cdot \text{S2[i,j,class,option]} \\
\text{R3[i,j,k,class,option]} &= \mathit{price[3,class,option]} \cdot \text{P3[class,option]} \cdot \text{S3[i,j,k,class,option]}
\end{aligned}
$$

Constraints

The following constraints are used in this example:

- bounds on variables
- for $(i, j, k) \in$ SCENARIOS3 and class $\in$ CLASSES,

$$\sum_{\text{option} \in \text{OPTIONS}} (\text{S1[i,class,option]} + \text{S2[i,j,class,option]} + \text{S3[i,j,k,class,option]}) + \text{TransferFrom[i,j,k,class]} - \text{TransferTo[i,j,k,class]} \leq \textit{num_seats[class]} \cdot \text{NumPlanes}$$

- for $(i, j, k) \in$ SCENARIOS3 and class $\in$ CLASSES,

$$\text{TransferFrom[i,j,k,class]} \leq \textit{transfer_fraction_ub} \cdot \textit{num_seats[class]}$$

- for $(i, j, k) \in$ SCENARIOS3 and class $\in$ CLASSES,

$$\text{TransferTo[i,j,k,class]} \leq \textit{transfer_fraction_ub} \cdot \textit{num_seats[class]}$$

- for $(i, j, k) \in$ SCENARIOS3,

$$\sum_{\text{class} \in \text{CLASSES}} \text{TransferFrom[i,j,k,class]} = \sum_{\text{class} \in \text{CLASSES}} \text{TransferTo[i,j,k,class]}$$

- for class $\in$ CLASSES,

$$\sum_{\text{option} \in \text{OPTIONS}} \text{P1[class,option]} = 1$$

- for $i \in$ SCENARIOS and class $\in$ CLASSES,

$$\sum_{\text{option} \in \text{OPTIONS}} \text{P2[i,class,option]} = 1$$

- for $(i, j) \in$ SCENARIOS2 and class $\in$ CLASSES,

$$\sum_{\text{option} \in \text{OPTIONS}} \text{P3[i,j,class,option]} = 1$$

- for $i \in$ SCENARIOS and class $\in$ CLASSES and option $\in$ OPTIONS,

$$\text{S1[i,class,option]} \leq \textit{demand[1,i,class,option]} \cdot \text{P1[class,option]}$$

- for $(i, j) \in$ SCENARIOS2 and class $\in$ CLASSES and option $\in$ OPTIONS,

$$\text{S2[i,j,class,option]} \leq \textit{demand[2,j,class,option]} \cdot \text{P2[i,class,option]}$$

- for $(i, j, k) \in$ SCENARIOS3 and class $\in$ CLASSES and option $\in$ OPTIONS,

  $$\text{S3[i,j,k,class,option]} \leq \textit{demand[3,k,class,option]} \cdot \text{P3[i,j,class,option]}$$

- for $i \in$ SCENARIOS and class $\in$ CLASSES and option $\in$ OPTIONS,

  $$\text{R1[i,class,option]} \leq \textit{price[1,class,option]} \cdot \text{S1[i,class,option]}$$

- for $i \in$ SCENARIOS and class $\in$ CLASSES and option $\in$ OPTIONS,

  $$\begin{aligned}&\textit{price[1,class,option]} \cdot \text{S1[i,class,option]} - \text{R1[i,class,option]} \\ &\leq \textit{price[1,class,option]} \cdot \textit{demand[1,i,class,option]} \cdot (1 - \text{P1[class,option]})\end{aligned}$$

- for $(i, j) \in$ SCENARIOS2 and class $\in$ CLASSES and option $\in$ OPTIONS,

  $$\text{R2[i,j,class,option]} \leq \textit{price[2,class,option]} \cdot \text{S2[i,j,class,option]}$$

- for $(i, j) \in$ SCENARIOS2 and class $\in$ CLASSES and option $\in$ OPTIONS,

  $$\begin{aligned}&\textit{price[2,class,option]} \cdot \text{S2[i,j,class,option]} - \text{R2[i,j,class,option]} \\ &\leq \textit{price[2,class,option]} \cdot \textit{demand[2,j,class,option]} \cdot (1 - \text{P2[i,class,option]})\end{aligned}$$

- for $(i, j, k) \in$ SCENARIOS3 and class $\in$ CLASSES and option $\in$ OPTIONS,

  $$\text{R3[i,j,k,class,option]} \leq \textit{price[3,class,option]} \cdot \text{S3[i,j,k,class,option]}$$

- for $(i, j, k) \in$ SCENARIOS3 and class $\in$ CLASSES and option $\in$ OPTIONS,

  $$\begin{aligned}&\textit{price[3,class,option]} \cdot \text{S3[i,j,k,class,option]} - \text{R3[i,j,k,class,option]} \\ &\leq \textit{price[3,class,option]} \cdot \textit{demand[3,k,class,option]} \cdot (1 - \text{P3[i,j,class,option]})\end{aligned}$$

Input Data

The following data sets and macro variables contain the input data that are used in this example:

```
data class_data;
   input class $9. num_seats;
   datalines;
First    37
Business 38
Economy  47
;

data price_data;
   input period class $9. price1-price3;
   datalines;
```

```
1 First    1200 1000  950
1 Business  900  800  600
1 Economy   500  300  200
2 First    1400 1300 1150
2 Business 1100  900  750
2 Economy   700  400  350
3 First    1500  900  850
3 Business  820  800  500
3 Economy   480  470  450
;

data scenario_data;
   input prob;
   datalines;
0.1
0.7
0.2
;

data demand_data;
   input period scenario class $9. demand1-demand3;
   datalines;
1 1 First    10 15 20
1 1 Business 20 25 35
1 1 Economy  45 55 60
1 2 First    20 25 35
1 2 Business 40 42 45
1 2 Economy  50 52 63
1 3 First    45 50 60
1 3 Business 45 46 47
1 3 Economy  55 56 64
2 1 First    20 25 35
2 1 Business 42 45 46
2 1 Economy  50 52 60
2 2 First    10 40 50
2 2 Business 50 60 80
2 2 Economy  60 65 90
2 3 First    50 55 80
2 3 Business 20 30 50
2 3 Economy  10 40 60
3 1 First    30 35 40
3 1 Business 40 50 55
3 1 Economy  50 60 80
3 2 First    30 40 60
3 2 Business 10 40 45
3 2 Economy  50 60 70
3 3 First    50 70 80
3 3 Business 40 45 60
3 3 Economy  60 65 70
;

data actual_demand_data;
   input period class $9. demand1-demand3;
   datalines;
```

```
1 First    25 30 40
1 Business 50 40 45
1 Economy  50 53 65
2 First    22 45 50
2 Business 45 55 75
2 Economy  50 60 80
3 First    45 60 75
3 Business 20 40 50
3 Economy  55 60 75
;

%let num_periods = 3;
%let num_planes = 6;
%let plane_cost = 50000;
%let transfer_fraction_ub = 0.10;
%let num_options = 3;
```

PROC OPTMODEL Statements and Output

The first several PROC OPTMODEL statements declare index sets and parameters and then read the input data:

```
proc optmodel;
   set PERIODS = 1..&num_periods;

   set <str> CLASSES;
   num num_seats {CLASSES};
   read data class_data into CLASSES=[class] num_seats;

   set OPTIONS = 1..&num_options;

   num price {PERIODS, CLASSES, OPTIONS};
   read data price_data into [period class]
      {option in OPTIONS} <price[period,class,option]=col('price'||option)>;

   set SCENARIOS;
   num prob {SCENARIOS};
   read data scenario_data into SCENARIOS=[_N_] prob;
   set SCENARIOS2 = SCENARIOS cross SCENARIOS;
   set SCENARIOS3 = SCENARIOS2 cross SCENARIOS;

   num demand {PERIODS, SCENARIOS, CLASSES, OPTIONS};
   read data demand_data into [period scenario class]
      {option in OPTIONS}
      <demand[period,scenario,class,option]=col('demand'||option)>;

   num actual_demand {PERIODS, CLASSES, OPTIONS};
   read data actual_demand_data into [period class]
      {option in OPTIONS}
      <actual_demand[period,class,option]=col('demand'||option)>;
```

```
num actual_price {PERIODS, CLASSES};
num actual_sales {PERIODS, CLASSES};
num actual_revenue {PERIODS, CLASSES};

num current_period;
```

The following VAR statements declare the decision variables:

```
var P1 {CLASSES, OPTIONS} binary;
var P2 {SCENARIOS, CLASSES, OPTIONS} binary;
var P3 {SCENARIOS2, CLASSES, OPTIONS} binary;

var S1 {SCENARIOS, CLASSES, OPTIONS} >= 0;
var S2 {SCENARIOS2, CLASSES, OPTIONS} >= 0;
var S3 {SCENARIOS3, CLASSES, OPTIONS} >= 0;

var R1 {SCENARIOS, CLASSES, OPTIONS} >= 0;
var R2 {SCENARIOS2, CLASSES, OPTIONS} >= 0;
var R3 {SCENARIOS3, CLASSES, OPTIONS} >= 0;

var TransferFrom {SCENARIOS3, CLASSES} >= 0;
var TransferTo {SCENARIOS3, CLASSES} >= 0;

var NumPlanes >= 0 <= &num_planes integer;
```

The following CON statement declares the constraints that enforce seat capacities:

```
con NumPlanes_con {<i,j,k> in SCENARIOS3, class in CLASSES}:
   sum {option in OPTIONS}
      (S1[i,class,option] + S2[i,j,class,option] + S3[i,j,k,class,option])
 + TransferFrom[i,j,k,class] - TransferTo[i,j,k,class]
<= num_seats[class] * NumPlanes;
```

The following statements declare the constraints that restrict adjustment between classes:

```
for {<i,j,k> in SCENARIOS3, class in CLASSES} do;
   TransferFrom[i,j,k,class].ub = &transfer_fraction_ub * num_seats[class];
   TransferTo[i,j,k,class].ub   = &transfer_fraction_ub * num_seats[class];
end;
con Balance_con {<i,j,k> in SCENARIOS3}:
   sum {class in CLASSES} TransferFrom[i,j,k,class]
 = sum {class in CLASSES} TransferTo[i,j,k,class];
```

The following CON statements declare the constraints that enforce one price level per class:

```
con P1_con {class in CLASSES}:
   sum {option in OPTIONS} P1[class,option] = 1;
con P2_con {i in SCENARIOS, class in CLASSES}:
   sum {option in OPTIONS} P2[i,class,option] = 1;
con P3_con {<i,j> in SCENARIOS2, class in CLASSES}:
   sum {option in OPTIONS} P3[i,j,class,option] = 1;
```

The following CON statements declare the constraints that sales cannot exceed demand:

```
con S1_con {i in SCENARIOS, class in CLASSES, option in OPTIONS}:
   S1[i,class,option] <= demand[1,i,class,option] * P1[class,option];
con S2_con {<i,j> in SCENARIOS2, class in CLASSES, option in OPTIONS}:
   S2[i,j,class,option] <= demand[2,j,class,option] * P2[i,class,option];
con S3_con {<i,j,k> in SCENARIOS3, class in CLASSES, option in OPTIONS}:
   S3[i,j,k,class,option] <= demand[3,k,class,option] *
      P3[i,j,class,option];
```

The following CON statements encode one possible linearization of the quadratic objective:

```
/* R1[i,class,option] =
   price[1,class,option] * P1[class,option] * S1[i,class,option] */
con R1_con_a {i in SCENARIOS, class in CLASSES, option in OPTIONS}:
   R1[i,class,option] <= price[1,class,option] * S1[i,class,option];
con R1_con_b {i in SCENARIOS, class in CLASSES, option in OPTIONS}:
   price[1,class,option] * S1[i,class,option] - R1[i,class,option]
<= price[1,class,option] * demand[1,i,class,option] *
      (1 - P1[class,option]);

/* R2[i,j,class,option] =
   price[2,class,option] * P2[i,class,option] * S2[i,j,class,option] */
con R2_con_a {<i,j> in SCENARIOS2, class in CLASSES, option in OPTIONS}:
   R2[i,j,class,option] <= price[2,class,option] * S2[i,j,class,option];
con R2_con_b {<i,j> in SCENARIOS2, class in CLASSES, option in OPTIONS}:
   price[2,class,option] * S2[i,j,class,option] - R2[i,j,class,option]
<= price[2,class,option] * demand[2,j,class,option] *
      (1 - P2[i,class,option]);

/* R3[i,j,k,class,option] =
   price[3,class,option] * P3[i,j,class,option] * S3[i,j,k,class,option] */
con R3_con_a {<i,j,k> in SCENARIOS3, class in CLASSES, option in OPTIONS}:
   R3[i,j,k,class,option] <= price[3,class,option] * S3[i,j,k,class,option];
con R3_con_b {<i,j,k> in SCENARIOS3, class in CLASSES, option in OPTIONS}:
   price[3,class,option] * S3[i,j,k,class,option] - R3[i,j,k,class,option]
<= price[3,class,option] * demand[3,k,class,option] *
      (1 - P3[i,j,class,option]);
```

An alternative "compact linearization" (not shown) involves fewer constraints, as in Chapter 10.

The following MAX statement declares the linearized objective, which depends on *current_period*:

```
max ExpectedYield =
   (if current_period <= 1
    then sum {i in SCENARIOS, class in CLASSES, option in OPTIONS}
            prob[i] * R1[i,class,option])
 + (if current_period <= 2
    then sum {<i,j> in SCENARIOS2, class in CLASSES, option in OPTIONS}
            prob[i] * prob[j] * R2[i,j,class,option])
 + (if current_period <= 3
    then sum {<i,j,k> in SCENARIOS3, class in CLASSES, option in OPTIONS}
            prob[i] * prob[j] * prob[k] * R3[i,j,k,class,option])
 + sum {period in 1..current_period-1, class in CLASSES}
      actual_revenue[period,class]
 - &plane_cost * NumPlanes;
```

The following NUM statements use the `.sol` variable suffix to compute the recommended prices from the optimal values of the decision variables:

```
num price_sol_1 {class in CLASSES} =
   sum {option in OPTIONS} price[1,class,option] * P1[class,option].sol;
num price_sol_2 {class in CLASSES, i in SCENARIOS} =
   sum {option in OPTIONS} price[2,class,option] * P2[i,class,option].sol;
num price_sol_3 {class in CLASSES, <i,j> in SCENARIOS2} =
   sum {option in OPTIONS} price[3,class,option] * P3[i,j,class,option].sol;
```

The following NUM statements use the `.sol` variable suffix to compute the recommended numbers of seats to sell:

```
num remaining_seats {class in CLASSES} =
   num_seats[class] * NumPlanes.sol
 - sum {period in 1..current_period-1} actual_sales[period,class];
num sell_up_to_1 {class in CLASSES} =
   min(
      max {i in SCENARIOS, option in OPTIONS} S1[i,class,option].sol,
      remaining_seats[class]);
num sell_up_to_2 {class in CLASSES} =
   min(
      max {<i,j> in SCENARIOS2, option in OPTIONS} S2[i,j,class,option].sol,
      remaining_seats[class]);
num sell_up_to_3 {class in CLASSES} =
   min(
      max {<i,j,k> in SCENARIOS3, option in OPTIONS}
      S3[i,j,k,class,option].sol, remaining_seats[class]);
```

The following statements call the mixed integer linear programming solver to determine the optimal prices for period 1:

```
current_period = 1;
solve;
for {i in SCENARIOS, class in CLASSES, option in OPTIONS}
   S1[i,class,option] = round(S1[i,class,option].sol);
print price_sol_1;
print sell_up_to_1;
print {i in SCENARIOS, class in CLASSES, option in OPTIONS:
   S1[i,class,option].sol > 0} S1;
print price_sol_2;
print price_sol_3;
print NumPlanes ExpectedYield;
```

The following statements fix the resulting prices for period 1 and use the MIN function to limit sales based on actual demand:

```
for {class in CLASSES, option in OPTIONS} do;
   if P1[class,option].sol > 0.5 then do;
      fix P1[class,option] = 1;
      actual_price[1,class] = price_sol_1[class];
      actual_sales[1,class] =
         min(sell_up_to_1[class], actual_demand[1,class,option]);
      for {i in SCENARIOS} fix S1[i,class,option] = actual_sales[1,class];
```

```
   end;
   else fix P1[class,option] = 0;
end;
for {class in CLASSES}
   actual_revenue[1,class] = actual_price[1,class] * actual_sales[1,class];
print actual_price actual_sales actual_revenue;
```

Figure 24.1 shows the output from the mixed integer linear programming solver for period 1.

