SAS/OR® 14.1 User's Guide: Mathematical Programming

The OPTQP Procedure

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Chapter 14
The OPTQP Procedure

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Overview: OPTQP Procedure

The OPTQP procedure solves quadratic programs—problems with quadratic objective function and a collection of linear constraints, including lower or upper bounds (or both) on the decision variables.

Mathematically, a quadratic programming (QP) problem can be stated as follows:

$$\begin{array}{rl} \min & \frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} + \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x}\ \{\geq, =, \leq\}\ \mathbf{b} \\ & \mathbf{l} \leq \mathbf{x} \leq \mathbf{u} \end{array}$$

where

$\mathbf{Q}$	$\in$	$\mathbb{R}^{n \times n}$	is the quadratic (also known as Hessian) matrix
$\mathbf{A}$	$\in$	$\mathbb{R}^{m \times n}$	is the constraints matrix
$\mathbf{x}$	$\in$	$\mathbb{R}^{n}$	is the vector of decision variables
$\mathbf{c}$	$\in$	$\mathbb{R}^{n}$	is the vector of linear objective function coefficients
$\mathbf{b}$	$\in$	$\mathbb{R}^{m}$	is the vector of constraints right-hand sides (RHS)
$\mathbf{l}$	$\in$	$\mathbb{R}^{n}$	is the vector of lower bounds on the decision variables
$\mathbf{u}$	$\in$	$\mathbb{R}^{n}$	is the vector of upper bounds on the decision variables

The quadratic matrix $\mathbf{Q}$ is assumed to be symmetric; that is,

$$q_{ij} = q_{ji}, \quad \forall i, j = 1, \ldots, n$$

Indeed, it is easy to show that even if $\mathbf{Q} \neq \mathbf{Q}^{\mathrm{T}}$, the simple modification

$$\tilde{\mathbf{Q}} = \frac{1}{2}(\mathbf{Q} + \mathbf{Q}^{\mathrm{T}})$$

produces an equivalent formulation $\mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} \equiv \mathbf{x}^{\mathrm{T}}\tilde{\mathbf{Q}}\mathbf{x}$; hence symmetry is assumed. When you specify a quadratic matrix, it suffices to list only lower triangular coefficients.

In addition to being symmetric, $\mathbf{Q}$ is also required to be positive semidefinite,

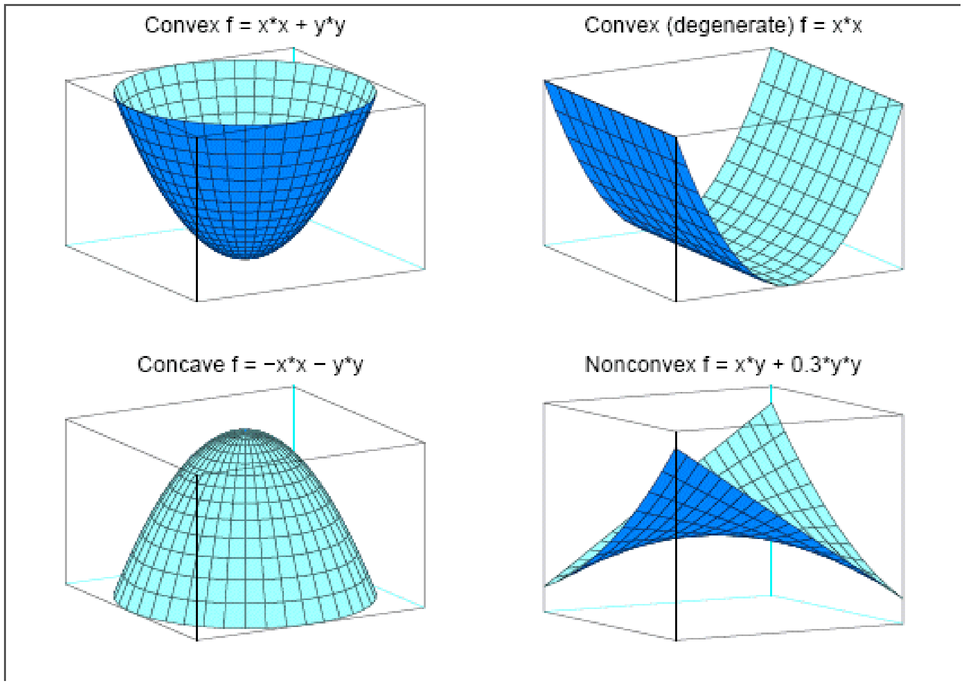
$$\mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} \geq 0, \quad \forall \mathbf{x} \in \mathbb{R}^{n}$$

for minimization type of models; it is required to be negative semidefinite for the maximization type of models. Convexity can come as a result of a matrix-matrix multiplication

$$\mathbf{Q} = \mathbf{L}\mathbf{L}^{\mathrm{T}}$$

or as a consequence of physical laws, and so on. See Figure 14.1 for examples of convex, concave, and nonconvex objective functions.

Figure 14.1 Examples of Convex, Concave, and Nonconvex Objective Functions



The order of constraints is insignificant. Some or all components of $\mathbf{l}$ or $\mathbf{u}$ (lower and upper bounds, respectively) can be omitted.

Getting Started: OPTQP Procedure

Consider a small illustrative example. Suppose you want to minimize a two-variable quadratic function $f(x_1, x_2)$ on the nonnegative quadrant, subject to two constraints:

$$
\begin{array}{rrcrcrcrcr}
\min & 2x_1 & + & 3x_2 & + & x_1^2 & + & 10x_2^2 & + & 2.5x_1x_2 \\
\text{subject to} & x_1 & - & x_2 & \leq & 1 & & & & \\
 & x_1 & + & 2x_2 & \geq & 100 & & & & \\
 & x_1 & & & \geq & 0 & & & & \\
 & & & x_2 & \geq & 0 & & & &
\end{array}
$$

The linear objective function coefficients, vector of right-hand sides, and lower and upper bounds are identified immediately as

$$
\mathbf{c} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 100 \end{bmatrix}, \quad \mathbf{l} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} +\infty \\ +\infty \end{bmatrix}
$$

Carefully construct the quadratic matrix $\mathbf{Q}$. Observe that you can use symmetry to separate the main-diagonal and off-diagonal elements:

$$
\frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} \equiv \frac{1}{2}\sum_{i,j=1}^{n} x_i\, q_{ij}\, x_j = \frac{1}{2}\sum_{i=1}^{n} q_{ii}\, x_i^2 + \sum_{i>j} x_i\, q_{ij}\, x_j
$$

The first expression

$$\frac{1}{2}\sum_{i=1}^{n} q_{ii}\, x_i^2$$

sums the main-diagonal elements. Thus, in this case you have

$$q_{11} = 2, \quad q_{22} = 20$$

Notice that the main-diagonal values are doubled in order to accommodate the 1/2 factor. Now the second term

$$\sum_{i>j} x_i\, q_{ij}\, x_j$$

sums the off-diagonal elements in the strict lower triangular part of the matrix. The only off-diagonal ($x_i\, x_j,\ i \neq j$) term in the objective function is $2.5\, x_1\, x_2$, so you have

$$q_{21} = 2.5$$

Notice that you do not need to specify the upper triangular part of the quadratic matrix.