Figure 24.1 Output from Mixed Integer Linear Programming Solver, Period 1

The OPTMODEL Procedure

Problem Summary	
Objective Sense	Maximization
Objective Function	ExpectedYield
Objective Type	Linear
Number of Variables	982
Bounded Above	0
Bounded Below	702
Bounded Below and Above	280
Free	0
Fixed	0
Binary	117
Integer	1
Number of Constraints	1200
Linear LE (<=)	1134
Linear EQ (=)	66
Linear GE (>=)	0
Linear Range	0
Constraint Coefficients	3708

Performance Information	
Execution Mode	Single-Machine
Number of Threads	4

Figure 24.1 *continued*

Solution Summary	
Solver	MILP
Algorithm	Branch and Cut
Objective Function	ExpectedYield
Solution Status	Optimal within Relative Gap
Objective Value	169543.624
Relative Gap	0.0000165314
Absolute Gap	2.8028356882
Primal Infeasibility	7.275958E-12
Bound Infeasibility	1.456613E-12
Integer Infeasibility	0
Best Bound	169546.42684
Nodes	27
Iterations	3595
Presolve Time	0.05
Solution Time	0.34

[1]	price_sol_1
Business	900
Economy	500
First	1200

[1]	sell_up_to_1
Business	45
Economy	55
First	45

[1]	[2]	[3]	S1
1	**Business**	**1**	20
1	**Economy**	**1**	45
1	**First**	**1**	10
2	**Business**	**1**	40
2	**Economy**	**1**	50
2	**First**	**1**	20
3	**Business**	**1**	45
3	**Economy**	**1**	55
3	**First**	**1**	45

	price_sol_2		
	1	**2**	**3**
Business	1100	1100	1100
Economy	700	700	700
First	1150	1150	1300

Figure 24.1 *continued*

[1]	[2]	[3]	price_sol_3
Business	1	1	800
Business	1	2	800
Business	1	3	800
Business	2	1	800
Business	2	2	800
Business	2	3	800
Business	3	1	800
Business	3	2	800
Business	3	3	800
Economy	1	1	480
Economy	1	2	480
Economy	1	3	450
Economy	2	1	480
Economy	2	2	480
Economy	2	3	450
Economy	3	1	480
Economy	3	2	480
Economy	3	3	450
First	1	1	1500
First	1	2	1500
First	1	3	1500
First	2	1	1500
First	2	2	1500
First	2	3	1500
First	3	1	1500
First	3	2	1500
First	3	3	1500

NumPlanes	ExpectedYield
3	169544

[1]	[2]	actual_price	actual_sales	actual_revenue
1	Business	900	45	40500
1	Economy	500	50	25000
1	First	1200	25	30000

The following statements drop the period 1 constraints and call the mixed integer linear programming solver to determine the optimal prices for period 2:

```
drop P1_con S1_con R1_con_a R1_con_b;
current_period = 2;
solve;
for {<i,j> in SCENARIOS2, class in CLASSES, option in OPTIONS}
   S2[i,j,class,option] = round(S2[i,j,class,option].sol);
print price_sol_2;
```

```
print sell_up_to_2;
print {<i,j> in SCENARIOS2, class in CLASSES, option in OPTIONS:
   i = 1 and S2[1,j,class,option].sol > 0} S2;
print price_sol_3;
print NumPlanes ExpectedYield;
```

The following statements fix the resulting prices for period 2 and use the MIN function to limit sales based on actual demand:

```
for {i in SCENARIOS, class in CLASSES, option in OPTIONS} do;
   if P2[i,class,option].sol > 0.5 then do;
      fix P2[i,class,option] = 1;
      actual_price[2,class] = price_sol_2[class,i];
      actual_sales[2,class] =
         min(sell_up_to_2[class], actual_demand[2,class,option]);
      for {j in SCENARIOS} fix S2[i,j,class,option] = actual_sales[2,class];
   end;
   else fix P2[i,class,option] = 0;
end;
for {class in CLASSES}
   actual_revenue[2,class] = actual_price[2,class] * actual_sales[2,class];
print actual_price actual_sales actual_revenue;
```

Figure 24.2 shows the output from the mixed integer linear programming solver for period 2.

Figure 24.2 Output from Mixed Integer Linear Programming Solver, Period 2

Problem Summary	
Objective Sense	Maximization
Objective Function	ExpectedYield
Objective Type	Linear
Number of Variables	982
Bounded Above	0
Bounded Below	693
Bounded Below and Above	271
Free	0
Fixed	18
Binary	117
Integer	1
Number of Constraints	1116
Linear LE (<=)	1053
Linear EQ (=)	63
Linear GE (>=)	0
Linear Range	0
Constraint Coefficients	3510

Performance Information	
Execution Mode	Single-Machine
Number of Threads	4

Figure 24.2 *continued*

Solution Summary	
Solver	MILP
Algorithm	Branch and Cut
Objective Function	ExpectedYield
Solution Status	Optimal within Relative Gap
Objective Value	172964.624
Relative Gap	0.0000361146
Absolute Gap	6.2467664414
Primal Infeasibility	1.455192E-11
Bound Infeasibility	2.941647E-11
Integer Infeasibility	3.330669E-16
Best Bound	172970.87077
Nodes	2
Iterations	2581
Presolve Time	0.03
Solution Time	0.19

	price_sol_2		
	1	**2**	**3**
Business	1100	1100	1100
Economy	700	700	700
First	1150	1150	1150

[1]	sell_up_to_2
Business	50
Economy	60
First	60

[1]	[2]	[3]	[4]	S2
1	**1**	**Business**	**1**	42
1	**1**	**Economy**	**1**	50
1	**1**	**First**	**3**	35
1	**2**	**Business**	**1**	50
1	**2**	**Economy**	**1**	60
1	**2**	**First**	**3**	50
1	**3**	**Business**	**1**	20
1	**3**	**Economy**	**1**	10
1	**3**	**First**	**3**	60

Figure 24.2 *continued*

[1]	[2]	[3]	price_sol_3
Business	1	1	800
Business	1	2	800
Business	1	3	800
Business	2	1	800
Business	2	2	800
Business	2	3	800
Business	3	1	800
Business	3	2	800
Business	3	3	800
Economy	1	1	470
Economy	1	2	480
Economy	1	3	450
Economy	2	1	480
Economy	2	2	480
Economy	2	3	450
Economy	3	1	480
Economy	3	2	480
Economy	3	3	450
First	1	1	1500
First	1	2	1500
First	1	3	1500
First	2	1	1500
First	2	2	1500
First	2	3	1500
First	3	1	1500
First	3	2	1500
First	3	3	1500

NumPlanes	ExpectedYield
3	172965

[1]	[2]	actual_price	actual_sales	actual_revenue
1	Business	900	45	40500
1	Economy	500	50	25000
1	First	1200	25	30000
2	Business	1100	45	49500
2	Economy	700	50	35000
2	First	1150	50	57500

The following statements drop the period 2 constraints and call the mixed integer linear programming solver to determine the optimal prices for period 3:

```
current_period = 3;
drop P2_con S2_con R2_con_a R2_con_b;
solve;

for {<i,j,k> in SCENARIOS3, class in CLASSES, option in OPTIONS}
   S3[i,j,k,class,option] = round(S3[i,j,k,class,option].sol);
print price_sol_3;
print sell_up_to_3;
print {<i,j,k> in SCENARIOS3, class in CLASSES, option in OPTIONS:
   <i,j> in {<1,1>} and S3[i,j,k,class,option].sol > 0} S3;
print NumPlanes ExpectedYield;
```

The following statements fix the resulting prices for period 3 and use the MIN function to limit sales based on actual demand:

```
for {<i,j> in SCENARIOS2, class in CLASSES, option in OPTIONS} do;
   if P3[i,j,class,option].sol > 0.5 then do;
      fix P3[i,j,class,option] = 1;
      actual_price[3,class] = price_sol_3[class,i,j];
      actual_sales[3,class] =
         min(sell_up_to_3[class], actual_demand[3,class,option]);
      for {k in SCENARIOS} fix S3[i,j,k,class,option]
         = actual_sales[3,class];
   end;
   else fix P3[i,j,class,option] = 0;
end;

for {class in CLASSES}
   actual_revenue[3,class] = actual_price[3,class] * actual_sales[3,class];
print actual_price actual_sales actual_revenue;
```

Figure 24.3 shows the output from the mixed integer linear programming solver for period 3.

Figure 24.3 Output from Mixed Integer Linear Programming Solver, Period 3

Problem Summary	
Objective Sense	Maximization
Objective Function	ExpectedYield
Objective Type	Linear
Number of Variables	982
Bounded Above	0
Bounded Below	666
Bounded Below and Above	244
Free	0
Fixed	72
Binary	117
Integer	1
Number of Constraints	864
Linear LE (<=)	810
Linear EQ (=)	54
Linear GE (>=)	0
Linear Range	0
Constraint Coefficients	2916

Performance Information	
Execution Mode	Single-Machine
Number of Threads	4

Solution Summary	
Solver	MILP
Algorithm	Branch and Cut
Objective Function	ExpectedYield
Solution Status	Optimal within Relative Gap
Objective Value	176392.4
Relative Gap	4.0173874E-6
Absolute Gap	0.7086394565
Primal Infeasibility	2.910383E-11
Bound Infeasibility	5.82645E-12
Integer Infeasibility	9.992007E-16
Best Bound	176393.10864
Nodes	2
Iterations	3461
Presolve Time	0.02
Solution Time	0.16

Figure 24.3 *continued*

[1]	[2]	[3]	price_sol_3
Business	1	1	800
Business	1	2	800
Business	1	3	800
Business	2	1	800
Business	2	2	800
Business	2	3	800
Business	3	1	800
Business	3	2	800
Business	3	3	800
Economy	1	1	480
Economy	1	2	480
Economy	1	3	480
Economy	2	1	480
Economy	2	2	480
Economy	2	3	480
Economy	3	1	480
Economy	3	2	480
Economy	3	3	480
First	1	1	1500
First	1	2	1500
First	1	3	1500
First	2	1	1500
First	2	2	1500
First	2	3	1500
First	3	1	1500
First	3	2	1500
First	3	3	1500

[1]	sell_up_to_3
Business	24
Economy	41
First	36

[1]	[2]	[3]	[4]	[5]	S3
1	1	1	Business	2	28
1	1	1	Economy	1	41
1	1	1	First	1	30
1	1	2	Business	2	28
1	1	2	Economy	1	41
1	1	2	First	1	30
1	1	3	Business	2	25
1	1	3	Economy	1	36
1	1	3	First	1	40

NumPlanes	ExpectedYield
3	176392

Figure 24.3 *continued*

[1]	[2]	actual_price	actual_sales	actual_revenue
1	Business	900	45	40500
1	Economy	500	50	25000
1	First	1200	25	30000
2	Business	1100	45	49500
2	Economy	700	50	35000
2	First	1150	50	57500
3	Business	800	24	19200
3	Economy	480	41	19680
3	First	1500	36	54000

The following statements print the expected yield that results from the optimal prices, with sales limited by actual demands:

```
   current_period = 4;
   print ExpectedYield;
quit;
```

Figure 24.4 shows the final expected yield.

Figure 24.4 Final Expected Yield

ExpectedYield
180380

Maximizing yield based on expected demands is much simpler than the stochastic programming approach and requires only one solver call. The first several PROC OPTMODEL statements are the same as before:

```
proc optmodel;
   set PERIODS = 1..&num_periods;

   set <str> CLASSES;
   num num_seats {CLASSES};
   read data class_data into CLASSES=[class] num_seats;

   set OPTIONS = 1..&num_options;

   num price {PERIODS, CLASSES, OPTIONS};
   read data price_data into [period class]
      {option in OPTIONS} <price[period,class,option]=col('price'||option)>;

   set SCENARIOS;
   num prob {SCENARIOS};
   read data scenario_data into SCENARIOS=[_N_] prob;

   num demand {PERIODS, SCENARIOS, CLASSES, OPTIONS};
   read data demand_data into [period scenario class]
      {option in OPTIONS}
      <demand[period,scenario,class,option]=col('demand'||option)>;
```

```
num actual_demand {PERIODS, CLASSES, OPTIONS};
read data actual_demand_data into [period class]
   {option in OPTIONS}
   <actual_demand[period,class,option]=col('demand'||option)>;

num actual_price {PERIODS, CLASSES};
num actual_sales {PERIODS, CLASSES};
num actual_revenue {PERIODS, CLASSES};
```

The following NUM statement declares the *expected_demand* parameter as a weighted sum of *demand*:

```
num expected_demand {period in PERIODS, class in CLASSES, option in OPTIONS}
   = sum {scenario in SCENARIOS}
      prob[scenario] * demand[period,scenario,class,option];
```

Note that the variables, constraints, and parameters are unified and simplified into fewer families than before:

```
var P {PERIODS, CLASSES, OPTIONS} binary;
var S {PERIODS, CLASSES, OPTIONS} >= 0;
var R {PERIODS, CLASSES, OPTIONS} >= 0;

var TransferFrom {CLASSES} >= 0;
var TransferTo {CLASSES} >= 0;

var NumPlanes >= 0 <= &num_planes integer;

con NumPlanes_con {class in CLASSES}:
   sum {period in PERIODS, option in OPTIONS} S[period,class,option]
 + TransferFrom[class] - TransferTo[class]
<= num_seats[class] * NumPlanes;

for {class in CLASSES} do;
   TransferFrom[class].ub = &transfer_fraction_ub * num_seats[class];
   TransferTo[class].ub   = &transfer_fraction_ub * num_seats[class];
end;
con Balance_con:
   sum {class in CLASSES} TransferFrom[class]
 = sum {class in CLASSES} TransferTo[class];

con P_con {period in PERIODS, class in CLASSES}:
   sum {option in OPTIONS} P[period,class,option] = 1;

con S_con {period in PERIODS, class in CLASSES, option in OPTIONS}:
   S[period,class,option]
<= expected_demand[period,class,option] * P[period,class,option];

/* R[period,class,option] =
   price[period,class,option] * P[period,class,option] *
      S[period,class,option] */
con R_con_a {period in PERIODS, class in CLASSES, option in OPTIONS}:
   R[period,class,option] <= price[period,class,option] *
      S[period,class,option];
con R_con_b {period in PERIODS, class in CLASSES, option in OPTIONS}:
   price[period,class,option] * S[period,class,option] -
      R[period,class,option]
```

```
<= price[period,class,option] * expected_demand[period,class,option]
   * (1 - P[period,class,option]);

max Yield =
   sum {period in PERIODS, class in CLASSES, option in OPTIONS}
      R[period,class,option]
 - &plane_cost * NumPlanes;

num price_sol {period in PERIODS, class in CLASSES} =
   sum {option in OPTIONS} price[period,class,option] *
      P[period,class,option].sol;

solve;
for {period in PERIODS, class in CLASSES, option in OPTIONS}
   S[period,class,option] = round(S[period,class,option].sol);
print price_sol;
print {period in PERIODS, class in CLASSES, option in OPTIONS:
   S[period,class,option].sol > 0} S;
print NumPlanes Yield;
for {period in PERIODS, class in CLASSES, option in OPTIONS} do;
   if P[period,class,option].sol > 0.5 then do;
      actual_price[period,class] = price_sol[period,class];
      actual_sales[period,class] =
         min(S[period,class,option], actual_demand[period,class,option]);
      actual_revenue[period,class] =
         actual_price[period,class] * actual_sales[period,class];
      R[period,class,option] = actual_revenue[period,class];
   end;
end;
print actual_price actual_sales actual_revenue;
print Yield;
```

Figure 24.5 shows the output from the mixed integer linear programming solver for the problem based on expected demands.