Finally, the matrix of constraints is as follows:

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$$

The SAS input data set with a quadratic programming system (QPS) format for the preceding problem can be expressed in the following manner:

```
data gsdata;
   input field1 $ field2 $ field3 $ field4 field5 $ field6 @;
   datalines;
NAME      .      EXAMPLE    .            .         .
ROWS      .      .          .            .         .
N         OBJ    .          .            .         .
L         R1     .          .            .         .
G         R2     .          .            .         .
COLUMNS   .      .          .            .         .
.         X1     R1          1.0         R2        1.0
.         X1     OBJ         2.0         .         .
.         X2     R1         -1.0         R2        2.0
.         X2     OBJ         3.0         .         .
RHS       .      .          .            .         .
.         RHS    R1          1.0         .         .
.         RHS    R2          100         .         .
RANGES    .      .          .            .         .
BOUNDS    .      .          .            .         .
QUADOBJ   .      .          .            .         .
.         X1     X1          2.0         .         .
.         X1     X2          2.5         .         .
.         X2     X2          20          .         .
ENDATA    .      .          .            .         .
;
```

For more details about the QPS-format data set, see Chapter 17, “The MPS-Format SAS Data Set.”

Alternatively, if you have a QPS-format flat file named gs.qps, then the following call to the SAS macro %MPS2SASD translates that file into a SAS data set, named gsdata:

```
%mps2sasd(mpsfile =gs.qps, outdata = gsdata);
```

NOTE: The SAS macro %MPS2SASD is provided in SAS/OR software. See “Converting an MPS/QPS-Format File: %MPS2SASD” on page 848 for details.

You can use the following call to PROC OPTQP:

```
proc optqp data=gsdata
   primalout = gspout
   dualout   = gsdout;
run;
```

The procedure output is displayed in Figure 14.2.

Figure 14.2 Procedure Output

The OPTQP Procedure

Performance Information	
Execution Mode	Single-Machine
Number of Threads	4

Problem Summary	
Problem Name	EXAMPLE
Objective Sense	Minimization
Objective Function	OBJ
RHS	RHS
Number of Variables	2
Bounded Above	0
Bounded Below	2
Bounded Above and Below	0
Free	0
Fixed	0
Number of Constraints	2
LE (<=)	1
EQ (=)	0
GE (>=)	1
Range	0
Constraint Coefficients	4
Hessian Diagonal Elements	2
Hessian Elements Above the Diagonal	1

Figure 14.2 *continued*

Solution Summary	
Solver	QP
Algorithm	Interior Point
Objective Function	OBJ
Solution Status	Optimal
Objective Value	15018
Primal Infeasibility	0
Dual Infeasibility	0
Bound Infeasibility	0
Duality Gap	3.633377E-16
Complementarity	0
Iterations	6
Presolve Time	0.00
Solution Time	0.39

The optimal primal solution is displayed in Figure 14.3.

Figure 14.3 Optimal Solution

Obs	Objective Function ID	RHS ID	Variable Name	Variable Type	Linear Objective Coefficient	Lower Bound	Upper Bound	Variable Value	Variable Status
1	OBJ	RHS	X1	N	2	0	1.7977E308	34	O
2	OBJ	RHS	X2	N	3	0	1.7977E308	33	O

The SAS log shown in Figure 14.4 provides information about the problem, convergence information after each iteration, and the optimal objective value.

Figure 14.4 Iteration Log

```
NOTE: The problem EXAMPLE has 2 variables (0 free, 0 fixed).
NOTE: The problem has 2 constraints (1 LE, 0 EQ, 1 GE, 0 range).
NOTE: The problem has 4 constraint coefficients.
NOTE: The objective function has 2 Hessian diagonal elements and 1 Hessian
      elements above the diagonal.
NOTE: The QP presolver value AUTOMATIC is applied.
NOTE: The QP presolver removed 0 variables and 0 constraints.
NOTE: The QP presolver removed 0 constraint coefficients.
NOTE: The presolved problem has 2 variables, 2 constraints, and 4 constraint
      coefficients.
NOTE: The QP solver is called.
NOTE: The Interior Point algorithm is used.
NOTE: The deterministic parallel mode is enabled.
NOTE: The Interior Point algorithm is using up to 4 threads.
                                           Primal         Bound          Dual
      Iter  Complement Duality Gap         Infeas         Infeas         Infeas    Time
         0  3.5863E+03  4.8823E+00  1.0251E+00  1.0354E+02  2.3142E-15       0
         1  1.9345E+03  9.6222E-01  4.4158E-01  4.4602E+01  5.7855E-16       0
         2  2.2140E+03  1.2297E-01  4.4158E-03  4.4602E-01  5.5926E-15       0
         3  5.0020E+01  3.2272E-03  4.4158E-05  4.4602E-03  9.6502E-15       0
         4  4.9973E-01  3.2332E-05  4.4158E-07  4.4602E-05  2.5148E-14       0
         5  4.9972E-03  3.2332E-07  4.4158E-09  4.4602E-07  3.0701E-14       0
         6  0.0000E+00  3.6334E-16  1.5730E-16  0.0000E+00  1.2342E-14       0
NOTE: Optimal.
NOTE: Objective = 15018.
NOTE: The Interior Point solve time is 0.00 seconds.
NOTE: The data set WORK.GSPOUT has 2 observations and 9 variables.
NOTE: The data set WORK.GSDOUT has 2 observations and 10 variables.
```

See the section "Interior Point Algorithm: Overview" on page 698 and the section "Iteration Log for the OPTQP Procedure" on page 700 for more details about convergence information given by the iteration log.

Syntax: OPTQP Procedure

The following statements are available in the OPTQP procedure:

PROC OPTQP < *options* > ;

    PERFORMANCE < *performance-options* > ;

Functional Summary

Table 14.1 outlines the options available for the OPTQP procedure classified by function.

Table 14.1 Options in the OPTQP Procedure

Description	Option
Data Set Options	
Specifies a QPS-format input SAS data set	DATA=
Specifies a dual solution output SAS data set	DUALOUT=
Specifies whether the QP model is a maximization or minimization problem	OBJSENSE=
Specifies the primal solution output SAS data set	PRIMALOUT=
Saves output data sets only if optimal	SAVE_ONLY_IF_OPTIMAL
Solver Options	
Enables or disables IIS detection	IIS=
Control Options	
Specifies the maximum number of iterations	MAXITER=
Specifies the time limit for the optimization process	MAXTIME=
Specifies the type of presolve	PRESOLVER=
Enables or disables iteration log	LOGFREQ=
Enables or disables printing summary	PRINTLEVEL=
Specifies the stopping criterion based on duality gap	STOP_DG=
Specifies the stopping criterion based on dual infeasibility	STOP_DI=
Specifies the stopping criterion based on primal infeasibility	STOP_PI=
Specifies units of CPU time or real time	TIMETYPE=

PROC OPTQP Statement

The following options can be specified in the PROC OPTQP statement.

DATA=*SAS-data-set*

specifies the input SAS data set. This data set can also be created from a QPS-format flat file by using the SAS macro %MPS2SASD. If the DATA= option is not specified, PROC OPTQP uses the most recently created SAS data set. See Chapter 17, "The MPS-Format SAS Data Set," for more details.

DUALOUT=*SAS-data-set*

DOUT=*SAS-data-set*

specifies the output data set to contain the dual solution. See the section "Output Data Sets" on page 695 for details.