Figure 24.5 Output from Mixed Integer Linear Programming Solver, Based on Expected Demands

The OPTMODEL Procedure

Problem Summary	
Objective Sense	Maximization
Objective Function	Yield
Objective Type	Linear
Number of Variables	88
Bounded Above	0
Bounded Below	54
Bounded Below and Above	34
Free	0
Fixed	0
Binary	27
Integer	1
Number of Constraints	94
Linear LE (<=)	84
Linear EQ (=)	10
Linear GE (>=)	0
Linear Range	0
Constraint Coefficients	258

Performance Information	
Execution Mode	Single-Machine
Number of Threads	4

Figure 24.5 *continued*

Solution Summary	
Solver	MILP
Algorithm	Branch and Cut
Objective Function	Yield
Solution Status	Optimal
Objective Value	180309
Relative Gap	0
Absolute Gap	0
Primal Infeasibility	3.637979E-12
Bound Infeasibility	0
Integer Infeasibility	0
Best Bound	180309
Nodes	1
Iterations	465
Presolve Time	0.01
Solution Time	0.05

	price_sol		
	Business	**Economy**	**First**
1	900	500	1200
2	1100	700	1150
3	800	480	1500

[1]	[2]	[3]	S
1	**Business**	**1**	39
1	**Economy**	**1**	51
1	**First**	**1**	24
2	**Business**	**1**	43
2	**Economy**	**1**	49
2	**First**	**3**	55
3	**Business**	**2**	35
3	**Economy**	**1**	37
3	**First**	**1**	34

NumPlanes	Yield
3	180309

Figure 24.5 *continued*

[1]	[2]	actual_price	actual_sales	actual_revenue
1	Business	900	39	35100
1	Economy	500	50	25000
1	First	1200	24	28800
2	Business	1100	43	47300
2	Economy	700	49	34300
2	First	1150	50	57500
3	Business	800	35	28000
3	Economy	480	37	17760
3	First	1500	34	51000

Yield
174760

As anticipated, the yield is smaller than in Figure 24.4 because there is no opportunity here to change prices based on actual demand from previous periods.

The following statements show the modifications to maximize yield based on actual demands:

```
drop S_con R_con_b;

con S_con_actual {period in PERIODS, class in CLASSES, option in OPTIONS}:
   S[period,class,option]
<= actual_demand[period,class,option] * P[period,class,option];

con R_con_b_actual {period in PERIODS, class in CLASSES, option in OPTIONS}:
   price[period,class,option] * S[period,class,option] -
      R[period,class,option]
<= price[period,class,option] * actual_demand[period,class,option]
   * (1 - P[period,class,option]);

solve;
for {period in PERIODS, class in CLASSES, option in OPTIONS}
   S[period,class,option] = round(S[period,class,option].sol);
print price_sol;
print {period in PERIODS, class in CLASSES, option in OPTIONS:
   S[period,class,option].sol > 0} S;
print NumPlanes Yield;
```

Since actual demand is considered directly in this formulation, the postprocessing steps do not need to reduce sales to actual demand:

```
for {period in PERIODS, class in CLASSES, option in OPTIONS} do;
   if P[period,class,option].sol > 0.5 then do;
      actual_price[period,class] = price_sol[period,class];
      actual_sales[period,class] = S[period,class,option];
      actual_revenue[period,class] =
         actual_price[period,class] * actual_sales[period,class];
      R[period,class,option] = actual_revenue[period,class];
   end;
end;
print actual_price actual_sales actual_revenue;
quit;
```

Figure 24.6 shows the output from the mixed integer linear programming solver for the problem based on actual demands.

Figure 24.6 Output from Mixed Integer Linear Programming Solver, Based on Actual Demands

Problem Summary	
Objective Sense	Maximization
Objective Function	Yield
Objective Type	Linear
Number of Variables	88
Bounded Above	0
Bounded Below	54
Bounded Below and Above	34
Free	0
Fixed	0
Binary	27
Integer	1
Number of Constraints	94
Linear LE (<=)	84
Linear EQ (=)	10
Linear GE (>=)	0
Linear Range	0
Constraint Coefficients	258

Performance Information	
Execution Mode	Single-Machine
Number of Threads	4

Solution Summary	
Solver	MILP
Algorithm	Branch and Cut
Objective Function	Yield
Solution Status	Optimal
Objective Value	193964
Relative Gap	0
Absolute Gap	0
Primal Infeasibility	4.121148E-12
Bound Infeasibility	0
Integer Infeasibility	3.176001E-14
Best Bound	193964
Nodes	4
Iterations	1268
Presolve Time	0.00
Solution Time	0.05

Figure 24.6 *continued*

	price_sol		
	Business	Economy	First
1	900	500	1200
2	1100	700	1300
3	820	480	1500

[1]	[2]	[3]	S
1	Business	1	50
1	Economy	1	50
1	First	1	25
2	Business	1	45
2	Economy	1	50
2	First	2	45
3	Business	1	20
3	Economy	1	36
3	First	1	45

NumPlanes	Yield
3	193964

[1]	[2]	actual_price	actual_sales	actual_revenue
1	Business	900	50	45000
1	Economy	500	50	25000
1	First	1200	25	30000
2	Business	1100	45	49500
2	Economy	700	50	35000
2	First	1300	45	58500
3	Business	820	20	16400
3	Economy	480	36	17280
3	First	1500	45	67500

As anticipated, the yield is highest here because optimal prices and sales are determined with perfect knowledge of demand.

The stochastic programming formulation shown earlier represents a natural compromise between the deterministic formulations based on expected demand or actual demand.

Features Demonstrated

The following features are demonstrated in this example:

- problem type: mixed integer linear programming (stochastic programming with recourse)
- numeric and string index sets
- multiple input data sets
- CROSS set operator
- sets of tuples
- **`.ub`** variable suffix
- linearization of product of continuous variable and binary variable
- compact linearization
- IF-THEN expression
- using a variable suffix (such as **`.sol`**) in the declaration of a numeric parameter
- MAX aggregation operator
- ROUND function
- IF-THEN statement
- using a colon (:) to select members of a set
- FIX statement
- MIN function
- DROP statement
- multiple SOLVE statements

Chapter 25
Car Rental 1

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Problem Statement

A small ('cut price') car rental company, renting one type of car, has depots in Glasgow, Manchester, Birmingham and Plymouth.[1] There is an estimated demand for each day of the week except Sunday when the company is closed. These estimates are given in Table 25.1. It is not necessary to meet all demand.

Table 25.1

	Glasgow	Manchester	Birmingham	Plymouth
Monday	100	250	95	160
Tuesday	150	143	195	99
Wednesday	135	80	242	55
Thursday	83	225	111	96
Friday	120	210	70	115
Saturday	230	98	124	80

Cars can be rented for one, two or three days and returned to either the depot from which rented or another depot at the start of the next morning. For example, a 2-day rental on Thursday means that the car has to be returned on Saturday morning; a 3-day rental on Friday means that the car has to be returned on Tuesday morning. A 1-day rental on Saturday means that the car has to be returned on Monday morning and a 2-day rental on Tuesday morning.

[1] Reproduced with permission of John Wiley & Sons Ltd. (Williams 2013, pp. 284–286).

Table 25.2

From	To			
	Glasgow	**Manchester**	**Birmingham**	**Plymouth**
Glasgow	60	20	10	10
Manchester	15	55	25	5
Birmingham	15	20	54	11
Plymouth	8	12	27	53

Table 25.3

From	To			
	Glasgow	**Manchester**	**Birmingham**	**Plymouth**
Glasgow	—	20	30	50
Manchester	20	—	15	35
Birmingham	30	15	—	25
Plymouth	50	35	25	—

The rental period is independent of the origin and destination. From past data, the company knows the distribution of rental periods: 55% of cars are hired for one day, 20% for two days and 25% for three days. The current estimates of percentages of cars hired from each depot and returned to a given depot (independent of day) are given in Table 25.2.

The marginal cost, to the company, of renting out a car ('wear and tear', administration etc.) is estimated as follows:

1-Day hire	£20
2-Day hire	£25
3-Day hire	£30

The 'opportunity cost' (interest on capital, storage, servicing, etc.) of owning a car is £15 per week.

It is possible to transfer undamaged cars from one depot to another depot, irrespective of distance. Cars cannot be rented out during the day in which they are transferred. The costs (£), per car, of transfer are given in Table 25.3.

Ten percent of cars returned by customers are damaged. When this happens, the customer is charged an excess of £100 (irrespective of the amount of damage that the company completely covers by its insurance). In addition, the car has to be transferred to a repair depot, where it will be repaired the following day. The cost of transferring a damaged car is the same as transferring an undamaged one (except when the repair depot is the current depot, when it is zero). Again the transfer of a damaged car takes a day, unless it is already at a repair depot. Having arrived at a repair depot, all types of repair (or replacement) take a day.

Only two of the depots have repair capacity. These are (cars/day) as follows:

Manchester	12
Birmingham	20

Having been repaired, the car is available for rental at the depot the next day or may be transferred to another depot (taking a day). Thus, a car that is returned damaged on a Wednesday morning is transferred to a repair depot (if not the current depot) during Wednesday, repaired on Thursday and is available for hire at the repair depot on Friday morning.

The rental price depends on the number of days for which the car is hired and whether it is returned to the same depot or not. The prices are given in Table 25.4 (in £).

Table 25.4

	Return to Same Depot	**Return to Another Depot**
1-Day hire	50	70
2-Day hire	70	100
3-Day hire	120	150

There is a discount of £20 for hiring on a Saturday so long as the car is returned on Monday morning. This is regarded as a 1-day hire.

For simplicity, we assume the following at the beginning of each day:

(1) Customers return cars that are due that day

(2) Damaged cars are sent to the repair depot

(3) Cars that were transferred from other depots arrive

(4) Transfers are sent out

(5) Cars are rented out

(6) If it is a repair depot, then the repaired cars are available for rental.

In order to maximise weekly profit, the company wants a ‘steady state’ solution in which the same expected number will be located at the same depot on the same day of subsequent weeks.

How many cars should the company own and where should they be located at the start of each day?

This is a case where the integrality of the cars is not worth modelling. Rounded fractional solutions are acceptable.

Mathematical Programming Formulation

Index Sets and Their Members

The following index sets and their members are used in this example:

- depot, i, $j \in$ DEPOTS
- day $\in$ DAYS $= \{0, \ldots, \text{num_days} - 1\}$
- length $\in$ LENGTHS: lengths of rental period in days

Parameters

Table 25.5 shows the parameters that are used in this example.

Table 25.5 Parameters

Parameter Name	Interpretation
day_name[day]	Name of day
num_days	Number of days in periodic planning horizon
demand[depot,day]	Estimated number of cars demanded per depot per day
length_prob[length]	Probability of each rental length
cost[length]	Marginal cost (in pounds) to company for each rental length
price_same[length]	Rental price (in pounds) if the car is returned to the same depot
price_diff[length]	Rental price (in pounds) if the car is returned to a different depot
transition_prob[i,j]	Transition probability from depot *i* to depot *j*
transfer_cost[i,j]	Cost (in pounds) to transfer a car from depot *i* to depot *j*
repair_capacity[depot]	Upper bound on number of cars per day repaired at each depot
opportunity_cost_per_week	Opportunity cost (in pounds) per week of owning a car
transfer_length	Number of days required to transfer a car from depot to another
repair_length	Number of days required to repair a car
damage_prob	Probability that a car is returned damaged
damage_charge	Charge (in pounds) to customer for returning a car damaged
saturday_discount	Discount (in pounds) for hiring on Saturday and returning on Monday
rental_price[i,j,day,length]	Price (in pounds) to rent a car at depot *i* and return to depot *j* for each starting day and length
max_length	$\max_{\text{length} \in \text{LENGTHS}} \text{length}$

Variables

Table 25.6 shows the variables that are used in this example.

Table 25.6 Variables

Variable Name	Interpretation
NumCars	Number of cars owned by the company
NumUndamagedCarsStart[depot,day]	Number of undamaged cars at each depot at the beginning of each day
NumDamagedCarsStart[depot,day]	Number of damaged cars at each depot at the beginning of each day
NumCarsRented_i_day[i,day]	Number of cars rented at depot *i* on each day
NumCarsRented[i,j,day,length]	Number of cars rented at depot *i* and returned to depot *j* for each starting day and length
NumUndamagedCarsIdle[depot,day]	Number of undamaged cars idle at each depot at the beginning of each day
NumDamagedCarsIdle[depot,day]	Number of damaged cars idle at each depot at the beginning of each day
NumCarsTransferred[i,j,day]	Number of cars transferred from depot *i* to depot *j* each day
NumUndamagedCarsTransferred[i,j,day]	Number of undamaged cars transferred from depot *i* to depot *j* each day
NumDamagedCarsTransferred[i,j,day]	Number of damaged cars transferred from depot *i* to depot *j* each day
NumDamagedCarsRepaired[i,day]	Number of damaged cars repaired at depot *i* each day

Objective

The objective is to maximize the following function:

$$\begin{aligned}\text{Profit} = & \sum_{\substack{i \in \text{DEPOTS},\\ j \in \text{DEPOTS},\\ \text{day} \in \text{DAYS},\\ \text{length} \in \text{LENGTHS}}} (\textit{rental_price[i,j,day,length]} - \textit{cost[length]}) \cdot \text{NumCarsRented[i,j,day,length]} \\ & + \sum_{\substack{i \in \text{DEPOTS},\\ \text{day} \in \text{DAYS}}} \textit{damage_prob} \cdot \textit{damage_charge} \cdot \text{NumCarsRented_i_day[i,day]} \\ & - \sum_{\substack{i \in \text{DEPOTS},\\ j \in \text{DEPOTS} \setminus \{i\},\\ \text{day} \in \text{DAYS}}} \textit{transfer_cost[i,j]} \cdot \text{NumCarsTransferred[i,j,day]} \\ & - \textit{opportunity_cost_per_week} \cdot \text{NumCars}\end{aligned}$$

Constraints

The following constraints are used in this example:

- bounds on variables
- for $i \in$ DEPOTS and $j \in$ DEPOTS and day $\in$ DAYS and length $\in$ LENGTHS,

  NumCarsRented[i,j,day,length] $=$ *transition_prob[i,j]*·*length_prob[length]*·NumCarsRented_i_day[i,day]

- for $i \in$ DEPOTS and $j \in$ DEPOTS $\setminus \{i\}$ and day $\in$ DAYS,

  NumCarsTransferred[i,j,day] $=$ NumUndamagedCarsTransferred[i,j,day]+NumDamagedCarsTransferred[i,j,day]

- for $i \in$ DEPOTS and day $\in$ DAYS,

  $$\begin{aligned}&\text{NumUndamagedCarsStart[i,day]}\\ &= (1 - \textit{damage_prob}) \cdot \sum_{\substack{j \in \text{DEPOTS},\\ \text{length} \in \text{LENGTHS}}} \text{NumCarsRented[j,i,day-length,length]}\\ &+ \sum_{j \in \text{DEPOTS} \setminus \{i\}} \text{NumUndamagedCarsTransferred[j,i,day-transfer_length]}\\ &+ \text{NumDamagedCarsRepaired[i,day-repair_length]}\\ &+ \text{NumUndamagedCarsIdle[i,day-1]}\end{aligned}$$

- for $i \in$ DEPOTS and day $\in$ DAYS,

  $$\begin{aligned}&\text{NumDamagedCarsStart[i,day]}\\ &= \textit{damage_prob} \cdot \sum_{\substack{j \in \text{DEPOTS},\\ \text{length} \in \text{LENGTHS}}} \text{NumCarsRented[j,i,day-length,length]}\\ &+ \sum_{j \in \text{DEPOTS} \setminus \{i\}} \text{NumDamagedCarsTransferred[j,i,day-transfer_length]}\\ &+ \text{NumDamagedCarsIdle[i,day-1]}\end{aligned}$$

- for $i \in$ DEPOTS and day $\in$ DAYS,

  $$\begin{aligned}&\text{NumUndamagedCarsStart[i,day]}\\ &= \text{NumCarsRented_i_day[i,day]}\\ &+ \sum_{j \in \text{DEPOTS} \setminus \{i\}} \text{NumUndamagedCarsTransferred[i,j,day]}\\ &+ \text{NumUndamagedCarsIdle[i,day]}\end{aligned}$$

- for $i \in$ DEPOTS and day $\in$ DAYS,

$$\begin{aligned}&\text{NumDamagedCarsStart[i,day]}\\&= \text{NumDamagedCarsRepaired[i,day]}\\&+ \sum_{j \in \text{DEPOTS} \setminus \{i\}} \text{NumDamagedCarsTransferred[i,j,day]}\\&+ \text{NumDamagedCarsIdle[i,day]}\end{aligned}$$

- $$\text{NumCars} = \sum_{i \in \text{DEPOTS}} \left(\textit{length_prob[3]} \cdot \text{NumCarsRented_i_day[i,0]} + \sum_{\text{length}=2}^{3} \textit{length_prob[length]} \cdot \text{NumCarsRented_i_day[i,1]} + \text{NumUndamagedCarsStart[i,2]} + \text{NumDamagedCarsStart[i,2]} \right)$$

Input Data

The following data sets and macro variables contain the input data that are used in this example:

```
data depot_data;
   input depot $10.;
   datalines;
Glasgow
Manchester
Birmingham
Plymouth
;

data demand_data;
   input day $10. Glasgow Manchester Birmingham Plymouth;
   datalines;
Monday    100 250  95 160
Tuesday   150 143 195  99
Wednesday 135  80 242  55
Thursday   83 225 111  96
Friday    120 210  70 115
Saturday  230  98 124  80
;

data length_data;
   input length prob cost price_same price_diff;
   datalines;
1 0.55 20  50  70
2 0.20 25  70 100
3 0.25 30 120 150
;
```

```
data transition_prob_data;
   input i $10. Glasgow Manchester Birmingham Plymouth;
   datalines;
Glasgow    60 20 10 10
Manchester 15 55 25  5
Birmingham 15 20 54 11
Plymouth    8 12 27 53
;

data transfer_cost_data;
   input i $10. Glasgow Manchester Birmingham Plymouth;
   datalines;
Glasgow     . 20 30 50
Manchester 20  . 15 35
Birmingham 30 15  . 25
Plymouth   50 35 25  .
;

data repair_data;
   input depot $10. repair_capacity;
   datalines;
Manchester 12
Birmingham 20
;

%let opportunity_cost_per_week = 15;
%let transfer_length = 1;
%let repair_length = 1;
%let damage_prob = 0.10;
%let damage_charge = 100;
%let saturday_discount = 20;
```

PROC OPTMODEL Statements and Output

The first several PROC OPTMODEL statements declare index sets and parameters, read the input data, and calculate additional parameters:

```
proc optmodel;
   set <str> DEPOTS;
   read data depot_data into DEPOTS=[depot];

   set DAYS;
   str day_name {DAYS};
   num demand {DEPOTS, DAYS};
   read data demand_data into DAYS=[_N_];
   num num_days = card(DAYS);
   DAYS = 0..num_days-1;
   read data demand_data into [_N_]
      {depot in DEPOTS} <demand[depot,_N_-1]=col(depot)>;

   set LENGTHS;
```

```
num length_prob {LENGTHS};
num cost {LENGTHS};
num price_same {LENGTHS};
num price_diff {LENGTHS};
read data length_data into LENGTHS=[length]
   length_prob=prob cost price_same price_diff;

num transition_prob {DEPOTS, DEPOTS};
read data transition_prob_data into [i]
   {j in DEPOTS} <transition_prob[i,j]=col(j)>;
for {i in DEPOTS, j in DEPOTS}
   transition_prob[i,j] = transition_prob[i,j] / 100;

num transfer_cost {DEPOTS, DEPOTS} init 0;
read data transfer_cost_data nomiss into [i]
   {j in DEPOTS} <transfer_cost[i,j]=col(j)>;

num repair_capacity {DEPOTS} init 0;
read data repair_data into [depot] repair_capacity;

num rental_price {i in DEPOTS, j in DEPOTS, day in DAYS, length in LENGTHS} =
   (if i = j then price_same[length] else price_diff[length])
 - (if day = 5 and length = 1 then &saturday_discount);
```

Because the company wants a periodic solution, all indices that correspond to days are taken modulo *num_days*. For simplicity, this notation is suppressed in the mathematical programming formulation that is described earlier. The following statements declare parameters to be used for that purpose in the subsequent constraint declarations:

```
num max_length = max {length in LENGTHS} length;
num mod {s in -max_length..num_days+max_length} = mod(s+num_days,num_days);
```

The following model declaration statements correspond directly to the mathematical programming formulation that is described earlier:

```
var NumCars >= 0;

var NumUndamagedCarsStart {DEPOTS, DAYS} >= 0;
var NumDamagedCarsStart {DEPOTS, DAYS} >= 0;

var NumCarsRented_i_day {i in DEPOTS, day in DAYS} >= 0 <= demand[i,day];
impvar NumCarsRented
   {i in DEPOTS, j in DEPOTS, day in DAYS, length in LENGTHS} =
   transition_prob[i,j] * length_prob[length] * NumCarsRented_i_day[i,day];

var NumUndamagedCarsIdle {DEPOTS, DAYS} >= 0;
var NumDamagedCarsIdle {DEPOTS, DAYS} >= 0;

var NumUndamagedCarsTransferred {i in DEPOTS, DEPOTS diff {i}, DAYS} >= 0;
var NumDamagedCarsTransferred {i in DEPOTS, DEPOTS diff {i}, DAYS} >= 0;
impvar NumCarsTransferred {i in DEPOTS, j in DEPOTS diff {i}, day in DAYS} =
   NumUndamagedCarsTransferred[i,j,day]
 + NumDamagedCarsTransferred[i,j,day];
```

```
var NumDamagedCarsRepaired {i in DEPOTS, DAYS} >= 0 <= repair_capacity[i];

max Profit =
   sum {i in DEPOTS, j in DEPOTS, day in DAYS, length in LENGTHS}
      (rental_price[i,j,day,length] - cost[length])
    * NumCarsRented[i,j,day,length]
 + sum {i in DEPOTS, day in DAYS}
      &damage_prob * &damage_charge * NumCarsRented_i_day[i,day]
 - sum {i in DEPOTS, j in DEPOTS diff {i}, day in DAYS}
      transfer_cost[i,j] * NumCarsTransferred[i,j,day]
 - &opportunity_cost_per_week * NumCars;

con Undamaged_Inflow_con {i in DEPOTS, day in DAYS}:
   NumUndamagedCarsStart[i,day]
 = (1 - &damage_prob) * sum {j in DEPOTS, length in LENGTHS}
      NumCarsRented[j,i,mod[day-length],length]
 + sum {j in DEPOTS diff {i}}
      NumUndamagedCarsTransferred[j,i,mod[day-&transfer_length]]
 + NumDamagedCarsRepaired[i,mod[day-&repair_length]]
 + NumUndamagedCarsIdle[i,mod[day-1]];

con Damaged_Inflow_con {i in DEPOTS, day in DAYS}:
   NumDamagedCarsStart[i,day]
 = &damage_prob * sum {j in DEPOTS, length in LENGTHS}
      NumCarsRented[j,i,mod[day-length],length]
 + sum {j in DEPOTS diff {i}}
      NumDamagedCarsTransferred[j,i,mod[day-&transfer_length]]
 + NumDamagedCarsIdle[i,mod[day-1]];

con Undamaged_Outflow_con {i in DEPOTS, day in DAYS}:
   NumUndamagedCarsStart[i,day]
 = NumCarsRented_i_day[i,day]
 + sum {j in DEPOTS diff {i}} NumUndamagedCarsTransferred[i,j,day]
 + NumUndamagedCarsIdle[i,day];

con Damaged_Outflow_con {i in DEPOTS, day in DAYS}:
   NumDamagedCarsStart[i,day]
 = NumDamagedCarsRepaired[i,day]
 + sum {j in DEPOTS diff {i}} NumDamagedCarsTransferred[i,j,day]
 + NumDamagedCarsIdle[i,day];

con NumCars_con:
   NumCars = sum {i in DEPOTS} (
      length_prob[3] * NumCarsRented_i_day[i,0]
    + sum {length in 2..3} length_prob[length] * NumCarsRented_i_day[i,1]
    + NumUndamagedCarsStart[i,2]
    + NumDamagedCarsStart[i,2]);
```

The NumCars_con constraint expresses the fact that every car is rented on Monday for three days, rented on Tuesday for two or three days, or at some depot at the beginning of Wednesday.

The following statements call the LP solver and use the generic problem symbols _NVAR_ and _VAR_ to round all variables to integer values, as in Williams (2013):

```
solve;
for {j in 1.._NVAR_} _VAR_[j] = round(_VAR_[j].sol);
```

The following statements print the solution and use the `.dual` variable suffix to print the shadow prices for repair capacity:

```
print NumCars;
print NumUndamagedCarsStart;
print NumDamagedCarsStart;
print NumCarsRented_i_day;
print {i in DEPOTS, j in DEPOTS diff {i}, day in DAYS:
   NumDamagedCarsTransferred[i,j,day].sol > 0} NumDamagedCarsTransferred;
print NumDamagedCarsRepaired.dual;
quit;
```

Figure 25.1 shows the output from the linear programming solver.

Figure 25.1 Output from Linear Programming Solver

The OPTMODEL Procedure

Problem Summary	
Objective Sense	Maximization
Objective Function	Profit
Objective Type	Linear
Number of Variables	289
Bounded Above	0
Bounded Below	241
Bounded Below and Above	36
Free	0
Fixed	12
Number of Constraints	97
Linear LE (<=)	0
Linear EQ (=)	97
Linear GE (>=)	0
Linear Range	0
Constraint Coefficients	1145

Performance Information	
Execution Mode	Single-Machine
Number of Threads	1

Figure 25.1 *continued*

Solution Summary	
Solver	LP
Algorithm	Dual Simplex
Objective Function	Profit
Solution Status	Optimal
Objective Value	119302.04331
Primal Infeasibility	2.415845E-13
Dual Infeasibility	4.547474E-13
Bound Infeasibility	0
Iterations	81
Presolve Time	0.00
Solution Time	0.00

NumCars
681

NumUndamagedCarsStart						
	0	**1**	**2**	**3**	**4**	**5**
Birmingham	183	195	126	127	145	182
Glasgow	111	82	70	79	79	71
Manchester	163	104	99	126	110	100
Plymouth	67	44	42	48	47	43

NumDamagedCarsStart						
	0	**1**	**2**	**3**	**4**	**5**
Birmingham	25	20	20	20	20	25
Glasgow	4	8	10	9	9	14
Manchester	12	12	12	12	12	12
Plymouth	3	5	5	10	16	5

NumCarsRented_i_day						
	0	**1**	**2**	**3**	**4**	**5**
Birmingham	95	195	126	111	70	81
Glasgow	100	82	70	79	79	0
Manchester	163	104	80	126	110	0
Plymouth	67	44	42	48	47	0

Figure 25.1 *continued*

[1]	[2]	[3]	NumDamagedCarsTransferred
Glasgow	**Birmingham**	**0**	3
Glasgow	**Birmingham**	**1**	4
Glasgow	**Birmingham**	**2**	8
Glasgow	**Birmingham**	**3**	8
Glasgow	**Birmingham**	**5**	8
Glasgow	**Manchester**	**0**	2
Glasgow	**Manchester**	**1**	2
Glasgow	**Manchester**	**2**	1
Glasgow	**Manchester**	**3**	1
Glasgow	**Manchester**	**4**	2
Glasgow	**Manchester**	**5**	6
Plymouth	**Birmingham**	**0**	3
Plymouth	**Birmingham**	**1**	5
Plymouth	**Birmingham**	**4**	16
Plymouth	**Birmingham**	**5**	5

	NumDamagedCarsRepaired.DUAL					
	0	**1**	**2**	**3**	**4**	**5**
Birmingham	591.46	590.99	591.46	591.46	591.46	591.46
Glasgow	625.49	637.39	635.16	632.06	625.49	625.49
Manchester	607.36	602.62	617.62	615.27	610.84	610.84
Plymouth	622.55	634.02	632.50	630.10	625.55	625.55

The optimal solution, objective value, and shadow prices differ from Williams (2013), because of errors that will be corrected in a subsequent printing by Wiley.

Features Demonstrated

The following features are demonstrated in this example:

- problem type: linear programming (generalized network flow, periodic inventory optimization)
- numeric and string index sets
- INIT option
- NOMISS option
- IMPVAR statement
- reading multiple data sets
- reading dense two-dimensional data
- set operator DIFF

- IF-THEN/ELSE expression
- MOD function
- CARD function
- generic problem symbols _NVAR_ and _VAR_
- printing sparse multi-dimensional data
- **`.dual`** variable suffix

Chapter 26
Car Rental 2

Contents

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Problem Statement

In the light of the solution to the problem stated in Chapter 25, the company wants to consider where it might be most worthwhile to expand repair capacity.[1] The weekly fixed costs, given below, include interest payments on the necessary loans for expansion.

The options are as follows:

(1) Expand repair capacity at Birmingham by 5 cars per day at a fixed cost per week of £18,000.

(2) Further expand repair capacity at Birmingham by 5 cars per day at a fixed cost per week of £8000.

(3) Expand repair capacity at Manchester by 5 cars per day at a fixed cost per week of £20,000.

(4) Further expand repair capacity at Manchester by 5 cars per day at a fixed cost per week of £5000.

(5) Create repair capacity at Plymouth of 5 cars per day at a fixed cost per week of £19,000.

If any of these options is chosen, it must be carried out in its entirety, that is, there can be no partial expansion. Also, a further expansion at a depot can be carried out only if the first expansion is also carried out, so for example option (2) at Birmingham cannot be chosen unless option (1) is also chosen. If option (2) is chosen, thereby also choosing option (1), these count as two options. Similar stipulations apply regarding the expansions at Manchester. At most three of the options can be carried out.

Mathematical Programming Formulation

This formulation builds on the formulation used in Chapter 25. This section includes only the new elements of the formulation.

[1] Reproduced with permission of John Wiley & Sons Ltd. (Williams 2013, p. 287).

Index Sets and Their Members

The following additional index set and its members are used in this example:

- expansion ∈ EXPANSIONS

Parameters

Table 26.1 shows the additional parameters that are used in this example.

Table 26.1 Parameters

Parameter Name	Interpretation
expansion_depot[expansion]	Depot for expansion
expansion_amount[expansion]	Additional number of cars per day provided by expansion
expansion_cost[expansion]	Cost (in pounds) per week for expansion
expansion_prerequisite[expansion]	Other expansion required by this expansion (missing value if none)
max_num_expansions	Number of expansions allowed

Variables

Table 26.2 shows the additional variables that are used in this example.

Table 26.2 Variables

Variable Name	Interpretation
ExpandCapacity[expansion]	1 if capacity expansion is carried out; 0 otherwise

Objective

The objective is to maximize the following function:

$$\text{Profit2} = \text{Profit} - \sum_{\text{expansion} \in \text{EXPANSIONS}} \textit{expansion_cost[expansion]} \cdot \text{ExpandCapacity[expansion]}$$

where Profit is defined in Chapter 25.