IIS=*number* | *string*

specifies whether PROC OPTQP attempts to identify a set of constraints and variables that form an irreducible infeasible set (IIS). Table 14.2 describes the valid values of the IIS= option.

Table 14.2 Values for IIS= Option

number	*string*	**Description**
0	OFF	Disables IIS detection.
1	ON	Enables IIS detection.

If an IIS is found, you can find information about infeasible constraints or variable bounds in the DUALOUT= and PRIMALOUT= data sets. The default value of this option is OFF. See the section "Irreducible Infeasible Set" on page 704 for details.

LOGFREQ=*k*

PRINTFREQ=*k*

specifies that the printing of the solution progress to the iteration log should occur after every k iterations. The print frequency, k, is an integer between zero and the largest four-byte, signed integer, which is $2^{31} - 1$. The value $k = 0$ disables the printing of the progress of the solution. The default value of this option is 1.

MAXITER=*k*

specifies the maximum number of predictor-corrector iterations performed by the interior point algorithm (see the section "Interior Point Algorithm: Overview" on page 698). The value k is an integer between 1 and the largest four-byte, signed integer, which is $2^{31} - 1$. If you do not specify this option, the procedure does not stop based on the number of iterations performed.

MAXTIME=*t*

specifies an upper limit of t seconds of time for reading in the data and performing the optimization process. The value of the TIMETYPE= option determines the type of units used. If you do not specify this option, the procedure does not stop based on the amount of time elapsed. The value of t can be any positive number; the default value is the positive number that has the largest absolute value that can be represented in your operating environment.

OBJSENSE=*option*

specifies whether the QP model is a minimization or a maximization problem. You specify OBJSENSE=MIN for a minimization problem and OBJSENSE=MAX for a maximization problem. Alternatively, you can specify the objective sense in the input data set; see the section "ROWS Section" on page 841 for details. If the objective sense is specified differently in these two places, this option supersedes the objective sense specified in the input data set. If the objective sense is not specified anywhere, then PROC OPTQP interprets and solves the quadratic program as a minimization problem.

PRESOLVER=*number* | *string*

PRESOL=*number* | *string*

specifies one of the following presolve options:

number	*string*	**Description**
0	NONE	Disables the presolver.
–1	AUTOMATIC	Applies the presolver by using default setting.

number	*string*	**Description**
1	BASIC	Applies the basic presolver.
2	MODERATE	Applies the moderate presolver.
3	AGGRESSIVE	Applies the aggressive presolver.

You can specify the option either by a word or by integers from –1 to 3. The default option is AUTOMATIC.

PRIMALOUT=*SAS-data-set*

POUT=*SAS-data-set*

specifies the output data set to contain the primal solution. See the section "Output Data Sets" on page 695 for details.

PRINTLEVEL=0 | 1 | 2

specifies whether a summary of the problem and solution should be printed. If PRINTLEVEL=1, then the Output Delivery System (ODS) tables ProblemSummary, SolutionSummary, and PerformanceInfo are produced and printed. If PRINTLEVEL=2, then the same tables are produced and printed along with an additional table called ProblemStatistics. If PRINTLEVEL=0, then no ODS tables are produced or printed. The default value is 1.

For details about the ODS tables created by PROC OPTQP, see the section "ODS Tables" on page 700.

SAVE_ONLY_IF_OPTIMAL

specifies that the PRIMALOUT= and DUALOUT= data sets be saved only if the final solution obtained by the solver at termination is optimal. If the PRIMALOUT= or DUALOUT= option is specified, and this option is not specified, then the output data sets will only contain solution values at optimality. If the SAVE_ONLY_IF_OPTIMAL option is not specified, the output data sets will not contain an intermediate solution.

STOP_DG=δ

specifies the desired relative duality gap, $\delta \in$[1E–9, 1E–4]. This is the relative difference between the primal and dual objective function values and is the primary solution quality parameter. The default value is 1E–6. See the section "Interior Point Algorithm: Overview" on page 698 for details.

STOP_DI=β

specifies the maximum allowed relative dual constraints violation, $\beta \in$[1E–9, 1E–4]. The default value is 1E–6. See the section "Interior Point Algorithm: Overview" on page 698 for details.

STOP_PI=α

specifies the maximum allowed relative bound and primal constraints violation, $\alpha \in$[1E–9, 1E–4]. The default value is 1E–6. See the section "Interior Point Algorithm: Overview" on page 698 for details.

TIMETYPE=*number* **|** *string*

specifies whether CPU time or real time is used for the MAXTIME= option and the _OROPTQP_ macro variable in a PROC OPTQP call. Table 14.4 describes the valid values of the TIMETYPE= option.

Table 14.4 Values for TIMETYPE= Option

number	*string*	**Description**
0	CPU	Specifies units of CPU time.
1	REAL	Specifies units of real time.

The default value of the TIMETYPE= option depends on the value of the NTHREADS= option in the PERFORMANCE statement. See the section “PERFORMANCE Statement” on page 21 for more information about the NTHREADS= option.

If you specify a value greater than 1 for the NTHREADS= option, the default value of the TIMETYPE= option is REAL. If you specify a value of 1 for the NTHREADS= option, the default value of the TIMETYPE= option is CPU.

PERFORMANCE Statement

PERFORMANCE < *performance-options* > **;**

The PERFORMANCE statement specifies *performance-options* for multithreaded (SMP) computing, passes variables around the distributed computing environment, and requests detailed results about the performance characteristics of the OPTQP procedure.

The PERFORMANCE statement for multithreaded computing mode is documented in the section “PERFORMANCE Statement” on page 21 in Chapter 4, “Shared Concepts and Topics.” The OPTQP procedure supports the deterministic and nondeterministic modes of the PARALLELMODE= option in the PERFORMANCE statement.

Details: OPTQP Procedure

Output Data Sets

This section describes the PRIMALOUT= and DUALOUT= output data sets. If the SAVE_ONLY_IF_OPTIMAL option is not specified, the output data sets do not contain an intermediate solution.

Definitions of Variables in the PRIMALOUT= Data Set

The PRIMALOUT= data set contains the primal solution to the quadratic programming (QP) model. The variables in the data set have the following names and meanings.

_OBJ_ID_

specifies the name of the objective function. Naming objective functions is particularly useful when there are multiple objective functions, in which case each objective function has a unique name. See the section “ROWS Section” on page 841 for details.

NOTE: PROC OPTQP does not support simultaneous optimization of multiple objective functions in this release.

_RHS_ID_

specifies the name of the variable that contains the right-hand-side value of each constraint. See the section “RHS Section (Optional)” on page 843 for details.

VAR
specifies the name of the decision variable.

TYPE
specifies the type of the decision variable. _TYPE_ can take one of the following values:

N nonnegative variable
D bounded variable with both finite lower and finite upper bound
F free variable
X fixed variable
O other

OBJCOEF
specifies the coefficient of the decision variable in the linear component of the objective function.

LBOUND
specifies the lower bound on the decision variable.

UBOUND
specifies the upper bound on the decision variable.

VALUE
specifies the value of the decision variable.

STATUS
specifies the status of the decision variable. _STATUS_ can indicate one of the following two cases:

O The QP problem is optimal.
I The QP problem could be infeasible or unbounded, or PROC OPTQP was not able to solve the problem.