Constraints

The following additional constraints are used in this example:

- for $i \in$ DEPOTS and day $\in$ DAYS,

$$\text{TotalManpower} = \sum_{i \in \text{INDUSTRIES}} \sum_{\text{year} \in \text{YEARS}} \Big(\textit{production_coeff['manpower',i]} \cdot \text{Production[i,year} + 1] + \textit{productive_capacity_coeff['manpower',i]} \cdot \text{ExtraCapacity[i,year} + 2] \Big)$$

- for expansion $\in$ EXPANSIONS such that *expansion_prerequisite[expansion]* is not missing,

$$\text{ExpandCapacity[expansion]} \leq \text{ExpandCapacity[expansion_prerequisite[expansion]]}$$

- $$\sum_{\text{expansion} \in \text{EXPANSIONS}} \text{ExpandCapacity[expansion]} \leq \textit{max_num_expansions}$$

Input Data

The following data set and macro variable contain the additional input data that are used in this example:

```
/* missing expansion_prerequisite indicates no prerequisite */
data expansion_data;
   input expansion expansion_depot $10. expansion_amount expansion_cost
      expansion_prerequisite;
   datalines;
1 Birmingham 5 18000 .
2 Birmingham 5  8000 1
3 Manchester 5 20000 .
4 Manchester 5  5000 3
5 Plymouth   5 19000 .
;

%let max_num_expansions = 3;
```

PROC OPTMODEL Statements and Output

For completeness, all statements are shown. Statements that are new or changed from Chapter 25 are indicated.

```
proc optmodel;
   set <str> DEPOTS;
   read data depot_data into DEPOTS=[depot];

   set DAYS;
   str day_name {DAYS};
   num demand {DEPOTS, DAYS};
   read data demand_data into DAYS=[_N_];
   num num_days = card(DAYS);
   DAYS = 0..num_days-1;
   read data demand_data into [_N_]
      {depot in DEPOTS} <demand[depot,_N_-1]=col(depot)>;

   set LENGTHS;
   num length_prob {LENGTHS};
   num cost {LENGTHS};
   num price_same {LENGTHS};
   num price_diff {LENGTHS};
   read data length_data into LENGTHS=[length]
      length_prob=prob cost price_same price_diff;

   num transition_prob {DEPOTS, DEPOTS};
   read data transition_prob_data into [i]
      {j in DEPOTS} <transition_prob[i,j]=col(j)>;
   for {i in DEPOTS, j in DEPOTS}
      transition_prob[i,j] = transition_prob[i,j] / 100;

   num transfer_cost {DEPOTS, DEPOTS} init 0;
   read data transfer_cost_data nomiss into [i]
      {j in DEPOTS} <transfer_cost[i,j]=col(j)>;

   num repair_capacity {DEPOTS} init 0;
   read data repair_data into [depot] repair_capacity;

   num rental_price {i in DEPOTS, j in DEPOTS, day in DAYS, length in LENGTHS} =
      (if i = j then price_same[length] else price_diff[length])
    - (if day = 5 and length = 1 then &saturday_discount);

   num max_length = max {length in LENGTHS} length;
   num mod {s in -max_length..num_days+max_length} = mod(s+num_days,num_days);

   var NumCars >= 0;

   var NumUndamagedCarsStart {DEPOTS, DAYS} >= 0;
   var NumDamagedCarsStart {DEPOTS, DAYS} >= 0;

   var NumCarsRented_i_day {i in DEPOTS, day in DAYS} >= 0 <= demand[i,day];
   impvar NumCarsRented
```

```
   {i in DEPOTS, j in DEPOTS, day in DAYS, length in LENGTHS} =
   transition_prob[i,j] * length_prob[length] * NumCarsRented_i_day[i,day];

var NumUndamagedCarsIdle {DEPOTS, DAYS} >= 0;
var NumDamagedCarsIdle {DEPOTS, DAYS} >= 0;

var NumUndamagedCarsTransferred {i in DEPOTS, DEPOTS diff {i}, DAYS} >= 0;
var NumDamagedCarsTransferred {i in DEPOTS, DEPOTS diff {i}, DAYS} >= 0;
impvar NumCarsTransferred {i in DEPOTS, j in DEPOTS diff {i}, day in DAYS} =
   NumUndamagedCarsTransferred[i,j,day]
 + NumDamagedCarsTransferred[i,j,day];

var NumDamagedCarsRepaired {i in DEPOTS, DAYS} >= 0 <= repair_capacity[i];

max Profit =
   sum {i in DEPOTS, j in DEPOTS, day in DAYS, length in LENGTHS}
      (rental_price[i,j,day,length] - cost[length])
    * NumCarsRented[i,j,day,length]
 + sum {i in DEPOTS, day in DAYS}
      &damage_prob * &damage_charge * NumCarsRented_i_day[i,day]
 - sum {i in DEPOTS, j in DEPOTS diff {i}, day in DAYS}
      transfer_cost[i,j] * NumCarsTransferred[i,j,day]
 - &opportunity_cost_per_week * NumCars;

con Undamaged_Inflow_con {i in DEPOTS, day in DAYS}:
   NumUndamagedCarsStart[i,day]
 = (1 - &damage_prob) * sum {j in DEPOTS, length in LENGTHS}
      NumCarsRented[j,i,mod[day-length],length]
 + sum {j in DEPOTS diff {i}}
      NumUndamagedCarsTransferred[j,i,mod[day-&transfer_length]]
 + NumDamagedCarsRepaired[i,mod[day-&repair_length]]
 + NumUndamagedCarsIdle[i,mod[day-1]];

con Damaged_Inflow_con {i in DEPOTS, day in DAYS}:
   NumDamagedCarsStart[i,day]
 = &damage_prob * sum {j in DEPOTS, length in LENGTHS}
      NumCarsRented[j,i,mod[day-length],length]
 + sum {j in DEPOTS diff {i}}
      NumDamagedCarsTransferred[j,i,mod[day-&transfer_length]]
 + NumDamagedCarsIdle[i,mod[day-1]];

con Undamaged_Outflow_con {i in DEPOTS, day in DAYS}:
   NumUndamagedCarsStart[i,day]
 = NumCarsRented_i_day[i,day]
 + sum {j in DEPOTS diff {i}} NumUndamagedCarsTransferred[i,j,day]
 + NumUndamagedCarsIdle[i,day];

con Damaged_Outflow_con {i in DEPOTS, day in DAYS}:
   NumDamagedCarsStart[i,day]
 = NumDamagedCarsRepaired[i,day]
 + sum {j in DEPOTS diff {i}} NumDamagedCarsTransferred[i,j,day]
 + NumDamagedCarsIdle[i,day];

con NumCars_con:
```

```
   NumCars = sum {i in DEPOTS} (
      length_prob[3] * NumCarsRented_i_day[i,0]
    + sum {length in 2..3} length_prob[length] * NumCarsRented_i_day[i,1]
    + NumUndamagedCarsStart[i,2]
    + NumDamagedCarsStart[i,2]);
```

The remaining statements are new in this example. The following statements declare an additional index set and parameters and then read the additional input data:

```
set EXPANSIONS;
str expansion_depot {EXPANSIONS};
num expansion_amount {EXPANSIONS};
num expansion_cost {EXPANSIONS};
num expansion_prerequisite {EXPANSIONS};
read data expansion_data into EXPANSIONS=[expansion]
   expansion_depot expansion_amount expansion_cost expansion_prerequisite;
```

The BINARY option in the following VAR statement declares ExpandCapacity to be a binary variable:

```
var ExpandCapacity {EXPANSIONS} binary;
```

The following statement uses the CONSTANT function to effectively remove the previously declared upper bound on the NumDamagedCarsRepaired variables by replacing it with the largest machine-representable number:

```
for {expansion in EXPANSIONS, day in DAYS}
   NumDamagedCarsRepaired[expansion_depot[expansion],day].ub =
   constant('BIG');
```

This large number does not cause any numerical difficulties for the solver, because PROC OPTMODEL recognizes this special constant and treats the variable as having no upper bound. The following Expansion_con constraint accounts for the new limit on the number of cars repaired, according to the expansion options that are chosen:

```
con Expansion_con {i in DEPOTS, day in DAYS}:
   NumDamagedCarsRepaired[i,day]
<= repair_capacity[i]
 + sum {expansion in EXPANSIONS: expansion_depot[expansion] = i}
      expansion_amount[expansion] * ExpandCapacity[expansion];
```

The following ExpansionPrerequisite_con constraint enforces the rule that ExpandCapacity[expansion] $= 1$ implies that ExpandCapacity[expansion_prerequisite[expansion]] $= 1$:

```
con ExpansionPrerequisite_con {expansion in EXPANSIONS:
      expansion_prerequisite[expansion] ne .}:
   ExpandCapacity[expansion]
<= ExpandCapacity[expansion_prerequisite[expansion]];
```

The following Cardinality constraint enforces the limit on the number of expansions that are chosen:

```
con Cardinality:
   sum {expansion in EXPANSIONS} ExpandCapacity[expansion]
<= &max_num_expansions;
```

The following objective declaration uses the previously declared objective function:

```
max Profit2 = Profit -
   sum {expansion in EXPANSIONS}
      expansion_cost[expansion] * ExpandCapacity[expansion];
```

The following statements call the mixed integer linear programming solver, round all variables, and print the specified parts of the solution:

```
solve;
for {j in 1.._NVAR_} _VAR_[j] = round(_VAR_[j].sol);
print expansion_depot ExpandCapacity;
print NumDamagedCarsRepaired;
print NumCars;
quit;
```

Figure 26.1 shows the output from the mixed integer linear programming solver.

Figure 26.1 Output from Mixed Integer Linear Programming Solver

The OPTMODEL Procedure

Problem Summary	
Objective Sense	Maximization
Objective Function	Profit2
Objective Type	Linear
Number of Variables	294
Bounded Above	0
Bounded Below	259
Bounded Below and Above	29
Free	0
Fixed	6
Binary	5
Integer	0
Number of Constraints	124
Linear LE (<=)	27
Linear EQ (=)	97
Linear GE (>=)	0
Linear Range	0
Constraint Coefficients	1208

Performance Information	
Execution Mode	Single-Machine
Number of Threads	4

Figure 26.1 *continued*

Solution Summary	
Solver	MILP
Algorithm	Branch and Cut
Objective Function	Profit2
Solution Status	Optimal
Objective Value	130792.95192
Relative Gap	0
Absolute Gap	0
Primal Infeasibility	1.705303E-13
Bound Infeasibility	0
Integer Infeasibility	0
Best Bound	130792.95192
Nodes	1
Iterations	132
Presolve Time	0.03
Solution Time	0.04

[1]	expansion_depot	ExpandCapacity
1	Birmingham	0
2	Birmingham	0
3	Manchester	1
4	Manchester	1
5	Plymouth	0

NumDamagedCarsRepaired						
	0	**1**	**2**	**3**	**4**	**5**
Birmingham	20	20	20	20	20	20
Glasgow	0	0	0	0	0	0
Manchester	22	22	22	22	22	22
Plymouth	0	0	0	0	0	0

NumCars
955

Note that the resulting profit is higher than in Chapter 25. This result is expected because the repair capacity expansion options allow more flexibility. The optimal solution and objective value differ from Williams (2013), because of errors that will be corrected in a subsequent printing by Wiley.

Features Demonstrated

The following features are demonstrated in this example:

- problem type: mixed integer linear programming
- numeric and string index sets
- INIT option
- NOMISS option
- IMPVAR statement
- reading multiple data sets
- reading dense two-dimensional data
- set operator DIFF
- IF-THEN/ELSE expression
- MOD function
- CARD function
- **`.ub`** variable suffix
- CONSTANT function
- fixed cost
- modeling if-then constraints by using binary variables
- cardinality constraint
- declaring an objective in terms of another objective
- generic problem symbols _NVAR_ and _VAR_

Chapter 27
Lost Baggage Distribution

Contents

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Problem Statement

A small company with six vans has a contract with a number of airlines to pick up lost or delayed baggage, belonging to customers in the London area, from Heathrow airport at 6 p.m. each evening.[1] The contract stipulates that each customer must have [his or her] baggage delivered by 8 p.m. The company requires a model, which they can solve quickly each evening, to advise them what is the minimum number of vans they need to use and to which customers each van should deliver and in what order. There is no practical capacity limitation on each van. All baggage that needs to be delivered in a two-hour period can be accommodated in a van. Having ascertained the minimum number of vans needed, a solution is then sought, which minimises the maximum time taken by any van.

On a particular evening, the places where deliveries need to be made and the times to travel between them (in minutes) are given in Table 27.1. No allowance is made for drop off times. For convenience, Heathrow will be regarded as the first location.

Formulate optimisation models that will minimise the number of vans that need to be used, and within this minimum, minimise the time taken for the longest time delivery.

[1] Reproduced with permission of John Wiley & Sons Ltd. (Williams 2013, pp. 287–289).

Table 27.1

Heathrow	20	25	35	65	90	85	80	86	25	35	20	44	35	82
	Harrow	15	35	60	55	57	85	90	25	35	30	37	20	40
		Ealing	30	50	70	55	50	65	10	25	15	24	20	90
			Holborn	45	60	53	55	47	12	22	20	12	10	21
				Sutton	46	15	45	75	25	11	19	15	25	25
					Dartford	15	15	25	45	65	53	43	63	70
						Bromley	17	25	41	25	33	27	45	30
							Greenwich	25	40	34	32	20	30	10
								Barking	65	70	72	61	45	13
									Hammersmith	20	8	7	15	25
										Kingston	5	12	45	65
											Richmond	14	34	56
												Battersea	30	40
													Islington	27
														Woolwich

Mathematical Programming Formulation

Index Sets and Their Members

The following index sets and their members are used in this example:

- $i, j \in$ NODES
- $(i, j) \in$ ARCS
- $v \in$ VEHICLES
- $i, j \in$ NODES_SOL
- $(i, j) \in$ ARCS_SOL
- $c_i \in$ COMPONENT_IDS
- $k \in$ COMPONENT[ci]
- $k \in$ SUBTOUR[s]

Parameters

Table 27.2 shows the parameters that are used in this example.

Table 27.2 Parameters

Parameter Name	Interpretation
travel_time[i,j]	Travel time (in minutes) between nodes *i* and *j*
num_vehicles	Number of vehicles
time_limit	Number of minutes in the planning horizon
depot	Node name of depot
num_subtours	Number of subtours in the current formulation
num_components[v]	Number of connected components in the solution for vehicle *v*
component_id[i]	Connected component that contains node *i*
ci	Dummy index for members of COMPONENT_IDS

Variables

Table 27.3 shows the variables that are used in this example.

Table 27.3 Variables

Variable Name	Interpretation
UseNode[i,v]	1 if node i is visited by vehicle v; 0 otherwise
UseArc[i,j,v]	1 if arc (i, j) is traversed by vehicle v; 0 otherwise
UseVehicle[v]	1 if vehicle v is used; 0 otherwise
TimeUsed[v]	Number of minutes used by vehicle v
MaxTimeUsed	$\max_{v \in \text{VEHICLES}}$ TimeUsed[v]

Objectives

The first objective is to minimize the following function:

$$\text{NumVehiclesUsed} = \sum_{v \in \text{VEHICLES}} \text{UseVehicle[v]}$$

The second objective is to minimize the following function:

$$\text{Makespan} = \max_{v \in \text{VEHICLES}} \text{TimeUsed[v]}$$

Constraints

The following constraints are used in this example:

- bounds on variables
- for $i \in$ NODES $\setminus$ {depot},

$$\sum_{v \in \text{VEHICLES}} \text{UseNode[i,v]} = 1$$

- for $i \in$ NODES and $v \in$ VEHICLES,

$$\sum_{(i,j) \in \text{ARCS}} \text{UseArc[i,j,v]} = \text{UseNode[i,v]}$$

- for $j \in$ NODES and $v \in$ VEHICLES,

$$\sum_{(i,j) \in \text{ARCS}} \text{UseArc[i,j,v]} = \text{UseNode[j,v]}$$

- for $i \in$ NODES and $v \in$ VEHICLES,

  $$\text{UseNode[i,v]} \leq \text{UseVehicle[v]}$$

- for $v \in$ VEHICLES,

  $$\text{UseVehicle[v]} \leq \text{UseNode[depot,v]}$$

- for $v \in$ VEHICLES,

  $$\text{TimeUsed[v]} = \sum_{(i,j) \in \text{ARCS}} \textit{travel_time[i,j]} \cdot \text{UseArc[i,j,v]}$$

- for $v \in$ VEHICLES $\setminus \{1\}$,

  $$\sum_{i \in \text{NODES}} \text{UseNode[i,v]} \leq \sum_{i \in \text{NODES}} \text{UseNode[i,v} - 1\text{]}$$

- for $s \in \{1, \ldots, \textit{num_subtours}\}$ and $k \in$ SUBTOUR[s] and $v \in$ VEHICLES,

  $$\sum_{\substack{i \in \text{NODES} \setminus \text{SUBTOUR[s]}, \\ j \in \text{SUBTOUR[s]}: \\ (i,j) \in \text{ARCS}}} \text{UseArc[i,j,v]} + \sum_{\substack{i \in \text{SUBTOUR[s]}, \\ j \in \text{NODES} \setminus \text{SUBTOUR[s]}: \\ (i,j) \in \text{ARCS}}} \text{UseArc[i,j,v]} \geq 2 \cdot \text{UseNode[k,v]}$$

- for $v \in$ VEHICLES,

  $$\text{MaxTimeUsed} \geq \text{TimeUsed[v]}$$

Input Data

The following data set and macro variables contain the input data that are used in this example:

```
data time_data;
   input location $11.
      Heathrow Harrow Ealing Holborn Sutton Dartford Bromley Greenwich
      Barking Hammersmith Kingston Richmond Battersea Islington Woolwich;
   datalines;
Heathrow    0 20 25 35 65 90 85 80 86 25 35 20 44 35 82
Harrow      .  0 15 35 60 55 57 85 90 25 35 30 37 20 40
Ealing      .  .  0 30 50 70 55 50 65 10 25 15 24 20 90
Holborn     .  .  .  0 45 60 53 55 47 12 22 20 12 10 21
Sutton      .  .  .  .  0 46 15 45 75 25 11 19 15 25 25
Dartford    .  .  .  .  .  0 15 15 25 45 65 53 43 63 70
Bromley     .  .  .  .  .  .  0 17 25 41 25 33 27 45 30
Greenwich   .  .  .  .  .  .  .  0 25 40 34 32 20 30 10
Barking     .  .  .  .  .  .  .  .  0 65 70 72 61 45 13
Hammersmith .  .  .  .  .  .  .  .  .  0 20  8  7 15 25
Kingston    .  .  .  .  .  .  .  .  .  .  0  5 12 45 65
```

```
Richmond    .  .  .  .  .  .  .  .  .  .  .  .  .  0 14 34 56
Battersea   .  .  .  .  .  .  .  .  .  .  .  .  .  .  0 30 40
Islington   .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  0 27
Woolwich    .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  0
;

%let num_vehicles = 6;
%let time_limit = 120;
%let depot = Heathrow;
```

PROC OPTMODEL Statements and Output

Although you could find the connected components of the support graph of each intermediate solution by using OPTMODEL's programming language capabilities as in Chapter 23, the following SAS macro instead uses the SOLVE WITH NETWORK statement together with the CONCOMP option:

```
%macro findConnectedComponents;
   if card(ARCS_SOL) > 0 then do;
      solve with NETWORK /
         direction = directed
         links     = (include=ARCS_SOL)
         subgraph  = (nodes=NODES_SOL)
         concomp
         out       = (concomp=component_id);
      COMPONENT_IDS = setof {i in NODES_SOL} component_id[i];
      for {c in COMPONENT_IDS} COMPONENT[c] = {};
      for {i in NODES_SOL} do;
         ci = component_id[i];
         COMPONENT[ci] = COMPONENT[ci] union {i};
      end;
   end;
   else COMPONENT_IDS = {};
%mend findConnectedComponents;
```

The following SAS macro contains a DO UNTIL loop that implements dynamic generation of subtour elimination constraints ("row generation"), as in Chapter 23:

```
%macro subtourEliminationLoop;
   /* loop until each vehicle's support graph is connected */
   do until (and {v in VEHICLES} num_components[v] <= 1);
      solve;
      /* find connected components for each vehicle */
      for {v in VEHICLES} do;
         NODES_SOL = {i in NODES: UseNode[i,v].sol > 0.5};
         ARCS_SOL = {<i,j> in ARCS: UseArc[i,j,v].sol > 0.5};
         %findConnectedComponents;
         num_components[v] = card(COMPONENT_IDS);
```

```
         /* create subtour from each component not containing depot node */
         for {k in COMPONENT_IDS: depot not in COMPONENT[k]} do;
            num_subtours = num_subtours + 1;
            SUBTOUR[num_subtours] = COMPONENT[k];
            put SUBTOUR[num_subtours]=;
         end;
      end;
      print UseVehicle TimeUsed num_components;
   end;
%mend subtourEliminationLoop;
```

The previous two macros are used within the main PROC OPTMODEL call. The first several PROC OPTMODEL statements declare index sets and parameters and then read the input data:

```
proc optmodel;
   num num_vehicles init &num_vehicles;
   set VEHICLES = 1..num_vehicles;
   str depot = "&depot";
   set <str> NODES;
   read data time_data into NODES=[location];
   set ARCS init NODES cross NODES;
   num travel_time {ARCS};
   read data time_data into [i=location]
      {j in NODES} <travel_time[i,j]=col(j)>;
```

The following statements make the travel times symmetric, except that travel time back to the depot is set to 0:

```
   for {<i,j> in ARCS: travel_time[i,j] = .}
      travel_time[i,j] = travel_time[j,i];
   /* ignore travel time back to depot */
   for {i in NODES}
      travel_time[i,depot] = 0;
   /* remove self-loops */
   ARCS = ARCS diff setof {i in NODES} <i,i>;
   print travel_time;
```

The PRINT statement results in the first section of output, shown in Figure 27.1.

Figure 27.1 *travel_time* Parameter

The OPTMODEL Procedure

	travel_time									
	Barking	**Battersea**	**Bromley**	**Dartford**	**Ealing**	**Greenwich**	**Hammersmith**	**Harrow**	**Heathrow**	**Holborn**
Barking		61	25	25	65	25	65	90	0	47
Battersea	61		27	43	24	20	7	37	0	12
Bromley	25	27		15	55	17	41	57	0	53
Dartford	25	43	15		70	15	45	55	0	60
Ealing	65	24	55	70		50	10	15	0	30
Greenwich	25	20	17	15	50		40	85	0	55
Hammersmith	65	7	41	45	10	40		25	0	12
Harrow	90	37	57	55	15	85	25		0	35
Heathrow	86	44	85	90	25	80	25	20		35
Holborn	47	12	53	60	30	55	12	35	0	
Islington	45	30	45	63	20	30	15	20	0	10
Kingston	70	12	25	65	25	34	20	35	0	22
Richmond	72	14	33	53	15	32	8	30	0	20
Sutton	75	15	15	46	50	45	25	60	0	45
Woolwich	13	40	30	70	90	10	25	40	0	21

	travel_time				
	Islington	**Kingston**	**Richmond**	**Sutton**	**Woolwich**
Barking	45	70	72	75	13
Battersea	30	12	14	15	40
Bromley	45	25	33	15	30
Dartford	63	65	53	46	70
Ealing	20	25	15	50	90
Greenwich	30	34	32	45	10
Hammersmith	15	20	8	25	25
Harrow	20	35	30	60	40
Heathrow	35	35	20	65	82
Holborn	10	22	20	45	21
Islington		45	34	25	27
Kingston	45		5	11	65
Richmond	34	5		19	56
Sutton	25	11	19		25
Woolwich	27	65	56	25	

The following model declaration statements correspond directly to the mathematical programming formulation that is described earlier:

```
var UseNode {NODES, VEHICLES} binary;
var UseArc {ARCS, VEHICLES} binary;
var UseVehicle {VEHICLES} binary;
var TimeUsed {v in VEHICLES} >= 0 <= &time_limit;

min NumVehiclesUsed =
```

```
   sum {v in VEHICLES} UseVehicle[v];

con NodeCover {i in NODES diff {depot}}:
   sum {v in VEHICLES} UseNode[i,v] = 1;

con Outflow {i in NODES, v in VEHICLES}:
   sum {<(i),j> in ARCS} UseArc[i,j,v] = UseNode[i,v];

con Inflow {j in NODES, v in VEHICLES}:
   sum {<i,(j)> in ARCS} UseArc[i,j,v] = UseNode[j,v];

con UseVehicle_con1 {i in NODES, v in VEHICLES}:
   UseNode[i,v] <= UseVehicle[v];

con UseVehicle_con2 {v in VEHICLES}:
   UseVehicle[v] <= UseNode[depot,v];

con TimeUsed_con {v in VEHICLES}:
   TimeUsed[v] = sum {<i,j> in ARCS} travel_time[i,j] * UseArc[i,j,v];
```

The following statements declare optional symmetry-breaking constraints to reduce the number of essentially identical branch-and-bound nodes that are explored by the mixed integer linear programming solver:

```
   /* several alternatives for symmetry-breaking constraints */
   con Symmetry {v in VEHICLES diff {1}}:
      sum {i in NODES} UseNode[i,v] <= sum {i in NODES} UseNode[i,v-1];
*  con Symmetry {v in VEHICLES diff {1}}:
      UseVehicle[v] <= UseVehicle[v-1];
*  con Symmetry {v in VEHICLES diff {1}}:
      TimeUsed[v] <= TimeUsed[v-1];
```

Williams (2013) breaks symmetry by using the first alternative, but you could use one of the other alternatives instead.

In SAS/OR 13.1, the mixed integer linear programming solver automatically detects and exploits symmetry without you having to explicitly declare such symmetry-breaking constraints. You can control the aggressiveness of symmetry detection by using the SYMMETRY= option in the SOLVE WITH MILP statement.

The following statements declare the subtour elimination constraints:

```
num num_subtours init 0;

/* subset of nodes not containing depot node */
set <str> SUBTOUR {1..num_subtours};

/* if node k in SUBTOUR[s] is used by vehicle v, then
   must use at least two arcs across partition induced by SUBTOUR[s] */
con Subtour_elimination
   {s in 1..num_subtours, k in SUBTOUR[s], v in VEHICLES}:
   sum {i in NODES diff SUBTOUR[s], j in SUBTOUR[s]: <i,j> in ARCS}
      UseArc[i,j,v]
 + sum {i in SUBTOUR[s], j in NODES diff SUBTOUR[s]: <i,j> in ARCS}
      UseArc[i,j,v]
>= 2 * UseNode[k,v];
```

The following statements declare the index sets and parameters that are needed to detect violated subtour elimination constraints:

```
num num_components {VEHICLES};
set <str> NODES_SOL;
set <str,str> ARCS_SOL;
num component_id {NODES_SOL};
set COMPONENT_IDS;
set <str> COMPONENT {COMPONENT_IDS};
num ci;
```

The following statements call the %subtourEliminationLoop macro to minimize the number of vehicles used and then use the `.sol` objective suffix to update the *num_vehicles* parameter to the resulting minimum value:

```
%subtourEliminationLoop;
num_vehicles = round(NumVehiclesUsed.sol);
```

Changing the value of *num_vehicles* automatically updates the VEHICLES index set and consequently all the model declarations that depend on VEHICLES.

The following statements declare the additional variables, objective, and constraints that are needed to minimize the makespan, given the minimum number of vehicles already found:

```
var MaxTimeUsed >= 0 <= &time_limit;

min Makespan = MaxTimeUsed;

con MaxTimeUsed_con {v in VEHICLES}:
   MaxTimeUsed >= TimeUsed[v];
```

The following statements call the %subtourEliminationLoop macro again to minimize the makespan and then print the nodes and arcs that are used by each vehicle in the final solution:

```
%subtourEliminationLoop;
for {v in VEHICLES: UseVehicle[v].sol > 0.5} do;
   print v;
   print {<i,j> in ARCS: UseArc[i,j,v].sol > 0.5} travel_time[i,j];
end;
quit;
```

Figure 27.2 through Figure 27.9 show the output from each iteration of subtour elimination.

Figure 27.2 Output from Subtour Elimination, Iteration 1

[1]	UseVehicle	TimeUsed	num_components
1	1	95	5
2	1	80	3
3	0	0	0
4	0	0	0
5	0	0	0
6	0	0	0

Figure 27.3 Output from Subtour Elimination, Iteration 2

[1]	UseVehicle	TimeUsed	num_components
1	1	115	2
2	1	85	3
3	0	0	0
4	0	0	0
5	0	0	0
6	0	0	0

Figure 27.4 Output from Subtour Elimination, Iteration 3

[1]	UseVehicle	TimeUsed	num_components
1	1	1.1300E+02	3
2	1	1.0100E+02	3
3	0	9.4143E-12	0
4	0	-1.1140E-12	0
5	0	-2.6702E-12	0
6	0	2.7265E-12	0

Figure 27.5 Output from Subtour Elimination, Iteration 4

[1]	UseVehicle	TimeUsed	num_components
1	1	120	2
2	1	100	2
3	0	0	0
4	0	0	0
5	0	0	0
6	0	0	0

Figure 27.6 Output from Subtour Elimination, Iteration 5

[1]	UseVehicle	TimeUsed	num_components
1	1	119	1
2	1	110	2
3	0	0	0
4	0	0	0
5	0	0	0
6	0	0	0

Figure 27.7 Output from Subtour Elimination, Iteration 6

[1]	UseVehicle	TimeUsed	num_components
1	1	114	2
2	1	120	1
3	0	0	0
4	0	0	0
5	0	0	0
6	0	0	0

Figure 27.8 Output from Subtour Elimination, Iteration 7

[1]	UseVehicle	TimeUsed	num_components
1	1	103	1
2	1	117	1
3	0	0	0
4	0	0	0
5	0	0	0
6	0	0	0

Figure 27.9 Output from Subtour Elimination, Iteration 8

[1]	UseVehicle	TimeUsed	num_components
1	1	99	1
2	1	100	1

Figure 27.10 shows the final problem and solution summaries from the mixed integer linear programming solver.

Figure 27.10 Final Problem and Solution Summaries from Mixed Integer Linear Programming Solver

Problem Summary	
Objective Sense	Minimization
Objective Function	Makespan
Objective Type	Linear
Number of Variables	455
Bounded Above	0
Bounded Below	0
Bounded Below and Above	455
Free	0
Fixed	0
Binary	452
Integer	0
Number of Constraints	203
Linear LE (<=)	33
Linear EQ (=)	76
Linear GE (>=)	94
Linear Range	0
Constraint Coefficients	8024

Performance Information	
Execution Mode	Single-Machine
Number of Threads	4

Solution Summary	
Solver	MILP
Algorithm	Branch and Cut
Objective Function	Makespan
Solution Status	Optimal
Objective Value	100
Relative Gap	0
Absolute Gap	0
Primal Infeasibility	5.107026E-15
Bound Infeasibility	3.330669E-15
Integer Infeasibility	3.330669E-15
Best Bound	100
Nodes	65
Iterations	12853
Presolve Time	0.45
Solution Time	1.02

Figure 27.11 shows the travel times for the arcs that are used by each vehicle in the final solution.

Figure 27.11 Travel Times for Arcs Used by Each Vehicle in the Final Solution

v
1

[1]	[2]	
Bromley	Dartford	15
Dartford	Heathrow	0
Ealing	Hammersmith	10
Hammersmith	Richmond	8
Harrow	Ealing	15
Heathrow	Harrow	20
Kingston	Sutton	11
Richmond	Kingston	5
Sutton	Bromley	15

v
2

[1]	[2]	
Barking	Heathrow	0
Battersea	Greenwich	20
Greenwich	Woolwich	10
Heathrow	Islington	35
Holborn	Battersea	12
Islington	Holborn	10
Woolwich	Barking	13

Features Demonstrated

The following features are demonstrated in this example:

- problem type: mixed integer linear programming (vehicle routing)
- reading dense (upper triangular) two-dimensional data
- sets of tuples
- sets indexed by other sets
- set operators UNION and DIFF
- multiple objectives
- INIT option
- CARD function

- using a colon (:) to select members of a set
- aggregation operators SETOF and AND
- modeling if-then constraints by using binary variables
- symmetry-breaking constraints
- MILP solver option SYMMETRY=
- calling a solver in a DO UNTIL loop
- row generation
- **`.sol`** variable suffix
- connected components
- SOLVE WITH NETWORK

Chapter 28
Protein Folding

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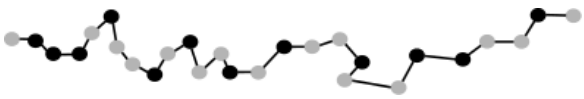
Problem Statement

This problem is based on one in the paper by Forrester and Greenberg (2008).[1] It is a simplification of a problem in molecular biology. We take a protein as consisting of a chain of amino acids. For the purpose of this problem, the amino acids come in two forms: *hydrophilic* (water-loving) and *hydrophobic* (water-hating). An example of such a chain is given in Figure 28.1, with the hydrophobic acids marked in bold.

Figure 28.1



Such a chain naturally folds so as to bring as many hydrophobic acids as possible close together. An optimum folding for the chain, in two dimensions, is given in Figure 28.2, with the new matches marked by dashed lines. The problem is to predict the optimum folding. (Forrester and Greenberg also impose a condition that the resultant protein be confined to a given lattice of points. We do not impose that condition here). This problem can be modelled by a number of integer programming formulations. Some of these are discussed in the above reference. Another formulation is suggested in section 13.28 [of Williams (2013)]. The problem posed here is to find the optimum folding for a chain of 50 amino acids with hydrophobic acids at positions 2, 4, 5, 6, 11, 12, 17, 20, 21, 25, 27, 28, 30, 31, 33, 37, 44 and 46 as shown in Figure 28.3.

[1] Reproduced with permission of John Wiley & Sons Ltd. (Williams 2013, pp. 289–290).