The following values can appear only if IIS=ON. See the section "Irreducible Infeasible Set" on page 704 for details.

I_L The lower bound of the variable is needed for the IIS.
I_U The upper bound of the variable is needed for the IIS.
I_F Both bounds of the variable are needed for the IIS (the variable is fixed or has conflicting bounds).

Definitions of Variables in the DUALOUT= Data Set

The DUALOUT= data set contains the dual solution to the QP model. Information about the objective rows of the QP problems is not included. The variables in the data set have the following names and meanings.

_OBJ_ID_
specifies the name of the objective function. Naming objective functions is particularly useful when there are multiple objective functions, in which case each objective function has a unique name. See the section "ROWS Section" on page 841 for details.

NOTE: PROC OPTQP does not support simultaneous optimization of multiple objective functions in this release.

_RHS_ID_
specifies the name of the variable that contains the right-hand-side value of each constraint. See the section "RHS Section (Optional)" on page 843 for details.

ROW
specifies the name of the constraint. See the section "ROWS Section" on page 841 for details.

TYPE
specifies the type of the constraint. _TYPE_ can take one of the following values:

- L "less than or equals" constraint
- E equality constraint
- G "greater than or equals" constraint
- R ranged constraint (both "less than or equals" and "greater than or equals")

See the sections "ROWS Section" on page 841 and "RANGES Section (Optional)" on page 844 for details.

RHS
specifies the value of the right-hand side of the constraint. It takes a missing value for a ranged constraint.

_L_RHS_
specifies the lower bound of a ranged constraint. It takes a missing value for a non-ranged constraint.

_U_RHS_
specifies the upper bound of a ranged constraint. It takes a missing value for a non-ranged constraint.

VALUE
specifies the value of the dual variable associated with the constraint.

STATUS
specifies the status of the constraint. _STATUS_ can indicate one of the following two cases:

- O The QP problem is optimal.
- I The QP problem could be infeasible or unbounded, or PROC OPTQP was not able to solve the problem.

The following values can appear only if option IIS=ON. See the section "Irreducible Infeasible Set" on page 704 for details.

- I_L The "GE" ($\geq$) condition of the constraint is needed for the IIS.
- I_U The "LE" ($\leq$) condition of the constraint is needed for the IIS.
- I_F Both conditions of the constraint are needed for the IIS (the constraint is an equality or a range constraint with conflicting bounds).

ACTIVITY

specifies the value of a constraint. In other words, the value of _ACTIVITY_ for the ith constraint is equal to $\mathbf{a}_i^{\mathrm{T}}\mathbf{x}$, where $\mathbf{a}_i$ refers to the ith row of the constraints matrix and $\mathbf{x}$ denotes the vector of current decision variable values.

Interior Point Algorithm: Overview

The interior point solver in PROC OPTQP implements an infeasible primal-dual predictor-corrector interior point algorithm. To illustrate the algorithm and the concepts of duality and dual infeasibility, consider the following QP formulation (the primal):

$$\begin{array}{rl} \min & \frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} + \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} \ge \mathbf{b} \\ & \mathbf{x} \ge \mathbf{0} \end{array}$$

The corresponding dual is as follows:

$$\begin{array}{rccccccc} \max & -\frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} & + & \mathbf{b}^{\mathrm{T}}\mathbf{y} & & & & \\ \text{subject to} & -\mathbf{Q}\mathbf{x} & + & \mathbf{A}^{\mathrm{T}}\mathbf{y} & + & \mathbf{w} & = & \mathbf{c} \\ & & & & & \mathbf{y} & \ge & \mathbf{0} \\ & & & & & \mathbf{w} & \ge & \mathbf{0} \end{array}$$

where $\mathbf{y} \in \mathbb{R}^m$ refers to the vector of dual variables and $\mathbf{w} \in \mathbb{R}^n$ refers to the vector of slack variables in the dual problem.

The dual makes an important contribution to the certificate of optimality for the primal. The primal and dual constraints combined with complementarity conditions define the first-order optimality conditions, also known as KKT (Karush-Kuhn-Tucker) conditions, which can be stated as follows:

$$\begin{array}{rcll} \mathbf{A}\mathbf{x} - \mathbf{s} & = & \mathbf{b} & \text{(primal feasibility)} \\ -\mathbf{Q}\mathbf{x} + \mathbf{A}^{\mathrm{T}}\mathbf{y} + \mathbf{w} & = & \mathbf{c} & \text{(dual feasibility)} \\ \mathbf{W}\mathbf{X}\mathbf{e} & = & \mathbf{0} & \text{(complementarity)} \\ \mathbf{S}\mathbf{Y}\mathbf{e} & = & \mathbf{0} & \text{(complementarity)} \\ \mathbf{x},\ \mathbf{y},\ \mathbf{w},\ \mathbf{s} & \ge & \mathbf{0} & \end{array}$$

where $\mathbf{e} \equiv (1, \ldots, 1)^{\mathrm{T}}$ is of appropriate dimension and $\mathbf{s} \in \mathbb{R}^m$ is the vector of primal slack variables.

NOTE: Slack variables (the s vector) are automatically introduced by the solver when necessary; it is therefore recommended that you not introduce any slack variables explicitly. This enables the solver to handle slack variables much more efficiently.

The letters $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{W}$, and $\mathbf{S}$ denote matrices with corresponding x, y, w, and s on the main diagonal and zero elsewhere, as in the following example:

$$\mathbf{X} \equiv \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_n \end{bmatrix}$$

If $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{w}^*, \mathbf{s}^*)$ is a solution of the previously defined system of equations that represent the KKT conditions, then $\mathbf{x}^*$ is also an optimal solution to the original QP model.

At each iteration the interior point algorithm solves a large, sparse system of linear equations as follows:

$$\begin{bmatrix} \mathbf{Y}^{-1}\mathbf{S} & \mathbf{A} \\ \mathbf{A}^{\mathrm{T}} & -\mathbf{Q} - \mathbf{X}^{-1}\mathbf{W} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{y} \\ \Delta\mathbf{x} \end{bmatrix} = \begin{bmatrix} \Xi \\ \Theta \end{bmatrix}$$

where $\Delta\mathbf{x}$ and $\Delta\mathbf{y}$ denote the vector of *search directions* in the primal and dual spaces, respectively, and Θ and Ξ constitute the vector of the right-hand sides.

The preceding system is known as the reduced KKT system. PROC OPTQP uses a preconditioned quasi-minimum residual algorithm to solve this system of equations efficiently.

An important feature of the interior point solver is that it takes full advantage of the sparsity in the constraint and quadratic matrices, thereby enabling it to efficiently solve large-scale quadratic programs.

The interior point algorithm works simultaneously in the primal and dual spaces. It attains optimality when both primal and dual feasibility are achieved and when complementarity conditions hold. Therefore, it is of interest to observe the following four measures where $\|v\|_2$ is the Euclidean norm of the vector v:

- relative primal infeasibility measure α:

$$\alpha = \frac{\|\mathbf{A}\mathbf{x} - \mathbf{b} - \mathbf{s}\|_2}{\|\mathbf{b}\|_2 + 1}$$

- relative dual infeasibility measure β:

$$\beta = \frac{\|\mathbf{Q}\mathbf{x} + \mathbf{c} - \mathbf{A}^{\mathrm{T}}\mathbf{y} - \mathbf{w}\|_2}{\|\mathbf{c}\|_2 + 1}$$

- relative duality gap δ:

$$\delta = \frac{|\mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} + \mathbf{c}^{\mathrm{T}}\mathbf{x} - \mathbf{b}^{\mathrm{T}}\mathbf{y}|}{|\frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} + \mathbf{c}^{\mathrm{T}}\mathbf{x}| + 1}$$

- absolute complementarity γ:

$$\gamma = \sum_{i=1}^{n} x_i w_i + \sum_{i=1}^{m} y_i s_i$$

These measures are displayed in the iteration log.

Parallel Processing

The interior point algorithm can be run in single-machine mode (in single-machine mode, the computation is executed by multiple threads on a single computer). You can specify options that control parallel processing in the PERFORMANCE statement, which is documented in the section "PERFORMANCE Statement" on page 21 in Chapter 4, "Shared Concepts and Topics."

Iteration Log for the OPTQP Procedure

The interior point solver in PROC OPTQP implements an infeasible primal-dual predictor-corrector interior point algorithm. The following information is displayed in the iteration log:

Iter	indicates the iteration number.
Complement	indicates the (absolute) complementarity.
Duality Gap	indicates the (relative) duality gap.
Primal Infeas	indicates the (relative) primal infeasibility measure.
Bound Infeas	indicates the (relative) bound infeasibility measure.
Dual Infeas	indicates the (relative) dual infeasibility measure.
Time	indicates the time elapsed (in seconds).

If the sequence of solutions converges to an optimal solution of the problem, you should see all columns in the iteration log converge to zero or very close to zero. Nonconvergence can be the result of insufficient iterations being performed to reach optimality. In this case, you might need to increase the value that you specify in the MAXITER= or MAXTIME= option. If the complementarity or the duality gap does not converge, the problem might be infeasible or unbounded. If the infeasibility columns do not converge, the problem might be infeasible.

ODS Tables

PROC OPTQP creates three Output Delivery System (ODS) tables by default. The first table, ProblemSummary, is a summary of the input QP problem. The second table, SolutionSummary, is a brief summary of the solution status. The third table, PerformanceInfo, is a summary of performance options. You can use ODS table names to select tables and create output data sets. For more information about ODS, see the *SAS Output Delivery System: User's Guide*.

If you specify a value of 2 for the PRINTLEVEL= option, then the ProblemStatistics table is produced. This table contains information about the problem data. See the section “Problem Statistics” on page 703 for more information.

If you specify the DETAILS option in the PERFORMANCE statement, then the Timing table is produced.

Table 14.5 lists all the ODS tables that can be produced by the OPTQP procedure, along with the statement and option specifications required to produce each table.

Table 14.5 ODS Tables Produced by PROC OPTQP

ODS Table Name	Description	Statement	Option
ProblemSummary	Summary of the input QP problem	PROC OPTQP	PRINTLEVEL=1 (default)
SolutionSummary	Summary of the solution status	PROC OPTQP	PRINTLEVEL=1 (default)
ProblemStatistics	Description of input problem data	PROC OPTQP	PRINTLEVEL=2
PerformanceInfo	List of performance options and their values	PROC OPTQP	PRINTLEVEL=1 (default)
Timing	Detailed solution timing	PERFORMANCE	DETAILS

A typical output of PROC OPTQP is shown in Output 14.5.

Figure 14.5 Typical OPTQP Output

The OPTQP Procedure

Performance Information	
Execution Mode	Single-Machine
Number of Threads	4

Problem Summary	
Problem Name	BANDM
Objective Sense	Minimization
Objective Function	1
RHS	ZZZZ0001
Number of Variables	472
Bounded Above	0
Bounded Below	472
Bounded Above and Below	0
Free	0
Fixed	0
Number of Constraints	305
LE (<=)	0
EQ (=)	305
GE (>=)	0
Range	0
Constraint Coefficients	2494
Hessian Diagonal Elements	25
Hessian Elements Above the Diagonal	16

Solution Summary	
Solver	QP
Algorithm	Interior Point
Objective Function	1
Solution Status	Optimal
Objective Value	16352.342037
Primal Infeasibility	1.270665E-11
Dual Infeasibility	3.556547E-16
Bound Infeasibility	0
Duality Gap	9.470938E-12
Complementarity	1.1778485E-8
Iterations	22
Presolve Time	0.00
Solution Time	0.23

You can create output data sets from these tables by using the ODS OUTPUT statement. This can be useful, for example, when you want to create a report to summarize multiple PROC OPTQP runs. The output data sets that correspond to the preceding output are shown in Output 14.6, where you can also find (in the row following the heading of each data set in the display) the variable names that are used in the table definition (template) of each table.

Figure 14.6 ODS Output Data Sets

Problem Summary

Obs	Label1	cValue1	nValue1
1	Problem Name	BANDM	.
2	Objective Sense	Minimization	.
3	Objective Function	1	.
4	RHS	ZZZZ0001	.
5			.
6	Number of Variables	472	472.000000
7	Bounded Above	0	0
8	Bounded Below	472	472.000000
9	Bounded Above and Below	0	0
10	Free	0	0
11	Fixed	0	0
12			.
13	Number of Constraints	305	305.000000
14	LE (<=)	0	0
15	EQ (=)	305	305.000000
16	GE (>=)	0	0
17	Range	0	0
18			.
19	Constraint Coefficients	2494	2494.000000
20			.
21	Hessian Diagonal Elements	25	25.000000
22	Hessian Elements Above the Diagonal	16	16.000000

Figure 14.6 *continued*

Solution Summary

Obs	Label1	cValue1	nValue1
1	Solver	QP	.
2	Algorithm	Interior Point	.
3	Objective Function	1	.
4	Solution Status	Optimal	.
5	Objective Value	16352.342037	16352
6			.
7	Primal Infeasibility	1.270665E-11	1.270665E-11
8	Dual Infeasibility	3.556547E-16	3.556547E-16
9	Bound Infeasibility	0	0
10	Duality Gap	9.470938E-12	9.470938E-12
11	Complementarity	1.1778485E-8	1.1778485E-8
12			.
13	Iterations	22	22.000000
14	Presolve Time	0.00	0
15	Solution Time	0.23	0.234001

Problem Statistics

Optimizers can encounter difficulty when solving poorly formulated models. Information about data magnitude provides a simple gauge to determine how well a model is formulated. For example, a model whose constraint matrix contains one very large entry (on the order of 10^9) can cause difficulty when the remaining entries are single-digit numbers. The PRINTLEVEL=2 option in the OPTQP procedure causes the ODS table ProblemStatistics to be generated. This table provides basic data magnitude information that enables you to improve the formulation of your models.

The example output in Output 14.7 demonstrates the contents of the ODS table ProblemStatistics.