Figure 28.2

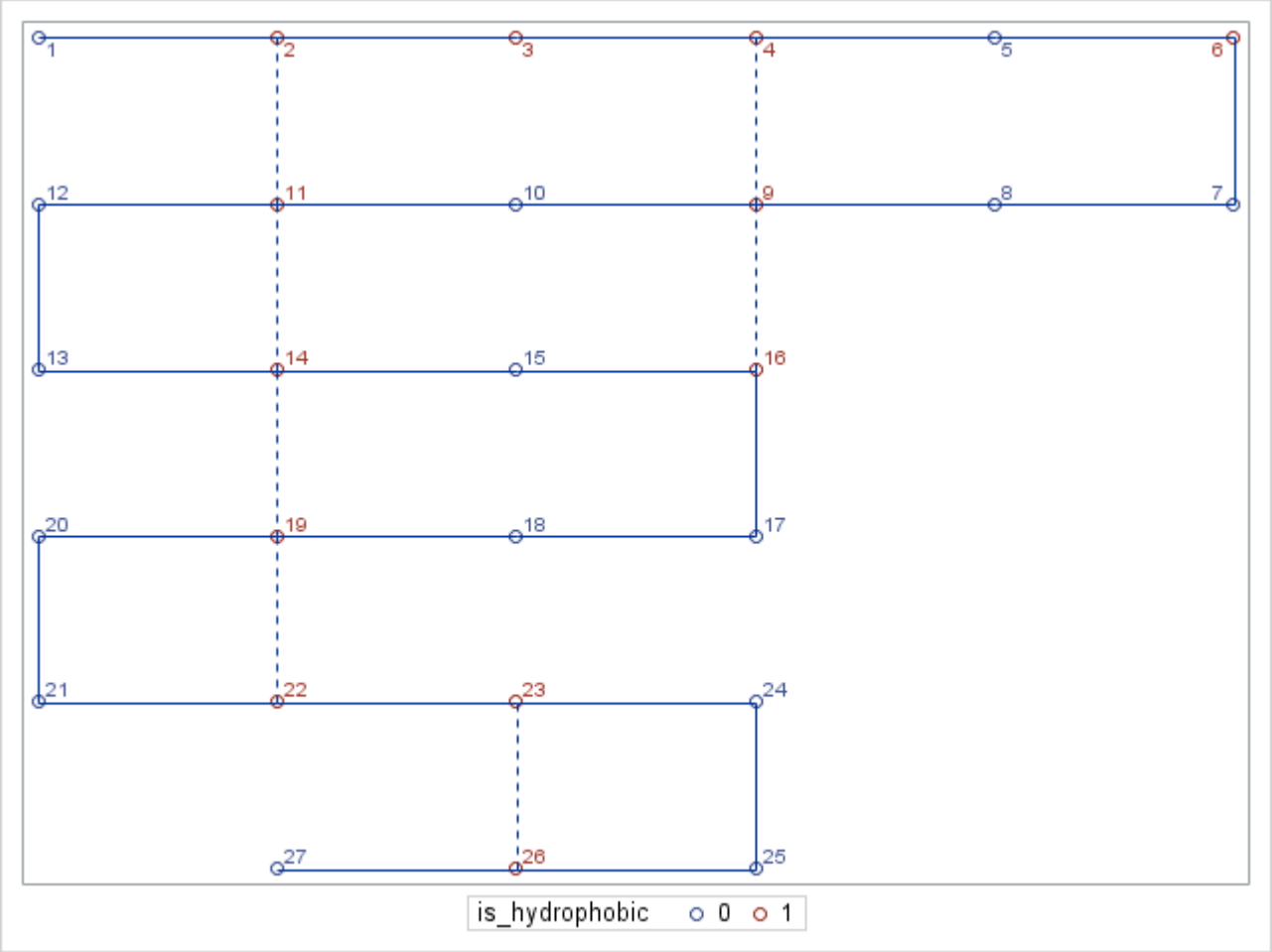
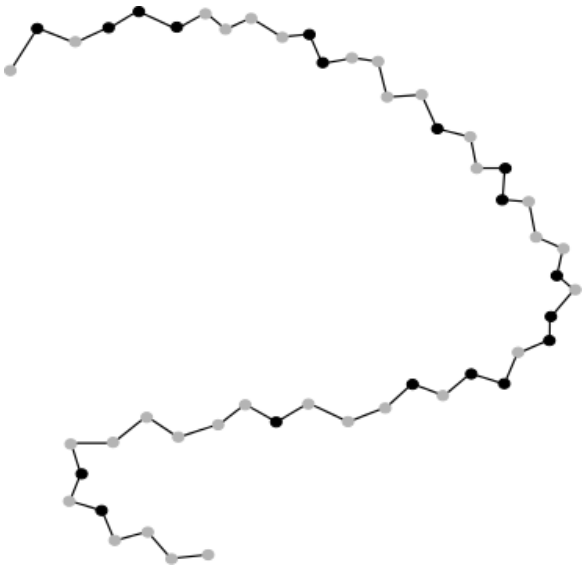


Figure 28.3



Mathematical Programming Formulation

The mixed integer linear programming formulation shown here matches Williams (2013). An alternative approach (not shown) formulates the model as a longest-path problem in a directed acyclic network, with node (i, j) corresponding to folds after positions i and j and not between, and with arcs of the form $(i, j) \rightarrow (j, k)$.

Index Sets and Their Members

The following index sets and their members are used in this example:

- $i \in$ POSITIONS
- $i, j \in$ HYDROPHOBIC
- $(i, j) \in$ PAIRS

Parameters

Table 28.1 shows the parameters that are used in this example.

Table 28.1 Parameters

Parameter Name	Interpretation
n	Number of amino acids in chain
$x[i]$	x coordinate of amino acid at position i, for plot
$y[i]$	y coordinate of amino acid at position i, for plot

Variables

Table 28.2 shows the variables that are used in this example.

Table 28.2 Variables

Variable Name	Interpretation
IsFold[i]	1 if fold occurs between positions i and $i + 1$; 0 otherwise
IsMatch[i,j]	1 if hydrophobic pair at positions i and j are matched; 0 otherwise

Objective

The objective is to maximize the following function:

$$\text{NumMatches} = \sum_{(i,j)\in \text{PAIRS}} \text{IsMatch[i,j]}$$

Constraints

The following constraints are used in this example:

- bounds on variables
- for $(i, j) \in$ PAIRS and $k \in \{i, \ldots, j-1\} \setminus \{(i+j-1)/2\}$,

  $$\text{IsMatch[i,j]} + \text{IsFold[k]} \leq 1$$

- for $(i, j) \in$ PAIRS,

  $$\text{IsMatch[i,j]} \leq \text{IsFold[(i+j-1)/2]}$$

Input Data

The following data set and macro variable contain the input data that are used in this example:

```
data hydrophobic_data;
   input position @@;
   datalines;
2 4 5 6 11 12 17 20 21 25 27 28 30 31 33 37 44 46
;

%let num_acids = 50;
```

PROC OPTMODEL Statements and Output

The first several PROC OPTMODEL statements declare index sets and parameters and then read the input data:

```
proc optmodel;
   num n = &num_acids;
   set POSITIONS = 1..n;
   set HYDROPHOBIC;
   read data hydrophobic_data into HYDROPHOBIC=[position];
```

The following statement uses the MOD function to declare the set of hydrophobic pairs that can possibly be matched (because they are not contiguous and are an even number of positions apart):

```
   set PAIRS
      = {i in HYDROPHOBIC, j in HYDROPHOBIC: i + 1 < j and mod(j-i-1,2) = 0};
```

The following model declaration statements correspond directly to the mathematical programming formulation that is described earlier:

```
   /* IsFold[i] = 1 if fold occurs between positions i and i + 1; 0 otherwise */
   var IsFold {1..n-1} binary;

   /* IsMatch[i,j] = 1 if hydrophobic pair at positions i and j are matched;
      0 otherwise */
   var IsMatch {PAIRS} binary;

   /* maximize number of matches */
   max NumMatches = sum {<i,j> in PAIRS} IsMatch[i,j];

   /* if IsMatch[i,j] = 1 then IsFold[k] = 0 */
   con DoNotFold {<i,j> in PAIRS, k in i..j-1 diff {(i+j-1)/2}}:
      IsMatch[i,j] + IsFold[k] <= 1;

   /* if IsMatch[i,j] = 1 then IsFold[k] = 1 */
   con FoldHalfwayBetween {<i,j> in PAIRS}:
      IsMatch[i,j] <= IsFold[(i+j-1)/2];
```

The following statements call the mixed integer linear programming solver and print the positive variables in the resulting optimal solution:

```
   solve;
   print {i in 1..n-1: IsFold[i].sol > 0.5} IsFold;
   print {<i,j> in PAIRS: IsMatch[i,j].sol > 0.5} IsMatch;
```

Figure 28.4 shows the output from the mixed integer linear programming solver.

Figure 28.4 Output from Mixed Integer Linear Programming Solver

The OPTMODEL Procedure

Problem Summary	
Objective Sense	Maximization
Objective Function	NumMatches
Objective Type	Linear
Number of Variables	124
Bounded Above	0
Bounded Below	0
Bounded Below and Above	124
Free	0
Fixed	0
Binary	124
Integer	0
Number of Constraints	1265
Linear LE (<=)	1265
Linear EQ (=)	0
Linear GE (>=)	0
Linear Range	0
Constraint Coefficients	2530

Performance Information	
Execution Mode	Single-Machine
Number of Threads	4

Solution Summary	
Solver	MILP
Algorithm	Branch and Cut
Objective Function	NumMatches
Solution Status	Optimal
Objective Value	10
Relative Gap	0
Absolute Gap	0
Primal Infeasibility	0
Bound Infeasibility	0
Integer Infeasibility	0
Best Bound	10
Nodes	1
Iterations	128
Presolve Time	0.09
Solution Time	0.10

Figure 28.4 *continued*

[1]	IsFold
3	1
8	1
14	1
18	1
22	1
26	1
28	1
30	1
38	1

[1]	[2]	IsMatch
2	5	1
5	12	1
6	11	1
12	17	1
17	20	1
20	25	1
25	28	1
27	30	1
31	46	1
33	44	1

The optimal solution and objective value differ from Williams (2013), because of an error that will be corrected in a subsequent printing by Wiley.

The following statements compute *x* and *y* coordinates for each position and write multiple output data sets to be used by the SGPLOT procedure:

```
   num x {POSITIONS};
   num y {POSITIONS};
   num xx init 0;
   num yy init 0;
   num dir init 1;
   for {i in POSITIONS} do;
      xx = xx + dir;
      x[i] = xx;
      y[i] = yy;
      if i = n or IsFold[i].sol > 0.5 then do;
         xx = xx + dir;
         dir = -dir;
         yy = yy - 1;
      end;
   end;
   create data plot_data from [i] x y is_hydrophobic=(i in HYDROPHOBIC);
   create data edge_data from [i]=(1..n-1)
      x1=x[i] y1=y[i] x2=x[i+1] y2=y[i+1] linepattern=1;
   create data match_data from [i j]={<i,j> in PAIRS: IsMatch[i,j].sol > 0.5}
      x1=x[i] y1=y[i] x2=x[j] y2=y[j] linepattern=2;
quit;
```

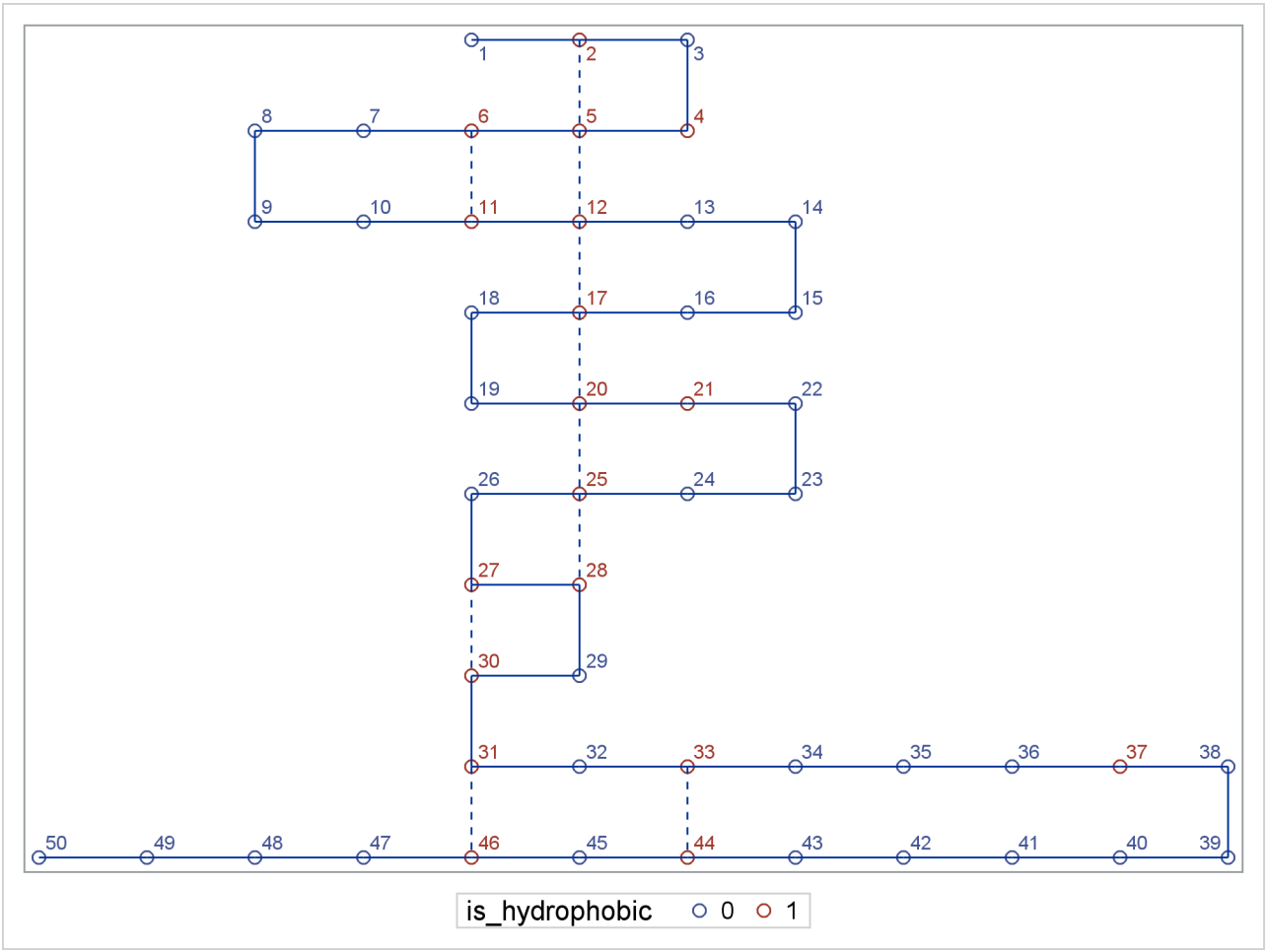
The following DATA step and PROC SGPLOT statements use the output data sets that are created by PROC OPTMODEL to display the optimal folding that corresponds to the MILP solution:

```
data sganno(drop=i j);
   retain drawspace "datavalue" linethickness 1;
   set edge_data match_data;
   function = 'line';
run;

proc sgplot data=plot_data sganno=sganno;
   scatter x=x y=y / group=is_hydrophobic datalabel=i;
   xaxis display=none;
   yaxis display=none;
run;
```

Figure 28.5 shows a plot of the optimal folding that corresponds to the MILP solution.

Figure 28.5 Plot of Optimal Folding



Features Demonstrated

The following features are demonstrated in this example:

- problem type: mixed integer linear programming
- set of tuples
- MOD function
- set operator DIFF
- modeling if-then constraints by using binary variables
- using a colon (:) to select members of a set
- `.sol` variable suffix
- creating multiple data sets
- SGPLOT procedure

Chapter 29
Protein Comparison

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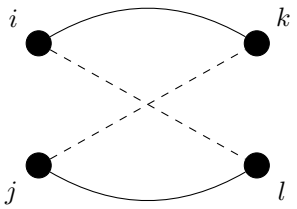
Problem Statement

This problem is also based on one in the paper by Forrester and Greenberg (2008).[1] It is concerned with measuring the similarities of two proteins. A protein can be represented by an (undirected) graph with the acids represented by the nodes and the edges being present when two acids are within a threshold distance of each other. This graphical representation is known as the *contact map* of the protein. Given two contact maps, representing proteins, we would like to find the largest (measured by number of corresponding edges) isomorphic subgraphs in each graph. The acids in each of the proteins are ordered. We need to preserve this ordering in each of the subgraphs, which implies that there can be no *crossovers* in the comparison. This is illustrated in Figure 29.1. If $i < k$ in the contact map for the first protein then we cannot have $l < j$ in the second protein, if i is to be associated with j and k with l in the comparison.