Figure 14.7 ODS Table ProblemStatistics

The OPTQP Procedure

Problem Statistics	
Number of Constraint Matrix Nonzeros	4
Maximum Constraint Matrix Coefficient	2
Minimum Constraint Matrix Coefficient	1
Average Constraint Matrix Coefficient	1.25
Number of Linear Objective Nonzeros	2
Maximum Linear Objective Coefficient	3
Minimum Linear Objective Coefficient	2
Average Linear Objective Coefficient	2.5
Number of Lower Triangular Hessian Nonzeros	1
Number of Diagonal Hessian Nonzeros	2
Maximum Hessian Coefficient	20
Minimum Hessian Coefficient	2
Average Hessian Coefficient	6.75
Number of RHS Nonzeros	2
Maximum RHS	100
Minimum RHS	1
Average RHS	50.5
Maximum Number of Nonzeros per Column	2
Minimum Number of Nonzeros per Column	2
Average Number of Nonzeros per Column	2
Maximum Number of Nonzeros per Row	2
Minimum Number of Nonzeros per Row	2
Average Number of Nonzeros per Row	2

Irreducible Infeasible Set

For a quadratic programming problem, an irreducible infeasible set (IIS) is an infeasible subset of constraints and variable bounds that becomes feasible if any single constraint or variable bound is removed. It is possible to have more than one IIS in an infeasible QP. Identifying an IIS can help isolate the structural infeasibility in a QP. The IIS=ON option directs the OPTQP procedure to search for an IIS in a specified QP.

Whether a quadratic programming problem is feasible or infeasible is determined by its constraints and variable bounds, which have nothing to do with its objective function. When you specify the IIS=ON option, the OPTQP procedure treats this problem as a linear programming problem by ignoring its objective function. Then finding IIS is the same as what PROC OPTLP does with the IIS=ON option. See the section "Irreducible Infeasible Set" on page 597 in Chapter 12, "The OPTLP Procedure," for more information about the irreducible infeasible set.

Macro Variable _OROPTQP_

The OPTQP procedure defines a macro variable named _OROPTQP_. This variable contains a character string that indicates the status of the procedure. The various terms of the variable are interpreted as follows.

STATUS

indicates the solver status at termination. It can take one of the following values:

OK	The procedure terminated normally.
SYNTAX_ERROR	Incorrect syntax was used.
DATA_ERROR	The input data were inconsistent.
OUT_OF_MEMORY	Insufficient memory was allocated to the procedure.
IO_ERROR	A problem occurred in reading or writing data.
ERROR	The status cannot be classified into any of the preceding categories.

ALGORITHM

indicates the algorithm that produced the solution data in the macro variable. This term only appears when STATUS=OK. It can take the following value:

IP	The interior point algorithm produced the solution data.

SOLUTION_STATUS

indicates the solution status at termination. It can take one of the following values:

OPTIMAL	The solution is optimal.
CONDITIONAL_OPTIMAL	The solution is optimal, but some infeasibilities (primal, dual or bound) exceed tolerances due to scaling or preprocessing.
INFEASIBLE	The problem is infeasible.
UNBOUNDED	The problem is unbounded.
INFEASIBLE_OR_UNBOUNDED	The problem is infeasible or unbounded.
ITERATION_LIMIT_REACHED	The maximum allowable number of iterations was reached.
TIME_LIMIT_REACHED	The maximum time limit was reached.
FAILED	The solver failed to converge, possibly due to numerical issues.
NONCONVEX	The quadratic matrix is nonconvex (minimization).
NONCONCAVE	The quadratic matrix is nonconcave (maximization).

OBJECTIVE

indicates the objective value obtained by the solver at termination.

PRIMAL_INFEASIBILITY

indicates the (relative) infeasibility of the primal constraints at the solution. See the section "Interior Point Algorithm: Overview" on page 698 for details.

DUAL_INFEASIBILITY
indicates the (relative) infeasibility of the dual constraints at the solution. See the section "Interior Point Algorithm: Overview" on page 698 for details.

BOUND_INFEASIBILITY
indicates the (relative) violation by the solution of the lower or upper bounds (or both). See the section "Interior Point Algorithm: Overview" on page 698 for details.

DUALITY_GAP
indicates the (relative) duality gap. See the section "Interior Point Algorithm: Overview" on page 698 for details.

COMPLEMENTARITY
indicates the (absolute) complementarity at the solution. See the section "Interior Point Algorithm: Overview" on page 698 for details.

ITERATIONS
indicates the number of iterations required to solve the problem.

PRESOLVE_TIME
indicates the time taken for preprocessing (in seconds).

SOLUTION_TIME
indicates the time (in seconds) taken to solve the problem, including preprocessing time.

NOTE: The time that is reported in PRESOLVE_TIME and SOLUTION_TIME is either CPU time or real time. The type is determined by the TIMETYPE= option.

Examples: OPTQP Procedure

This section contains examples that illustrate the use of the OPTQP procedure. Example 14.1 illustrates how to model a linear least squares problem and solve it by using PROC OPTQP. Example 14.2 and Example 14.3 explain in detail how to model the portfolio optimization and selection problems.

Example 14.1: Linear Least Squares Problem

The linear least squares problem arises in the context of determining a solution to an overdetermined set of linear equations. In practice, these equations could arise in data fitting and estimation problems. An overdetermined system of linear equations can be defined as

$$\mathbf{Ax} = \mathbf{b}$$

where $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^m$, and $m > n$. Since this system usually does not have a solution, you need to be satisfied with some sort of approximate solution. The most widely used approximation is the least squares solution, which minimizes $\|\mathbf{Ax} - \mathbf{b}\|_2^2$.

This problem is called a least squares problem for the following reason. Let $\mathbf{A}$, $\mathbf{x}$, and $\mathbf{b}$ be defined as previously. Let $k_i(x)$ be the kth component of the vector $\mathbf{Ax} - \mathbf{b}$:

$$k_i(x) = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n - b_i,\ i = 1, 2, \ldots, m$$

By definition of the Euclidean norm, the objective function can be expressed as follows:

$$\|\mathbf{Ax} - \mathbf{b}\|_2^2 = \sum_{i=1}^{m} k_i(x)^2$$

Therefore, the function you minimize is the sum of squares of m terms $k_i(x)$; hence the term least squares. The following example is an illustration of the *linear* least squares problem; that is, each of the terms k_i is a linear function of x.function $\sum_{ij} a_{ij}x_j$ plus a constant, $-b_i$.

Consider the following least squares problem defined by

$$\mathbf{A} = \begin{bmatrix} 4 & 0 \\ -1 & 1 \\ 3 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

This translates to the following set of linear equations:

$$4x_1 = 1, \quad -x_1 + x_2 = 0, \quad 3x_1 + 2x_2 = 1$$

The corresponding least squares problem is

$$\text{minimize} \quad (4x_1 - 1)^2 + (-x_1 + x_2)^2 + (3x_1 + 2x_2 - 1)^2$$

The preceding objective function can be expanded to

$$\text{minimize} \quad 26x_1^2 + 5x_2^2 + 10x_1x_2 - 14x_1 - 4x_2 + 2$$

In addition, you impose the following constraint so that the equation $3x_1 + 2x_2 = 1$ is satisfied within a tolerance of 0.1:

$$0.9 \leq 3x_1 + 2x_2 \leq 1.1$$

You can create the QPS-format input data set by using the following SAS statements:

```
data lsdata;
   input field1 $ field2 $ field3 $ field4 field5 $ field6 @;
   datalines;
NAME      .      LEASTSQ     .           .         .
ROWS      .      .           .           .         .
N         OBJ    .           .           .         .
G         EQ3    .           .           .         .
COLUMNS   .      .           .           .         .
.         X1     OBJ        -14          EQ3       3
.         X2     OBJ        -4           EQ3       2
RHS       .      .           .           .         .
.         RHS    OBJ        -2           EQ3       0.9
RANGES    .      .           .           .         .
.         RNG    EQ3         0.2         .         .
BOUNDS    .      .           .           .         .
FR        BND1   X1          .           .         .
FR        BND1   X2          .           .         .
QUADOBJ   .      .           .           .         .
.         X1     X1          52          .         .
.         X1     X2          10          .         .
.         X2     X2          10          .         .
ENDATA    .      .           .           .         .
;
```

The decision variables x_1 and x_2 are free, so they have bound type FR in the BOUNDS section of the QPS-format data set.

You can use the following SAS statements to solve the least squares problem:

```
proc optqp data=lsdata
   printlevel = 0
   primalout = lspout;
run;
```

The optimal solution is displayed in Output 14.1.1.

Output 14.1.1 Solution to the Least Squares Problem

Primal Solution

Obs	Objective Function ID	RHS ID	Variable Name	Variable Type	Linear Objective Coefficient	Lower Bound	Upper Bound	Variable Value	Variable Status
1	OBJ	RHS	X1	F	-14	-1.7977E308	1.7977E308	0.23810	O
2	OBJ	RHS	X2	F	-4	-1.7977E308	1.7977E308	0.16190	O

The iteration log is shown in Output 14.1.2.

Output 14.1.2 Iteration Log

```
NOTE: The problem LEASTSQ has 2 variables (2 free, 0 fixed).
NOTE: The problem has 1 constraints (0 LE, 0 EQ, 0 GE, 1 range).
NOTE: The problem has 2 constraint coefficients.
NOTE: The objective function has 2 Hessian diagonal elements and 1 Hessian
      elements above the diagonal.
NOTE: The QP presolver value AUTOMATIC is applied.
NOTE: The QP presolver removed 0 variables and 0 constraints.
NOTE: The QP presolver removed 0 constraint coefficients.
NOTE: The presolved problem has 2 variables, 1 constraints, and 2 constraint
      coefficients.
NOTE: The QP solver is called.
NOTE: The Interior Point algorithm is used.
NOTE: The deterministic parallel mode is enabled.
NOTE: The Interior Point algorithm is using up to 4 threads.
                                         Primal        Bound         Dual
      Iter  Complement Duality Gap       Infeas       Infeas       Infeas    Time
         0  1.9181E-02  5.8936E-03  1.9637E-08  0.0000E+00  3.5390E-04       0
         1  9.0486E-04  2.8311E-04  8.6896E-10  1.1565E-17  1.3055E-05       0
         2  1.5370E-05  4.9441E-06  6.4151E-11  2.3130E-17  1.3055E-07       0
         3  1.5357E-07  4.9397E-08  1.7428E-12  5.7824E-18  1.3056E-09       0
NOTE: Optimal.
NOTE: Objective = 0.0095238095.
NOTE: The Interior Point solve time is 0.02 seconds.
NOTE: The data set WORK.LSPOUT has 2 observations and 9 variables.
```

Example 14.2: Portfolio Optimization

Consider a portfolio optimization example. The two competing goals of investment are (1) long-term growth of capital and (2) low risk. A good portfolio grows steadily without wild fluctuations in value. The Markowitz model is an optimization model for balancing the return and risk of a portfolio. The decision variables are the amounts invested in each asset. The objective is to minimize the variance of the portfolio's total return, subject to the constraints that (1) the expected growth of the portfolio reaches at least some target level and (2) you do not invest more capital than you have.

Let $x_1, \ldots, x_n$ be the amount invested in each asset, $\mathcal{B}$ be the amount of capital you have, $\mathbf{R}$ be the random vector of asset returns over some period, and $\mathbf{r}$ be the expected value of $\mathbf{R}$. Let G be the minimum growth you hope to obtain, and $\mathcal{C}$ be the covariance matrix of $\mathbf{R}$. The objective function is $\mathrm{Var}\left(\sum_{i=1}^{n} x_i R_i\right)$, which can be equivalently denoted as $\mathbf{x}^{\mathrm{T}}\mathcal{C}\mathbf{x}$.

Assume, for example, $n = 4$. Let $\mathcal{B} = 10{,}000$, $G = 1000$, $\mathbf{r} = [0.05, -0.2, 0.15, 0.30]$, and

$$\mathcal{C} = \begin{bmatrix} 0.08 & -0.05 & -0.05 & -0.05 \\ -0.05 & 0.16 & -0.02 & -0.02 \\ -0.05 & -0.02 & 0.35 & 0.06 \\ -0.05 & -0.02 & 0.06 & 0.35 \end{bmatrix}$$

The QP formulation can be written as follows:

$$
\begin{array}{ll}
\min & 0.08x_1^2 - 0.1x_1x_2 - 0.1x_1x_3 - 0.1x_1x_4 + 0.16x_2^2 \\
 & -0.04x_2x_3 - 0.04x_2x_4 + 0.35x_3^2 + 0.12x_3x_4 + 0.35x_4^2
\end{array}
$$

$$
\begin{array}{lrcl}
\text{subject to} & & & \\
\text{(budget)} & x_1 + x_2 + x_3 + x_4 & \leq & 10000 \\
\text{(growth)} & 0.05x_1 - 0.2x_2 + 0.15x_3 + 0.30x_4 & \geq & 1000 \\
 & x_1,\ x_2,\ x_3,\ x_4 & \geq & 0
\end{array}
$$

The corresponding QPS-format input data set is as follows:

```
data portdata;
   input field1 $ field2 $ field3 $ field4 field5 $ field6 @;
datalines;
NAME    .         PORT      .          .          .
ROWS    .         .         .          .          .
N      OBJ.FUNC   .         .          .          .
L      BUDGET     .         .          .          .
G      GROWTH     .         .          .          .
COLUMNS .         .         .          .          .
.      X1         BUDGET    1.0        GROWTH     0.05
.      X2         BUDGET    1.0        GROWTH     -.20
.      X3         BUDGET    1.0        GROWTH     0.15
.      X4         BUDGET    1.0        GROWTH     0.30
RHS     .         .         .          .          .
.      RHS        BUDGET    10000      .          .
.      RHS        GROWTH    1000       .          .
RANGES .          .         .          .          .
BOUNDS .          .         .          .          .
QUADOBJ .          .        .           .         .
.      X1         X1        0.16       .          .
.      X1         X2        -.10       .          .
.      X1         X3        -.10       .          .
.      X1         X4        -.10       .          .
.      X2         X2        0.32       .          .
.      X2         X3        -.04       .          .
.      X2         X4        -.04       .          .
.      X3         X3        0.70       .          .
.      X3         X4        0.12       .          .
.      X4         X4        0.70       .          .
ENDATA .          .         .          .          .
;
```

Use the following SAS statements to solve the problem:

```
proc optqp data=portdata
  primalout = portpout
  printlevel = 0
  dualout   = portdout;
run;
```

The optimal solution is shown in Output 14.2.1.