Figure 29.1

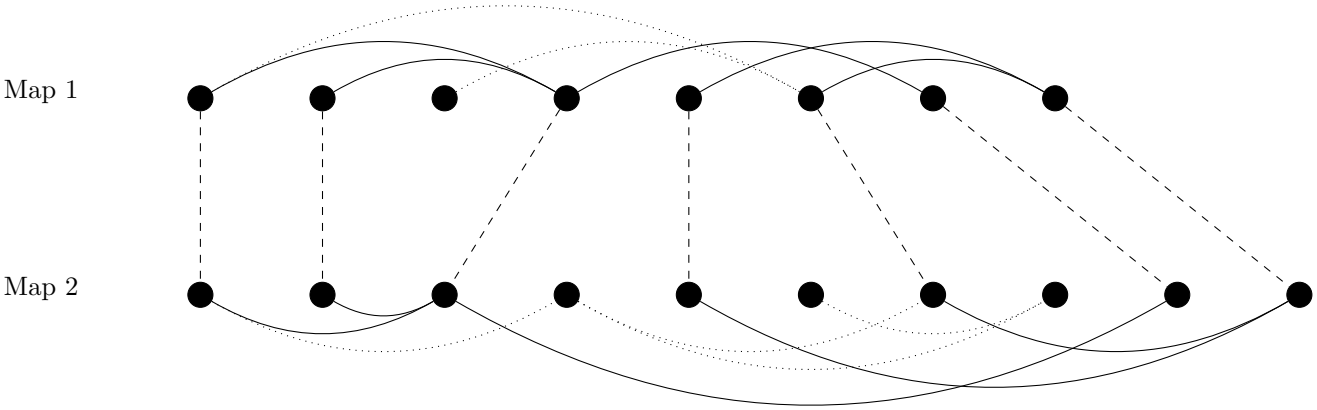


This problem is well known for being very difficult to solve for even modestly sized proteins.

[1] Reproduced with permission of John Wiley & Sons Ltd. (Williams 2013, pp. 290–291).

In Figure 29.2, we give an optimal comparison between two small contact maps leading to 5 corresponding edges.

Figure 29.2



The problem we present here is to compare the contact maps given in Figure 29.3 and Figure 29.4.

Figure 29.3

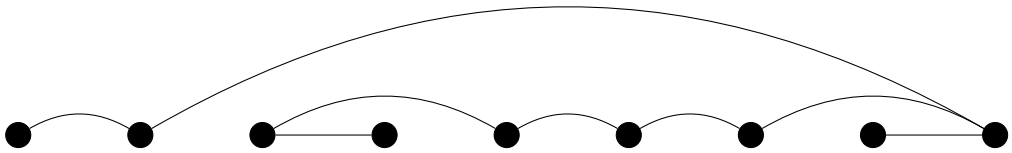
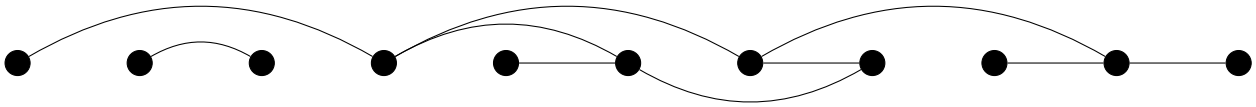


Figure 29.4



Mathematical Programming Formulation

Index Sets and Their Members

The following index sets and their members are used in this example:

- $g \in \{1, 2\}$
- $i \in$ NODES[g]
- $(i, k), (j, l) \in$ EDGES[g]
- $(i, j), (k, l) \in$ IJ $=$ NODES[1] $\times$ NODES[2]
- $(i, j, k, l) \in$ EDGE_PAIRS: pairs of edges that can possibly correspond to each other

Variables

Table 29.1 shows the variables that are used in this example.

Table 29.1 Variables

Variable Name	Interpretation
Assign[i,j]	1 if node $i \in$ NODES[1] is assigned to node $j \in$ NODES[2]; 0 otherwise
IsCorrespondingEdge[i,j,k,l]	1 if edge $(i,k) \in$ EDGES[1] corresponds to edge $(j,l) \in$ EDGES[2]; 0 otherwise

Objective

The objective is to maximize the following function:

$$\text{NumCorrespondingEdges} = \sum_{(i,j,k,l) \in \text{EDGE_PAIRS}} \text{IsCorrespondingEdge[i,j,k,l]}$$

Constraints

The following constraints are used in this example:

- bounds on variables
- for $i \in$ NODES[1],

$$\sum_{(i,j) \in \text{IJ}} \text{Assign[i,j]} \leq 1$$

- for $j \in$ NODES[2],

$$\sum_{(i,j) \in \text{IJ}} \text{Assign[i,j]} \leq 1$$

- for $(i, j) \in$ IJ and $(k, l) \in$ IJ such that $i < k$ and $j > l$,

$$\text{Assign[i,j]} + \text{Assign[k,l]} \leq 1$$

- for $(i, j, k, l) \in$ EDGE_PAIRS,

$$\text{IsCorrespondingEdge[i,j,k,l]} \leq \text{Assign[i,j]}$$

- for $(i, j, k, l) \in$ EDGE_PAIRS,

$$\text{IsCorrespondingEdge[i,j,k,l]} \leq \text{Assign[k,l]}$$

Input Data

The following data sets contain the input data that are used in this example:

```
data edge_data1;
   input i j;
   datalines;
1 2
2 9
3 4
3 5
5 6
6 7
7 9
8 9
;

data edge_data2;
   input i j;
   datalines;
1 4
2 3
4 6
4 7
5 6
6 8
7 8
7 10
9 10
10 11
;
```

PROC OPTMODEL Statements and Output

The first several PROC OPTMODEL statements declare index sets and read the input data:

```
proc optmodel;
   set <num,num> EDGES {1..2};
   read data edge_data1 into EDGES[1]=[i j];
   read data edge_data2 into EDGES[2]=[i j];
```

The following statements declare and initialize the NODES[g] sets and then use the INTER operator to store their elements in increasing order:

```
   set NODES {g in 1..2} init union {<i,j> in EDGES[g]} {i,j};
   for {g in 1..2} NODES[g] = 1..card(NODES[g]) inter NODES[g];
```

The following statements declare the remaining sets:

```
set IJ = NODES[1] cross NODES[2];
set EDGE_PAIRS = {<i,j> in IJ, <k,l> in IJ: i < k and j ne l and
   (<i,k> in EDGES[1]) and (<j,l> in EDGES[2])};
```

The following model declaration statements correspond directly to the mathematical programming formulation that is described earlier:

```
/* Assign[i,j] = 1 if node i in NODES[1] assigned to node j in NODES[2] */
var Assign {IJ} binary;

/* IsCorrespondingEdge[i,j,k,l] = 1 if edge <i,k> in EDGES[1]
   corresponds to edge <j,l> in EDGES[2] */
var IsCorrespondingEdge {EDGE_PAIRS} binary;

/* maximize number of corresponding edges */
max NumCorrespondingEdges =
   sum {<i,j,k,l> in EDGE_PAIRS} IsCorrespondingEdge[i,j,k,l];

/* assign each i to at most one j */
con Assign_i {i in NODES[1]}:
   sum {<(i),j> in IJ} Assign[i,j] <= 1;

/* assign at most one i to each j */
con Assign_j {j in NODES[2]}:
   sum {<i,(j)> in IJ} Assign[i,j] <= 1;

/* disallow crossing edges */
con NoCrossover {<i,j> in IJ, <k,l> in IJ: i < k and j > l}:
   Assign[i,j] + Assign[k,l] <= 1;

/* if IsCorrespondingEdge[i,j,k,l] = 1 then Assign[i,j] = Assign[k,l] = 1 */
con Corresponding1 {<i,j,k,l> in EDGE_PAIRS}:
   IsCorrespondingEdge[i,j,k,l] <= Assign[i,j];
con Corresponding2 {<i,j,k,l> in EDGE_PAIRS}:
   IsCorrespondingEdge[i,j,k,l] <= Assign[k,l];
```

The following statements call the mixed integer linear programming solver and use the FILE and PUT statements to write log output to the listing:

```
   solve;
   file print;
   for {<i,j> in IJ: Assign[i,j].sol > 0.5}
      put ('Node '||i||' in graph 1 corresponds to node '||j||' in graph 2.');
   for {<i,j,k,l> in EDGE_PAIRS: IsCorrespondingEdge[i,j,k,l].sol > 0.5} do;
      put ('Edge ('||i||','||k||') in graph 1 corresponds to') @@;
      put ('edge ('||j||','||l||') in graph 2.');
   end;
quit;
```

Figure 29.5 shows the output from the mixed integer linear programming solver.

Figure 29.5 Output from Mixed Integer Linear Programming Solver

The OPTMODEL Procedure

Problem Summary	
Objective Sense	Maximization
Objective Function	NumCorrespondingEdges
Objective Type	Linear
Number of Variables	179
Bounded Above	0
Bounded Below	0
Bounded Below and Above	179
Free	0
Fixed	0
Binary	179
Integer	0
Number of Constraints	2160
Linear LE (<=)	2160
Linear EQ (=)	0
Linear GE (>=)	0
Linear Range	0
Constraint Coefficients	4478

Performance Information	
Execution Mode	Single-Machine
Number of Threads	4

Solution Summary	
Solver	MILP
Algorithm	Branch and Cut
Objective Function	NumCorrespondingEdges
Solution Status	Optimal
Objective Value	5
Relative Gap	0
Absolute Gap	0
Primal Infeasibility	0
Bound Infeasibility	0
Integer Infeasibility	0
Best Bound	5
Nodes	1
Iterations	111
Presolve Time	0.07
Solution Time	0.08

Figure 29.6 shows the output from the PUT statements.

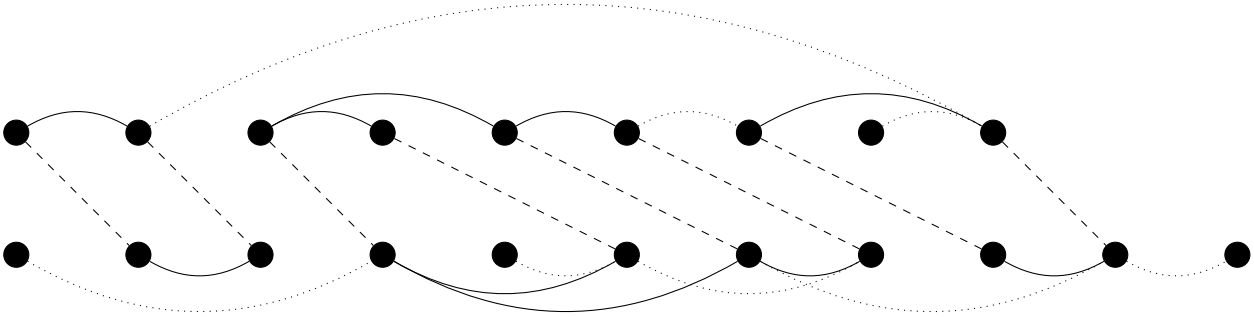
Figure 29.6 Output from PUT Statements

```
The OPTMODEL Procedure

Node 1 in graph 1 corresponds to node 2 in graph 2.
Node 2 in graph 1 corresponds to node 3 in graph 2.
Node 3 in graph 1 corresponds to node 4 in graph 2.
Node 4 in graph 1 corresponds to node 6 in graph 2.
Node 5 in graph 1 corresponds to node 7 in graph 2.
Node 6 in graph 1 corresponds to node 8 in graph 2.
Node 7 in graph 1 corresponds to node 9 in graph 2.
Node 9 in graph 1 corresponds to node 10 in graph 2.
Edge (1,2) in graph 1 corresponds to edge (2,3) in graph 2.
Edge (3,4) in graph 1 corresponds to edge (4,6) in graph 2.
Edge (3,5) in graph 1 corresponds to edge (4,7) in graph 2.
Edge (5,6) in graph 1 corresponds to edge (7,8) in graph 2.
Edge (7,9) in graph 1 corresponds to edge (9,10) in graph 2.
```

Figure 29.7 shows the optimal solution as a comparison between contact maps, with the matched nodes indicated by dashed lines and the edges of the isomorphic subgraphs highlighted.

Figure 29.7 Optimal Comparison between Contact Maps



Features Demonstrated

The following features are demonstrated in this example:

- problem type: mixed integer linear programming
- sets of tuples
- sets indexed by other sets
- reading multiple input data sets
- set operators UNION, INTER, and CROSS

- CARD function
- using a colon (:) to select members of a set
- modeling if-then constraints by using binary variables
- FILE statement
- PUT statement

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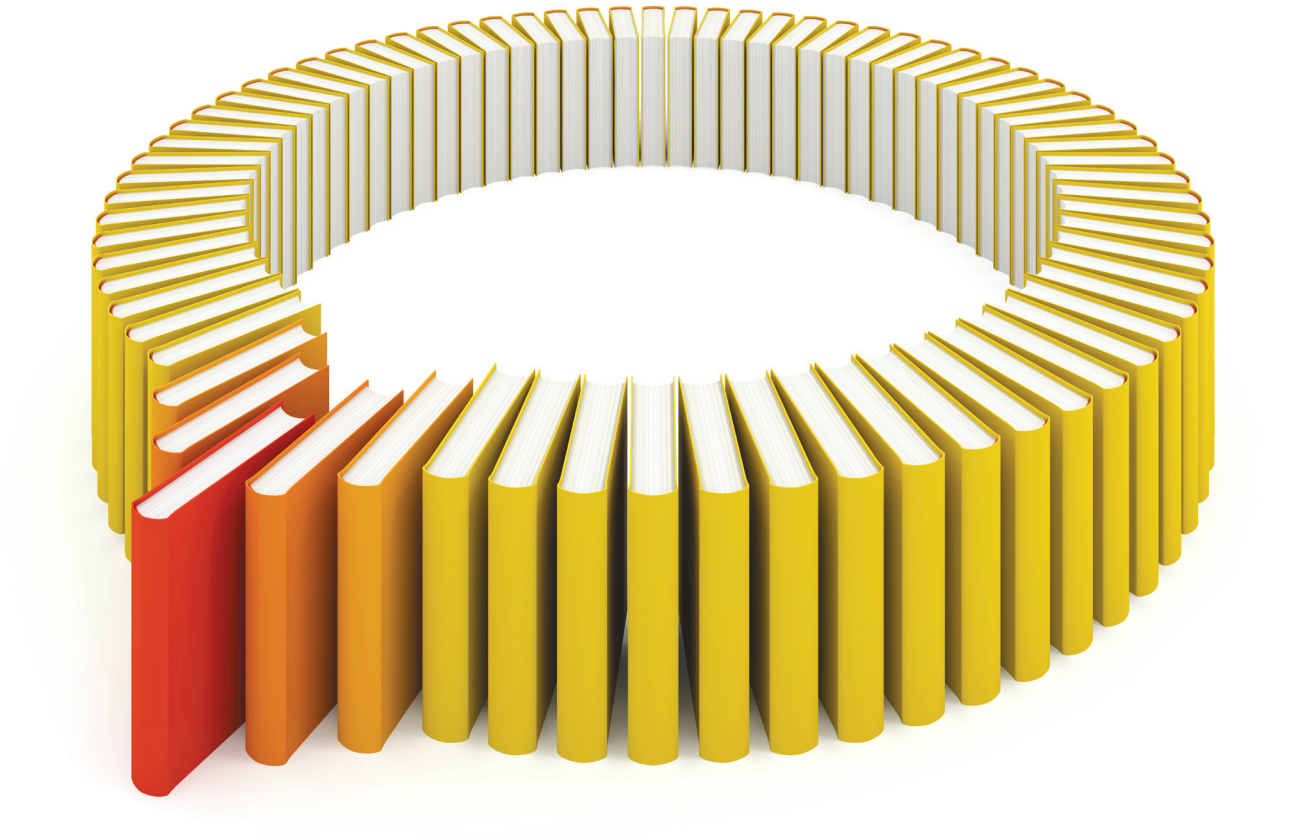

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