Output 14.2.1 Portfolio Optimization

The OPTQP Procedure
Primal Solution

Obs	Objective Function ID	RHS ID	Variable Name	Variable Type	Linear Objective Coefficient	Lower Bound	Upper Bound	Variable Value	Variable Status
1	OBJ.FUNC	RHS	X1	N	0	0	1.7977E308	3452.86	O
2	OBJ.FUNC	RHS	X2	N	0	0	1.7977E308	0.00	O
3	OBJ.FUNC	RHS	X3	N	0	0	1.7977E308	1068.81	O
4	OBJ.FUNC	RHS	X4	N	0	0	1.7977E308	2223.45	O

Thus, the minimum variance portfolio that earns an expected return of at least 10% is $x_1 = 3452.86$, $x_2 = 0$, $x_3 = 1068.81$, $x_4 = 2223.45$. Asset 2 gets nothing, because its expected return is -20% and its covariance with the other assets is not sufficiently negative for it to bring any diversification benefits. What if you drop the nonnegativity assumption? You need to update the BOUNDS section in the existing QPS-format data set to indicate that the decision variables are free.

```
...
RANGES .          .        .          .         .
BOUNDS .          .        .          .         .
FR     BND1       X1       .          .         .
FR     BND1       X2       .          .         .
FR     BND1       X3       .          .         .
FR     BND1       X4       .          .         .
QUADOBJ .          .       .          .         .
...
```

Financially, that means you are allowed to short-sell—that is, sell low-mean-return assets and use the proceeds to invest in high-mean-return assets. In other words, you put a negative portfolio weight in low-mean assets and “more than 100%” in high-mean assets. You can see in the optimal solution displayed in Output 14.2.2 that the decision variable x_2, denoting Asset 2, is equal to -1563.61, which means short sale of that asset.

Output 14.2.2 Portfolio Optimization with Short-Sale Option

The OPTQP Procedure
Primal Solution

Obs	Objective Function ID	RHS ID	Variable Name	Variable Type	Linear Objective Coefficient	Lower Bound	Upper Bound	Variable Value	Variable Status
1	OBJ.FUNC	RHS	X1	F	0	-1.7977E308	1.7977E308	1684.35	O
2	OBJ.FUNC	RHS	X2	F	0	-1.7977E308	1.7977E308	-1563.61	O
3	OBJ.FUNC	RHS	X3	F	0	-1.7977E308	1.7977E308	682.51	O
4	OBJ.FUNC	RHS	X4	F	0	-1.7977E308	1.7977E308	1668.95	O

Example 14.3: Portfolio Selection with Transactions

Consider a portfolio selection problem with a slight modification. You are now required to take into account the current position and transaction costs associated with buying and selling assets. The objective is to find the minimum variance portfolio. In order to understand the scenario better, consider the following data.

You are given three assets. The current holding of the three assets is denoted by the vector **c** = [200, 300, 500], the amount of asset bought and sold is denoted by b_i and s_i, respectively, and the net investment in each asset is denoted by x_i and is defined by the following relation:

$$x_i - b_i + s_i = c_i, \quad i = 1, 2, 3$$

Suppose you pay a transaction fee of 0.01 every time you buy or sell. Let the covariance matrix $\mathcal{C}$ be defined as

$$\mathcal{C} \quad = \quad \begin{bmatrix} 0.027489 & -0.00874 & -0.00015 \\ -0.00874 & 0.109449 & -0.00012 \\ -0.00015 & -0.00012 & 0.000766 \end{bmatrix}$$

Assume that you hope to obtain at least 12% growth. Let **r** = [1.109048, 1.169048, 1.074286] be the vector of expected return on the three assets, and let $\mathcal{B}$=1000 be the available funds. Mathematically, this problem can be written in the following manner:

$$\begin{array}{ll} \min & 0.027489x_1^2 - 0.01748x_1x_2 - 0.0003x_1x_3 + 0.109449x_2^2 \\ & -0.00024x_2x_3 + 0.000766x_3^2 \end{array}$$

$$\begin{array}{rrcll} \text{subject to} & & & & \\ \text{(return)} & \sum_{i=1}^{3} r_i x_i & \geq & 1.12\mathcal{B} & \\ \text{(budget)} & \sum_{i=1}^{3} x_i + \sum_{i=1}^{3} 0.01(b_i + s_i) & = & \mathcal{B} & \\ \text{(balance)} & x_i - b_i + s_i & = & c_i, & i = 1, 2, 3 \\ & x_i,\ b_i,\ s_i & \geq & 0, & i = 1, 2, 3 \end{array}$$

The QPS-format input data set is as follows:

```
data potrdata;
   input field1 $ field2 $ field3 $ field4 field5 $ field6 @;
datalines;
NAME     .         POTRAN    .           .          .
ROWS     .         .         .           .          .
N        OBJ.FUNC  .         .           .          .
G        RETURN    .         .           .          .
E        BUDGET    .         .           .          .
E        BALANC1   .         .           .          .
E        BALANC2   .         .           .          .
E        BALANC3   .         .           .          .
COLUMNS  .         .         .           .          .
.        X1        RETURN    1.109048    BUDGET     1.0
.        X1        BALANC1   1.0         .          .
.        X2        RETURN    1.169048    BUDGET     1.0
.        X2        BALANC2   1.0         .          .
.        X3        RETURN    1.074286    BUDGET     1.0
.        X3        BALANC3   1.0         .          .
.        B1        BUDGET    .01         BALANC1   -1.0
.        B2        BUDGET    .01         BALANC2   -1.0
.        B3        BUDGET    .01         BALANC3   -1.0
.        S1        BUDGET    .01         BALANC1    1.0
.        S2        BUDGET    .01         BALANC2    1.0
.        S3        BUDGET    .01         BALANC3    1.0
RHS      .         .         .           .          .
.        RHS       RETURN    1120        .          .
.        RHS       BUDGET    1000        .          .
.        RHS       BALANC1   200         .          .
.        RHS       BALANC2   300         .          .
.        RHS       BALANC3   500         .          .
RANGES   .         .         .           .          .
BOUNDS   .         .         .           .          .
QUADOBJ  .         .         .           .          .
.        X1        X1        0.054978    .          .
.        X1        X2        -.01748     .          .
.        X1        X3        -.0003      .          .
.        X2        X2        0.218898    .          .
.        X2        X3        -.00024     .          .
.        X3        X3        0.001532    .          .
ENDATA   .         .         .           .          .
;
```

Use the following SAS statements to solve the problem:

```
proc optqp data=potrdata
  primalout = potrpout
  printlevel = 0
  dualout    = potrdout;
run;
```

The optimal solution is displayed in Output 14.3.1.

Output 14.3.1 Portfolio Selection with Transactions

The OPTQP Procedure
Primal Solution

Obs	Objective Function ID	RHS ID	Variable Name	Variable Type	Linear Objective Coefficient	Lower Bound	Upper Bound	Variable Value	Variable Status
1	OBJ.FUNC	RHS	X1	N	0	0	1.7977E308	397.584	O
2	OBJ.FUNC	RHS	X2	N	0	0	1.7977E308	406.115	O
3	OBJ.FUNC	RHS	X3	N	0	0	1.7977E308	190.165	O
4	OBJ.FUNC	RHS	B1	N	0	0	1.7977E308	197.584	O
5	OBJ.FUNC	RHS	B2	N	0	0	1.7977E308	106.115	O
6	OBJ.FUNC	RHS	B3	N	0	0	1.7977E308	0.000	O
7	OBJ.FUNC	RHS	S1	N	0	0	1.7977E308	0.000	O
8	OBJ.FUNC	RHS	S2	N	0	0	1.7977E308	0.000	O
9	OBJ.FUNC	RHS	S3	N	0	0	1.7977E308	309.835	O

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